Part III Algebraic Geometry 2017 **P.M.H. Wilson**

Introductory Reading

[R] M. Reid, Undergraduate Algebraic Geometry, Cambridge University Press (1988).

[W] P.M.H. Wilson, Part IIB Algebraic Curves Notes, Lent 1998.
https://www.dpmms.cam.ac.uk/∼pmhw/AlgC98.pdf

Standard Reference for Commutative Algebra

[AM] M. Atiyah and I. MacDonald, Introduction to Commutative Algebra, Addison–Wesley (1969)

Main references for course

[K] G.R. Kempf, Algebraic Varieties, Cambridge University Press (1993) (primary reference).

[S] I. Shafarevich, Basic Algebraic Geometry, Springer (1974) (useful for some parts of the course).

More advanced but the standard text

[H] R. Hartshorne, Algebraic Geometry, Springer (1977).

Historical reference for material in course (beautifully written)

[FAC] J.-P. Serre: Faisceaux algébriques cohérents. Annals of Math. **61** (1955), 197-278.

Plan of course

§0 Preliminaries on classical Algebraic Geometry and commutative algebra [R, W, AM]
§1 Sheaf Theory [K, H]
§2 Construction and properties of abstract varieties [K, H]
§3 Locally free and coherent modules [K, H, FAC]
§4 Sheaf cohomology [K, H, FAC]
§5 Differentials and Riemann-Roch for curves [K, H]

§0. Preliminaries on classical Algebraic Geometry and Commutative Algebra

In this section, I shall make explicit basic concepts and results that I am assuming from elsewhere. For more details of the Algebraic Geometry, the reader is referred to my Algebraic Curves Notes [W] on the web for a rapid introduction to the basic material; more details than that will not be required. The first chapter on Sheaf Theory will take about 5 lectures and will only incidentally need any algebraic geometry, and so the reader has a couple of weeks to familiarise himself/herself with the material in §0.

A little classical algebraic geometry.
(Throughout the course, we shall take the base field k to be **algebraically closed**.)

Affine varieties: An *affine variety* $V \subseteq \mathbf{A}^n(k)$ (where, once one has chosen coordinates, $\mathbf{A}^n(k) = k^n$) is given by the vanishing of polynomials $f_1, \dots, f_r \in k[X_1, \dots, X_n]$.
If $I = \langle f_1, \dots, f_r \rangle \lhd k[X_1, \dots, X_n]$ is any ideal, we set

$$V = V(I) := \{z \in \mathbf{A}^n \;:\; f(z) = 0 \;\forall f \in I\}.$$

Projective varieties: First set $\mathbf{P}^n(k) := (k^{n+1} \setminus \{\mathbf{0}\})/k^*$ with *homogeneous coordinates* $(x_0 : x_1 : \ldots : x_n)$. A *projective variety* $V \subseteq \mathbf{P}^n$ is given by the vanishing of homogeneous polynomials $F_1, \dots, F_r \in k[X_0, X_1, \dots, X_n]$. If I is the ideal generated by the F_i (a *homogeneous* ideal, i.e. if $F \in I$, then so are all its homogeneous parts), we set

$$V = V^h(I) := \{z \in \mathbf{P}^n \;:\; F(z) = 0 \;\forall \text{ homogeneous } F \in I\}.$$

Coordinate ring of an affine variety.

If $V = V(I) \subseteq \mathbf{A}^n$, set

$$I(V) := \{f \in k[X_1, \dots, X_n] \;:\; f(x) = 0 \;\forall x \in V\}.$$

Observe: $V(I(V)) = V$ (tautology) and $I(V(I)) \supseteq \sqrt{I}$ (obvious). Recall that the radical $\sqrt{I}$ of the ideal I is defined by $f \in \sqrt{I} \iff \exists m > 0$ s.t. $f^m \in I$.

Hilbert's Nullstellensatz (note $k = \bar{k}$): $I(V(I)) = \sqrt{I}$. ([R] §3, [AM] pp 82-3).

Coordinate ring: $k[V] := k[X_1, \dots, X_n]/I(V)$. This may be regarded as the ring of polynomial functions on V, and it is a finitely generated reduced k-algebra. Recall that a k-algebra is a commutative ring containing k as a subring; it is finitely generated if it is the quotient of a polynomial ring over k, and *reduced* if $a^m = 0 \Rightarrow a = 0$.

Given an affine subvariety $W \subseteq V$, have $I(W) \supseteq I(V)$ defining an ideal of $k[V]$, also denoted $I(W) \triangleleft k[V]$.

Corollary of 0-satz: If $\mathbf{m}$ is a maximal ideal of $k[V]$, then $\mathbf{m} = \mathbf{m}_P$ for some $P \in V$, where $\mathbf{m}_P$ is the maximal ideal $\{f \in k[V] \ : \ f(P) = 0\}$.

Proof. 0-satz implies $I(V(\mathbf{m})) = \sqrt{\mathbf{m}} = \mathbf{m} \neq k[V]$. So $V(\mathbf{m}) \neq \emptyset$, since otherwise $I(V(\mathbf{m})) = k[V]$. Choose $P \in V(\mathbf{m})$; then $\mathbf{m}_p \supseteq \mathbf{m}$. Since $\mathbf{m}$ maximal, this implies $\mathbf{m}_P = \mathbf{m}$.

Observe that $\{P\} = V(\mathbf{m}_P) = V(\mathbf{m})$, and so there exists a natural bijection

$$\{\text{points of affine variety } V\} \longleftrightarrow \{\text{maximal ideals of } k[V]\} \qquad (\dagger)$$

Definition. A variety W is *irreducible* if there do not exist proper subvarieties W_1, W_2 of W with $W = W_1 \cup W_2$.

Lemma 0.1. A subvariety W of an affine variety V is irreducible $\iff$ $\mathcal{P} = I(W)$ is prime, i.e. $\iff$ $k[W]$ is an ID (integral domain).

Proof. ($\Rightarrow$) If $I(W)$ not prime, there exist $f, g \notin I(W)$ such that $fg \in I(W)$. Set $W_1 := V(f) \cap W$ and $W_2 := V(g) \cap W$; then W_1, W_2 are proper subvarieties with $W = W_1 \cup W_2$, i.e. W not irreducible.

($\Leftarrow$) If W_1, W_2 are proper subvarieties with $W = W_1 \cup W_2$, choose $f \in I(W_1) \setminus I(W)$ and $g \in I(W_2) \setminus I(W)$; then $fg \in I(W)$, i.e. $I(W)$ not prime.

For a projective variety $V \subseteq \mathbf{P}^n$, we let $I^h(V) \triangleleft k[X_0, X_1, \ldots, X_n]$ be the *homogeneous ideal* of V, by definition generated by the homogeneous polynomials vanishing on V.

Exercise. Show that a projective variety V is irreducible $\iff$ $I^h(V)$ is prime.
(($\Leftarrow$) as in (0.1), ($\Rightarrow$) by considering homogeneous parts of polynomials.)

Generalizing ($\dagger$), for V an affine variety, we have a bijection given by $W \mapsto I(W)$,

$$\{\text{irreducible subvarieties } W \text{ of an affine variety } V\} \longleftrightarrow \{\text{prime ideals of } k[V]\}.$$

Proof. Given a prime ideal $\mathcal{P} \triangleleft k[V]$, the Nullstellensatz implies $I(V(\mathcal{P})) = \sqrt{\mathcal{P}} = \mathcal{P}$ in $k[V]$, so there is an inverse map.

Projective Nullstellensatz. Suppose I is a homogeneous ideal in $k[X_0, X_1, \ldots, X_n]$ and $V = V^h(I) \subseteq \mathbf{P}^n$. The Projective Nullstellensatz ([R] p82, [W]) says:
If $\sqrt{I} \neq \langle X_0, X_1, \ldots, X_n \rangle$ (the *irrelevant* ideal), then $I^h(V) = \sqrt{I}$.

Proof. An easy deduction from the Affine Nullstellensatz, noting that I also defines an affine variety in $\mathbf{A}^{n+1}$, the *affine cone* on the projective variety $V \subseteq \mathbf{P}^n$.

Decomposition of variety into irreducible components.

For V an affine or projective variety, there is a decomposition $V = V_1 \cup \ldots \cup V_N$ with the V_i irreducible subvarieties and the decomposition is essentially unique.

Proof. Suppose V is affine (similar argument for V projective): If there does not exist such a finite decomposition in the above form, then there exists a strictly decreasing sequence of subvarieties

$$V = V_0 \supset V_1 \supset V_2 \supset \ldots.$$

(If $V = W \cup W'$, then at least one of W, W' has no such decomposition and let this be V_1; continue in same way using Countable Axiom of Choice to obtain sequence.)

Hence in $k[V]$, $0 = I(V_0) \subseteq I(V_1) \subseteq \ldots.$ Hilbert's Basis Theorem implies that there exists N such that $I(V_{N+r}) = I(V_N)$ for all $r \geq 0$. Hence $V_{N+r} = V(I(V_{N+r})) = V(I(V_N)) = V_N$ for all $r \geq 0$, a contradiction.

An easy "topological" argument ([R] Exercise 3.8, [W]) with the Zariski topology (see below) shows that the decomposition is essentially unique.

Zariski topology. Let V be a variety (affine or projective), then the *Zariski topology* is the topology on V whose closed sets are the subvarieties. This is the underlying topology for this course

We check this is a topology. Wlog take V affine. Clearly V and $\emptyset$ are closed. Observe that for ideals $(I_\alpha)_{\alpha \in A}$ of $k[V]$, we have $V(\sum_\alpha I_\alpha) = \bigcap_\alpha V(I_\alpha)$ is closed. Finally we claim for ideals I, J of $k[V]$ that $V(IJ) = V(I) \cup V(J)$ $(= V(I \cap J))$ is closed.

Proof. Clearly $V(IJ) \supseteq V(I \cap J) \supseteq V(I) \cup V(J)$. Suppose however there exists a point $P \in V(IJ) \setminus (V(I) \cup V(J))$: we can choose $f \in I$ such that $f(P) \neq 0$ and $g \in J$ such that $g(P) \neq 0$. Then $fg \in IJ$ with non-zero value at P, a contradiction.

Note that V being irreducible as a topological space corresponds to the previous definition. Also, we have a well-defined concept of connectedness.

When V is affine, we have a basis of open sets of the form $D(f)$ for $f \in k[V]$, where $D(f) := \{x \in V \ : \ f(x) \neq 0\}$; any *open* set is of the form $V \setminus V(f_1, \ldots, f_r) = \bigcup_{i=1}^r D(f_i)$. If $V = \mathbf{A}^1$, get *cofinite* topology; in fact Zariski topology is only Hausdorff for a finite set of points. For V projective, we have a basis of open sets of the form $D^h(F) = V \setminus V^h(F)$, for F a homogeneous polynomial.

Exercise. The Zariski topology is compact in the usual sense (called precompact in some terminology since it is not Hausdorff), i.e. any open cover of V has a finite subcover.

Function fields of irreducible varieties

If V is an *irreducible* affine variety, then the *field of rational functions* or the *function field* $k(V) := \text{fof}\, k[V]$. Here $k[V]$ is an integral domain and fof denotes the field of fractions. In fact, we define the *dimension* of V by $\dim V := \text{tr deg}_k\, k(V)$.

For $V \subseteq \mathbf{P}^n$ an irreducible projective variety, we define

$$k(V) := \{F/G \ : \ F, G \text{ homogeneous polynomials of the same degree, } G \notin I^h(V)\}/\sim$$

where the zero polynomial has any degree and where $F_1/G_1 \sim F_2/G_2 \iff F_1G_2 - F_2G_1 \in I^h(V)$. Need V irreducible here, i.e. $I^h(V)$ prime, to show that $\sim$ is transitive, and hence an equivalence relation.

If $V \subseteq \mathbf{P}^n$ an irreducible projective variety and U a non-empty affine piece of V (say $U = V \cap \{X_0 \neq 0\}$), then U is an affine variety, $U \subseteq \mathbf{A}^n$ with affine coordinates $x_i = X_i/X_0$ for $i = 1, \ldots, n$, the equations for U coming from those for V by "putting $X_0 = 1$". (The property of being covered by open affine varieties will in due course generalise to abstract varieties.) It is an easy check now that U is irreducible and $k(V) \cong k(U)$, the isomorphism being given by "putting $X_0 = 1$".

We say that $h \in k(V)$ is *regular* at $P \in V$ if it can be written as a quotient f/g with $f, g \in k[V], g(P) \neq 0$ (affine case), or F/G with F, G homogeneous polynomials of the same degree, $G(P) \neq 0$ (projective case).

Define $\mathcal{O}_{V,P} := \{h \in k(V) \ : \ h \text{ regular at } P\}$, the *local ring of V at P,* with maximal ideal $\mathbf{m}_{V,P} := \{h \in \mathcal{O}_{V,P} \ : \ h(P) = 0\}$, the kernel of the *evaluation map* $\mathcal{O}_{V,P} \to k$ given by evaluation at P. $\mathcal{O}_{V,P}$ is a *local ring,* i.e. $\mathbf{m}_{V,P}$ is the unique maximal ideal. Since $\mathcal{O}_{V,P} \setminus \mathbf{m}_{V,P}$ consists of units of $\mathcal{O}_{V,P}$ and any proper ideal consists of non-units, any proper ideal is contained in $\mathbf{m}_{V,P}$, and hence $\mathbf{m}_{V,P}$ is the unique maximal ideal.

Morphisms of affine varieties

For $V \subseteq \mathbf{A}^n$, $W \subseteq \mathbf{A}^m$, a morphism $\phi : V \to W$ is a map given by elements $\phi_1, \ldots, \phi_m \in k[V]$. This yields a k-algebra homomorphism $\phi^* : k[W] \to k[V]$ (where $\phi^*(f) = f \circ \phi$; so if y_j a coordinate function on W induced from polynomial Y_j, we have $\phi^*(y_j) = \phi_j$). Conversely, given a k-algebra homomorphism $\alpha : k[W] \to k[V]$, we define a morphism $\alpha^* = \psi : V \to W$ given by elements $\alpha(y_1), \ldots, \alpha(y_m) \in k[V]$. Note that $\psi(P)$ is in W, since for all $g \in I(W)$, $g(\psi(P)) = g(\alpha(y_1), \ldots, \alpha(y_m))(P) = (\alpha(g(y_1, \ldots, y_m)))(P) = 0$ since $g(y_1, \ldots, y_m) = 0$ in $k[W]$.

Observe: For $\phi : V \to W$, we have $\phi^{**} = \phi$; for $\alpha : k[W] \to k[V]$, we have $\alpha^{**} = \alpha$. For $\psi : U \to V$ also a morphism of affine varieties, we have $\phi\psi$ a morphism $U \to W$ with $(\phi\psi)^* = \psi^*\phi^*$. For $\beta : k[V] \to k[U]$ a morphism of k-algebras, we have $(\beta\alpha)^* = \alpha^*\beta^*$.

We deduce that affine varieties V, W are *isomorphic* (i.e. there is an invertible morphism between them) $V \cong W \iff k[W] \cong k[V]$ as k-algebras. Recall: the k-algebras which occur as coordinate rings are the finitely generated reduced k-algebras. So formally, there is an equivalence of categories between the category of affine varieties over k and their morphisms, and the opposite of the category of finitely generated reduced k-algebras and their morphisms, i.e. there is a contravariant equivalence between the category of affine varieties and the category of finitely generated reduced k-algebras.

Thus affine algebraic geometry over k is a branch of commutative algebra. Commutative Algebra may be interpreted as affine algebraic geometry once one has generalized varieties to *schemes.*

For (irreducible) *affine* varieties, we can reconstruct the variety (up to isomorphism) from its ring of everywhere regular rational functions by (0.2) below; for irreducible projective varieties, the only everywhere regular rational functions are the constants (see Corollary 2 to Proposition 2.2).

Lemma 0.2. For V an irreducible affine variety,

$$\{f \in k(V) \ : \ f \text{ regular everywhere}\} = k[V].$$

Proof. See Example Sheet 1.

Remark. A projective variety $V \subseteq \mathbf{P}^n$ is covered by finitely many Zariski open sets which are affine varieties, e.g. the open sets $U_i := V \cap \{X_i \neq 0\} \subseteq \mathbf{A}^n$, with affine coordinates

$X_0/X_i, \dots, X_{i-1}/X_i, X_{i+1}/X_i, \dots, X_n/X_i$. This idea of patching together affine varieties leads to the concept of general abstract varieties — definition later via sheaf theory.

A little Commutative Algebra

Let A be a commutative ring (with a 1).

Definition. A module M over A is *finitely generated* if $\exists n > 0$ and $x_1, \dots, x_n \in M$ such that $M = Ax_1 + \dots + Ax_n$ ($\iff$ M is a quotient of the free module A^n).

Nakayama's lemma ([AM] p21)
If M is a finitely generated module over a local ring $(A, \mathbf{m})$, where $\mathbf{m}$ is the unique maximal ideal of A, such that $M = \mathbf{m}M$, then $M = 0$.

A useful corollary of this is with above notation and $N \subseteq M$ a submodule with $M = \mathbf{m}M + N$, then $M = N$ (apply Nakayama to quotient module M/N).

Rings and modules of fractions. Let A be a commutative ring, $S \subseteq A$ a *multiplicative subset* (i.e. $1 \in S$ and $s, t \in S \Rightarrow st \in S$). We can define an equivalence relation $\sim$ on $A \times S$ by $(a, s) \sim (a', s') \iff t(as' - a's) = 0$ for some $t \in S$ (easy check that $\sim$ is an equivalence relation). Let a/s denote the equivalence class of (a, s) and $S^{-1}A$ the set of such classes a/s. Define addition and multiplication in the obvious way. Then $S^{-1}A$ is a commutative ring and there exists a natural ring homomorphism $\phi : A \to S^{-1}A$, namely $\phi(a) = a/1$. $S^{-1}A$ is called the *ring of fractions* of A w.r.t. S.

There is a universal property: If $g : A \to B$ is a homorphism of rings with $g(S) \subseteq U(B)$ (units of B), then $\exists!\ g' : S^{-1}A \to B$ with $g'\phi = g$ (namely $g'(a/s) = g(a)g(s)^{-1} \in B$).

$S^{-1}A$ has a 1 ($= 1/1$) and a zero ($= 0/1$). Then $a/s = 0 \iff ta = 0$ for some $t \in S$; hence $S^{-1}A = 0 \iff 1/1 = 0/1 \iff 0 \in S$.

The map $A \to S^{-1}A$ is an isomorphism $\iff$ $S \subseteq U(A)$ (for ($\Leftarrow$), take $B = A$ in universal property).

Let $T \subset A$ be the set of non divisors of zero, a multiplicative subset. Set $T^{-1}A = \mathrm{tot}(A)$, the *total ring of fractions* — we have an injection $A \hookrightarrow \mathrm{tot}(A)$. If A is an integral domain (ID), then $\mathrm{tot}(A) = \mathrm{fof}(A)$ (taking $T = A \setminus \{0\}$). For a *reducible* affine variety V, we should replace the function field $k(V)$ by the *ring* $\mathrm{Rat}(V) := \mathrm{tot}(k[V])$ of rational functions on V.

Relevant examples

(1) If $f \in A$, let $f^{\mathbf{N}} = \{1, f, f^2, \ldots\} = S$. Write A_f for $S^{-1}A$ in this case.

(2) If $\mathcal{P}$ is a prime ideal of A, then $S = A \backslash \mathcal{P}$ is a multiplicative subset. Write $A_{\mathcal{P}}$ for $S^{-1}A$, called the *localisation* of A at $\mathcal{P}$, a local ring with unique maximal ideal $\mathcal{P}A_{\mathcal{P}}$ consisting of elements a/s with $a \in \mathcal{P}$, $s \notin \mathcal{P}$ (all the other elements of $A_{\mathcal{P}}$ are units).

If now M is an A-module, $S \subseteq A$ a multiplicative subset, the *module of fractions* $S^{-1}M$ (both an A-module and an $S^{-1}A$-module) is defined analogously, with $m/s = m'/s' \iff t(s'm - sm') = 0$ for some $t \in S$. The $S^{-1}A$-module structure is defined via $(a/s).(m/t) = (am)/(st)$.

Tensor products

Definition, The tensor product $M \otimes_A N$ of A-modules M and N is an A-module equipped with an A-bilinear map $g : M \times N \to M \otimes_A N$ with the following universal property:

Given any A-bilinear map $f : M \times N \to P$, $\exists!$ morphism of A-modules $h : M \otimes_A N \to P$ which factorizes $f = hg$.

$M \otimes_A N$ is defined up to isomorphism by this property (easy application of universal property). The existence of such a module is straightforward and unenlightening (see [AM] p 24) — take the free module F over A on the set $M \times N$ and quotient out by the appropriate submodule of bilinear relations. We omit the subscript A where no confusion would result in doing so. We denote by $x \otimes y$ the image of (x, y) in $M \otimes_A N$.

Elementary properties (all proved from universal property, [AM] p 26)

If M, N, P are A-modules, there exist isomorphisms of A-modules

- $M \otimes N \cong N \otimes M$, where $x \otimes y \mapsto y \otimes x$.
- $(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$, where $(x \otimes y) \otimes z \mapsto x \otimes (y \otimes z)$.
- $(M \oplus N) \otimes P \cong (M \otimes P) \oplus (N \otimes P)$, where $(x, y) \otimes z \mapsto (x \otimes z, y \otimes z)$.
- $A \otimes M \cong M$, where $a \otimes x \mapsto ax$.

Change of ring: Given a morphism of rings $f : A \to B$ (NB $f(1) = 1$), we call B an A-algebra – this generalises previous concept of k-algebras. Given an A-algebra structure on B, $f : A \to B$, and an A-module M, set $M_B := B \otimes_A M$; this is also a B-modules in an obvious way with B acting on the first factor.

Proposition 0.2. Let M be an A-module.

(a) If $I \lhd A$ and $B = A/I$, then $B \otimes_A M \cong M/IM$.

(b) If $S \subseteq A$ is a multiplicative subset and $B = S^{-1}A$, then $B \otimes_A M \cong S^{-1}M$ (this is therefore an alternative definition).

Proof. (a) The obvious bilinear map $(A/I) \times M \to M/IM$ induces (using universal property) a morphism of A-modules $(A/I) \otimes_A M \to M/IM$, where for any $a \in A$, $x \in M$, we have $\bar{a} \otimes x \mapsto \overline{ax}$. The inverse morphism $M/IM \to (A/I) \otimes_A M$ is given by $\bar{x} \mapsto 1 \otimes x$ (check well-defined).

(b) Use universal properties of both S^{-1} and $\otimes_A$ — see [AM] p 40.

Proposition 0.3. If M, N are A-modules, $I \lhd A$, S a multiplicative subset of A, then

(a) $(A/I) \otimes_A (M \otimes_A N) \cong (M/IM) \otimes_{A/I} (N/IN)$,

(b) $S^{-1}(M \otimes_A N) \cong S^{-1}M \otimes_{S^{-1}A} S^{-1}N$.

Proof. Example Sheet 1.

For instance, if $\mathcal{P}$ a prime ideal of A, then $(M \otimes_A N)_{\mathcal{P}} \cong M_{\mathcal{P}} \otimes_{A_{\mathcal{P}}} N_{\mathcal{P}}$ (where we define $M_{\mathcal{P}} = (A \setminus \mathcal{P})^{-1}M$, etc.).

***R*-algebras.** Given a commutative ring R and R-algebras $\theta_1 : R \to A$, $\theta_2 : R \to B$, a morphism $A \to B$ of R-algebras is given by morphism of rings $f : A \to B$ such that $f\theta_1 = \theta_2$. Given R-algebras A and B, the tensor product $A \otimes_R B$ has the structure of an R-algebra:

- Multiplication given by $(a \otimes b)(a' \otimes b') = (aa') \otimes (bb')$, and extend linearly.
- The ring homomorphism $R \to A \otimes_R B$ given by $r \mapsto \theta_1(r) \otimes 1 = 1 \otimes \theta_2(r)$.

Also have R-algebra morphisms $\alpha : A \to A \otimes_R B$ and $\beta : B \to A \otimes_R B$ given by $a \mapsto a \otimes 1$, respectively $b \mapsto 1 \otimes b$. These satisfy a *universal property* that, given any R-algebra morphisms $\alpha' : A \to C$ and $\beta' : B \to C$, $\exists!$ R-algebra morphism $\phi : A \otimes_R B \to C$ such that $\alpha' = \phi\alpha$ and $\beta' = \phi\beta$. Moreover $A \otimes_R B$ is determined (up to isomorphism) by this universal property (check).

Using this, we can deduce for R-algebras A, B, C that $A \otimes_R (B \otimes_R C) \cong (A \otimes_R B) \otimes_R C$ etc. are naturally isomorphic as R-algebras (rather than just R-modules).