 P.Sousi@statslab.cam.ac.uk

Example Sheet 1

**1**. Let $(a_n)_{n\geq 1}$ be a sequence of real numbers satisfying $a_{n+m} \leq a_n + a_m$ for all $n, m \geq 1$. Prove that the limit as $n \to \infty$ of $a_n/n$ exists and

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf\left\{\frac{a_k}{k} : k \geq 1\right\} \in [-\infty, \infty).$$

**2**. Let $\sigma_n$ be the number of self-avoiding paths of length $n$ starting from 0. Show that the limit

$$\kappa = \lim_{n\to\infty} (\sigma_n)^{1/n}$$

exists and satisfies $d \leq \kappa \leq 2d - 1$.

**3**. A self-avoiding walk $\omega$ in $\mathbb{Z}^d$ is a **self-avoiding bridge** if the $d$-th coordinate of $\omega$ is uniquely minimised by its starting point and is maximised (not necessarily uniquely) by its endpoint.
Let $b_n$ be the number of SAB of length $n$ starting from the origin and $\sigma_n$ the number of self-avoiding paths of length $n$. Let $\Omega$ be the set of all self-avoiding paths. Define

$$\chi(z) = \sum_{n\geq 0} z^n \sigma_n \quad \text{and} \quad B(z) = \sum_{n\geq 0} z^n b_n.$$

1. Show that $\chi(z) \leq z^{-1} \exp\left(2B(z) - 2\right)$ for all $z \geq 0$.
2. Let $a(z, n) = \sum_{\omega \text{ SAB}} z^{|\omega|} 1(\omega_d = n)$. Show that the following limit exists

$$a(z) = \lim_{n\to\infty} -\frac{1}{n} \log a(z, n)$$

   and that $a(z, n) \leq e^{-a(z)n}$ for all $n \geq 0$.
3. Recall $\kappa$ is the connective constant of $\mathbb{Z}^d$. Show that $a(1/\kappa) \geq 0$.
4. Show that $B((1 - \epsilon)/\kappa) \leq \epsilon^{-1}$.
5. Conclude that there exists a positive constant $c$ so that $\sigma_n \leq \exp\left(c\sqrt{n}\right) \kappa^n$ for all $n \geq 0$.

[This theorem was originally proved by Hammersley and Welsh. The proof given above was found by Hutchcroft.]

**4**. Let $\sigma_n$ denote the number of circuits of the dual lattice of $\mathbb{Z}^2$ that surround the origin and have length $n \geq 4$. Show that $\sigma_n \leq n \cdot 4^n$.

**5**. Let $\Omega = \{0, 1\}^{E(\mathbb{Z}^d)}$ be endowed with the product $\sigma$-algebra $\mathcal{F}$ generated by the cylinder sets and equipped with the product probability measure $\mathbb{P}_p$ for $p \in [0, 1]$. Let $\tau$ be a translation (other than the identity) given by $\tau(\omega)(e) = \omega(\tau^{-1}(e))$. Let $A$ be an event that is invariant under $\tau$. Show that $\mathbb{P}_p(A) \in \{0, 1\}$.

(*Hint:* As a first step show that for all $A \in \mathcal{F}$ and for all $\epsilon > 0$, there exists a cylinder set $B$ for which $\mathbb{P}_p(A \triangle B) \leq \epsilon$.)

**6**. Let $G$ be an infinite connected graph of maximal vertex degree $\Delta$. Show that the critical probabilities for bond and site percolation on $G$ satisfy

$$p_c^{\text{bond}} \leq p_c^{\text{site}} \leq 1 - (1 - p_c^{\text{bond}})^{\Delta}.$$

**7**. Consider bond percolation on the $d$-regular tree $\mathcal{T}_d$ with $d \geq 3$. Let $N$ be the number of infinite clusters. Show that for all $p < 1$ either $\mathbb{P}_p(N = \infty) = 1$ or $\mathbb{P}_p(N = 0) = 1$.

**8**. Let $f, g : [0,1] \to \mathbb{R}$ be two functions with the same type of monotonicity, i.e. either they are both increasing or both decreasing. Show that

$$\int_0^1 f(x)g(x)\,dx \geq \int_0^1 f(x)\,dx \cdot \int_0^1 g(x)\,dx.$$

**9**. (a) Let $X$ be a nonnegative random variable. Prove that

$$\mathbb{P}(X > 0) \geq \frac{(\mathbb{E}[X])^2}{\mathbb{E}[X^2]}.$$

(b) A spherically symmetric tree $T$ is a tree which has a root $\rho$ with $a_0$ children, each of which has $a_1$ children, etc. So, all vertices in generation $k$ have $a_k$ children. Let $A_n$ be the number of vertices in generation $n$, which is of course equal to $\prod_{i=0}^{n-1} a_i$. Show that the critical probability $p_c$ of $T$ satisfies

$$p_c = \frac{1}{\liminf_n A_n^{1/n}}.$$

**10**. Using the BK inequality prove that

$$\mathbb{P}_{\frac{1}{2}}\left(0 \leftrightarrow \partial[-n,n]^2\right) \geq \frac{1}{2\sqrt{n}}.$$

**11**. Show that the function $\theta(p)$ is continuous on $(p_c, 1]$.

**12**. Let $x$ and $y$ be two vertices of any graph $G$ and define $f_p(x,y) := \mathbb{P}_p(x \leftrightarrow y)$. Show that $f_p$ is continuous from the left as a function of $p$.

**13**. Let $D_n$ denote the largest diameter (in the sense of graph theory) of the open clusters of bond percolation on $\mathbb{Z}^d$ that intersect the box $[-n,n]^d$. Show that when $p < p_c$, then $D_n / \log n \to \alpha(p)$ almost surely, for some $\alpha(p) \in (0, \infty)$.

**14**. Dynamical percolation on $\mathbb{Z}^d$ is defined as follows: at time 0 start with bond percolation on $\mathbb{Z}^d$ with parameter $p$ and assign i.i.d. clocks to the edges of $\mathbb{Z}^d$ following the exponential distribution with parameter 1. When a clock rings, the edge refreshes its state to open with probability $p$ and closed with probability $1 - p$ independently of everything else. Call $I(t)$

the event that there exists an infinite connected component at time $t$. Let $p < p_c(\mathbb{Z}^d)$. Prove that for all times $t$

$$\mathbb{P}_p(I(t)) = 0.$$

Now show that

$$\mathbb{P}_p(\exists\, t:\; I(t)) = 0.$$

Next show that when $p > p_c(\mathbb{Z}^d)$, then for all $t \geq 0$

$$\mathbb{P}_p(I(t)) = 1.$$

Finally show that

$$\mathbb{P}_p(\forall\, t:\; I(t)) = 1.$$

**15** ($n$-th root trick). Let $A_1, \ldots, A_n$ be increasing events all having the same probability. Then

$$\mathbb{P}_p(A_1) \geq 1 - (1 - \mathbb{P}_p(\cup_{i=1}^n A_i))^{\frac{1}{n}}.$$

**16**. Show that if $T$ is any tree and $p < 1$, then bond percolation with parameter $p$ on $T$ has either no infinite clusters a.s. or infinitely many clusters a.s.