METRIC AND TOPOLOGICAL SPACES

Part IB of the Mathematical Tripos of Cambridge

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Course schedules

• *Metrics*: Definition and examples. Limits and continuity. Open sets and neighbourhoods. Characterizing limits and continuity using neighbourhoods and open sets. [3]

• *Topology*: Definition of a topology. Metric topologies. Further examples. Neighbourhoods, closed sets, convergence and continuity. Hausdorff spaces. Homeomorphisms. Topological and non-topological properties. Completeness. Subspace, quotient and product topologies. [3]

• *Connectedness*: Definition using open sets and integer-valued functions. Examples, including intervals. Components. The continuous image of a connected space is connected. Path-connectedness. Path-connected spaces are connected but not conversely. Connected open sets in Euclidean space are path-connected. [3]

• *Compactness*: Definition using open covers. Examples: finite sets and [0, 1]. Closed subsets of compact spaces are compact. Compact subsets of a Hausdorff space must be closed. The compact subsets of the real line. Continuous images of compact sets are compact. Quotient spaces. Continuous real-valued functions on a compact space are bounded and attain their bounds. The product of two compact spaces is compact. The compact subsets of Euclidean space. Sequential compactness. [3]

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§1 METRIC SPACES

1.1 Introduction

Consider the Euclidean space $\mathbb{R}^n$ equipped with the standard Euclidean inner product: Given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with coordinates x_i, y_i respectively, we define $(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^n x_i y_i$, sometimes denoted by the dot product $\mathbf{x} \cdot \mathbf{y}$.

From this we have the **Euclidean norm** on $\mathbb{R}^n$, $\|\mathbf{x}\| := (\mathbf{x}, \mathbf{x})^{1/2}$, representing the length of the vector $\mathbf{x}$. We have a **distance function** $d_2(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\| = \left(\sum_i (x_i - y_i)^2\right)^{1/2}$, very often written as d simply. This is an example of a metric.

> **Definition 1.1.** A **metric space** (X, d) consists of a set X and a function, called the **metric**, $d : X \times X \to \mathbb{R}$ such that, for all $P, Q, R \in X$:
>
> (i) $d(P, Q) \geqslant 0$, with equality iff $P = Q$;
> (ii) $d(P, Q) = d(Q, P)$;
> (iii) $d(P, Q) + d(Q, R) \geqslant d(P, R)$.

Condition (iii) is called the **triangle inequality**.

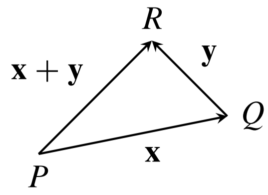
With the Euclidean metric, for any (possibly degenerate) triangle with vertices P, Q, R, the sum of the lengths of two sides of the triangle is at least the length of the third side.

Proposition 1.2. The Euclidean distance function d_2 on $\mathbb{R}^n$ is a metric in the sense of Definition 1.1.

Proof. (i) and (ii) are immediate. For (iii), use Cauchy-Schwarz inequality which says

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leqslant \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right),$$

or in inner product notation, $(\mathbf{x}, \mathbf{y})^2 \leqslant \|\mathbf{x}\|^2\|\mathbf{y}\|^2$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. (See Lemma 1.3)



We take P to be the origin, Q with position vector $\mathbf{x}$ with respect to P, and R with position vector $\mathbf{y}$ with respect to Q. Then R has position vector $\mathbf{x}+\mathbf{y}$ with respect to P. Now

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|^2 &= \|\mathbf{x}\|^2 + 2(\mathbf{x}, \mathbf{y}) + \|\mathbf{y}\|^2 \\ &\leqslant \|\mathbf{x}\|^2 + 2\|\mathbf{x}\| \cdot \|\mathbf{y}\| + \|\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2.\end{aligned}$$

This implies $d(P, R) = \|\mathbf{x}+\mathbf{y}\| \leqslant \|\mathbf{x}\| + \|\mathbf{y}\| = d(P, Q) + d(Q, R)$. ∎

Lemma 1.3. ***(Cauchy-Schwarz).*** $(\mathbf{x}, \mathbf{y})^2 \leqslant \|\mathbf{x}\|^2\|\mathbf{y}\|^2$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Proof. For $\mathbf{x} \neq \mathbf{0}$, the quadratic polynomial in the real variable λ

$$\|\lambda\mathbf{x}+\mathbf{y}\|^2 = \lambda^2\|\mathbf{x}\|^2 + 2\lambda(\mathbf{x}, \mathbf{y}) + \|\mathbf{y}\|^2$$

is non-negative for all λ. Considering the discriminant, we have $4(\mathbf{x}, \mathbf{y})^2 \leqslant 4\|\mathbf{x}\|^2\|\mathbf{y}\|^2$. ∎

Remarks.

1. In the Euclidean case, equality in the triangle inequality $\Leftrightarrow$ Q lies on the straight line segment PR. (See Example Sheet 1, Question 2)
2. The argument for Cauchy-Schwarz above generalises to integrals. For example, if f, g are continuous functions on $[0, 1]$, consider

$$\int_0^1 (\lambda f + g)^2 \geqslant 0 \Rightarrow \left(\int_0^1 fg\right)^2 \leqslant \int_0^1 f^2 \int_0^1 g^2.$$

More examples of metric spaces

(i) Let $X := \mathbb{R}^n$, and $d_1(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^n |x_i - y_i|$ or $d_\infty(\mathbf{x}, \mathbf{y}) := \max_i |x_i - y_i|$. These are both metrics.

(ii) Let X be any set, and for $x, y \in X$, define the **discrete metric** to be

$$d_{\text{disc}}(x, y) := \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

(iii) Let $X := C[0,1] := \{f : [0,1] \to \mathbb{R} \text{ where } f \text{ is continuous}\}$. We can define metrics d_1, d_2, d_∞ on X by

$$\begin{aligned} d_1(f, g) &:= \int_0^1 |f - g|, \\ d_2(f, g) &:= \left(\int_0^1 (f - g)^2 \right)^{\frac{1}{2}}, \\ d_\infty(f, g) &:= \sup_{x \in [0,1]} |f(x) - g(x)|. \end{aligned}$$

For d_2, the triangle inequality follows from Cauchy-Schwarz for integrals, i.e. $\left(\int fg\right)^2 \leqslant \left(\int f^2\right)\left(\int g^2\right)$. See Remark 2 after Lemma 1.3, and use the same argument as in Proposition 1.2.

(iv) **British rail metric.** Consider $\mathbb{R}^n$ with the Euclidean metric d, and let O denote the origin. Define a new metric ρ on $\mathbb{R}^n$ by

$$\rho(P, Q) := \begin{cases} d(P, O) + d(O, Q) & \text{if } P \neq Q, \\ 0 & \text{if } P = Q, \end{cases}$$

i.e. all journeys from P to $Q \neq P$ go via O. (All rail journeys go via London.)

Some metrics in fact satisfy a stronger triangle inequality. A metric space (X, d) is called **ultra-metric** if d satisfies condition (iii)$'$:

$$d(P, R) \leqslant \max\{d(P, Q), d(Q, R)\}$$

for all $P, Q, R \in X$.

Example. Let $X := \mathbb{Z}$ and p be a prime. The ***p*-adic metric** is defined by

$$d_p(m, n) := \begin{cases} 0 & \text{if } m = n, \\ 1/p^r & \text{if } m \neq n \text{ where } r = \max\{s \in \mathbb{N} : p^s \mid (m - n)\}. \end{cases}$$

We claim that d is an ultra metric. Indeed, suppose that $d_p(m, n) = 1/p^{r_1}$ and $d_p(n, q) = 1/p^{r_2}$ for distinct $m, n, q \in \mathbb{Z}$. Then $p^{r_1} \mid (m - n)$ and $p^{r_2} \mid (n - q)$ together imply $p^{\min\{r_1, r_2\}} \mid (m - q)$. So, for some $r \geqslant \min\{r_1, r_2\}$,

$$\begin{aligned} d_p(m, q) = 1/p^r &\leqslant 1/p^{\min\{r_1, r_2\}} \\ &= \max\{1/p^{r_1}, 1/p^{r_2}\} \\ &= \max\{d_p(m, n), d_p(n, q)\}. \end{aligned}$$

This extends to a p-adic metric on $\mathbb{Q}$: For any rational $x \neq y$, we can write $x - y = p^r m/n$, where $r \in \mathbb{Z}$, m, n are coprime to p, and define $d_p(x, y) = 1/p^r$ similarly. Then we also have $(\mathbb{Q}, d_p)$ as an ultra-metric space.

Example. The sequence $a_n := 1 + p + \cdots + p^{n-1}$, where p is prime, is convergent in $(\mathbb{Q}, d_p)$ with limit $a := (1-p)^{-1}$. This is because $d_p(a_n, a) = 1/p^n$ for all n, and so $d_p(a_n, a) \to 0$ as $n \to \infty$.

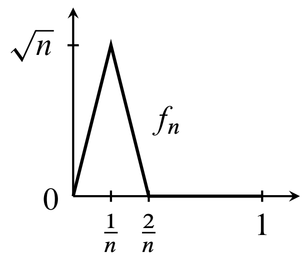
Lipschitz equivalence

Definition 1.4. Two metrics ρ_1, ρ_2 on a set X are **Lipschitz equivalent** if $\exists 0 < \lambda_1 \leqslant \lambda_2 \in \mathbb{R}$ such that $\lambda_1 \rho_1 \leqslant \rho_2 \leqslant \lambda_2 \rho_1$.

Remark. For metrics d_1, d_2, d_∞ on $\mathbb{R}^n$, one can show that $d_1 \geqslant d_2 \geqslant d_\infty \geqslant d_2/\sqrt{n} \geqslant d_1/n$, so they are all Lipschitz equivalent. (See Example Sheet 1, Question 9)

Proposition 1.5. d_1, d_∞ on $C[0,1]$ are not Lipschitz equivalent.

Proof. For $n > 1$, let $f_n \in C[0,1]$ be as follows:



$d_1(f_n, 0) =$ Area of the triangle $= 1/\sqrt{n} \to 0$ as $n \to \infty$, while $d_\infty(f_n, 0) = \sqrt{n} \to \infty$ as $n \to \infty$. ∎

Exercise. Show that $d_2(f_n, 0) = \sqrt{\frac{2}{3}}$ for all n, and so d_2 is not Lipschitz equivalent to either d_1 or d_∞ on $C[0,1]$.

1.2 Open balls and open sets

Let (X, d) be a metric space, and let $P \in X, \delta > 0$. We define the **open ball** by $B_d(P, \delta) := \{Q \in X : d(P, Q) < \delta\}$. This is often written as $B(P, \delta)$ or $B_\delta(P)$ simply.

Examples.

1. In $(\mathbb{R}, d_1)$, we get open intervals of the form $(P - \delta, P + \delta)$.
2. In $(\mathbb{R}^2, d_2)$, we get open discs of radius δ; With $(\mathbb{R}^2, d_\infty)$ we get squares; With $(\mathbb{R}^2, d_1)$, we get tilted squares. (See Figures 1 (i), (ii) and (iii) below.)
3. In $(C[0,1], d_\infty)$, see the example in Figure 1 (iv) below.
4. In (X, d_{disc}) where X is any set, $B\left(P, \frac{1}{2}\right) = \{P\}$ for all $P \in X$.

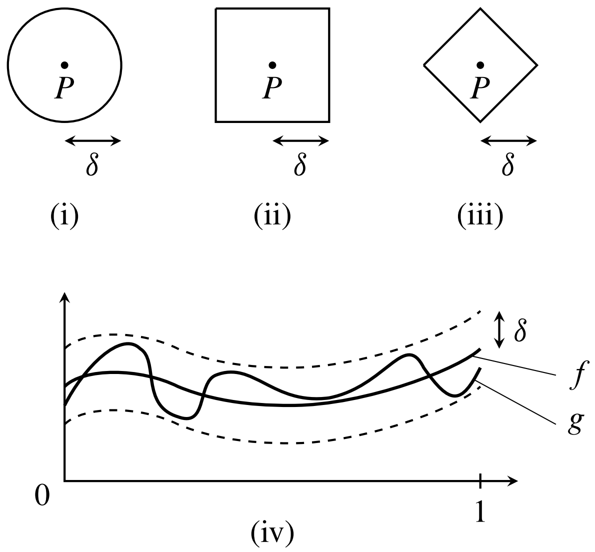
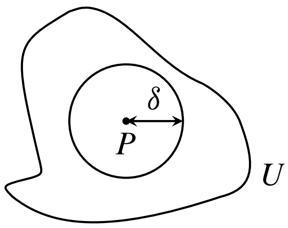


Figure 1: $B(P, \delta)$ in (i) $(\mathbb{R}^2, d_2)$, (ii) $(\mathbb{R}^2, d_\infty)$ and (iii) $(\mathbb{R}^2, d_1)$. In (iv), $g \in B(f, \delta)$, where $f, g \in C[0,1]$ and we use the metric d_∞.

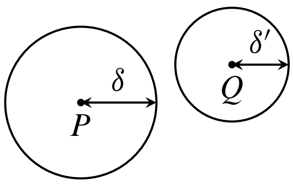
Definition 1.6. A subset $U \subseteq X$ of a metric space (X, d) is an **open subset** if $\forall P \in U, \exists$ open ball $B(P, \delta) \subseteq U$ for some $\delta > 0$.

Note that an open subset is just a union of (usually infinitely many) open balls. Here is an example of an open subset U of the space $(\mathbb{R}^2, d_2)$:



From Definition 1.6, we also define that a subset $F \subseteq X$ is **closed** if $X \setminus F$ is open.

Example. The **closed ball** $\overline{B}(P, \delta) := \{Q \in X : d(P, Q) \leqslant \delta\}$ is a closed subset of X. Indeed, we show its complement is open.



If $Q \notin \overline{B}(P, \delta)$, then $d(P, Q) > \delta$. We take some $0 < \delta' < d(P, Q) - \delta$. Suppose $R \in B(Q, \delta')$. Then using the triangle inequality, $d(P, R) \geqslant d(P, Q) - d(R, Q) > d(P, Q) - \delta' > \delta$. So $B(Q, \delta') \subseteq X \setminus \overline{B}(P, \delta)$, as required.

Lemma 1.7. Let (X, d) be a metric space. Then:

(i) X and $\emptyset$ are open subsets of (X, d);
(ii) If, $\forall i \in I$, U_i are open subsets of (X, d), then so too is $\bigcup_{i \in I} U_i$;
(iii) If $U_1, U_2 \subseteq X$ are open, then so too is $U_1 \cap U_2$.

Proof. (i) and (ii) are immediate.

For (iii): Let $P \in U_1 \cap U_2$, then $\exists$ open balls $B(P, \delta_1) \subseteq U_1$ and $B(P, \delta_2) \subseteq U_2$. Take $\delta = \min\{\delta_1, \delta_2\}$. Then $B(P, \delta) \subseteq U_1 \cap U_2$. ∎

Definition 1.8. Let P be a point in (X, d). An **open neighbourhood (nbhd)** of P is an open subset $N \ni P$, for example, the open balls centred at P.

Example. Open neighbourhoods around $P \in \mathbb{R}^2$ with the British rail metric ρ:

If $P \neq O$, and $0 < \delta < d(P, O)$, we have $B_\rho(P, \delta) = \{P\}$. If $P = O$, then $B_\rho(P, \delta)$ are the open Euclidean discs of radius δ. So $U \subseteq (\mathbb{R}^2, \rho)$ open means either $O \notin U$, then U can be arbitrary, or $O \in U$, then for some $\delta > 0$ the Euclidean disc $B_{\text{Eucl}}(O, \delta) \subseteq U$.

1.3 Limits and continuity

Suppose $x_1, x_2, \ldots$ is a sequence of points in the metric space (X, d). We define $x_n \to x \in X$, and say x_n **converges** to **limit** x, to mean $d(x_n, x) \to 0$ as $n \to \infty$. Equivalently, we have $\forall \epsilon > 0$, $\exists N$ such that $x_n \in B(x, \epsilon)$ $\forall n \geqslant N$. See an example about convergence in $(\mathbb{Q}, d_p)$ on page 5.

Example. Consider $f_n \in X := C[0, 1]$ in the proof of Proposition 1.5 on page 5. $f_n \to 0$ in (X, d_1), but not so in (X, d_2) or (X, d_∞).

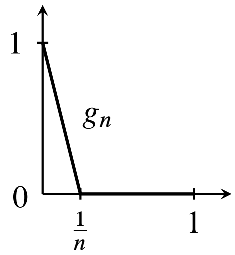
Proposition. $x_n \to x$ in $(X, d) \Leftrightarrow \forall$ open neighbourhoods $U \ni x$, $\exists N$ such that $x_n \in U$ for all $n \geqslant N$.

Proof. ($\Leftarrow$): Take $U = B(x, \epsilon)$ for any given $\epsilon > 0$.

($\Rightarrow$): Given an open set $U \ni x$, $\exists \epsilon > 0$ such that $B(x, \epsilon) \subseteq U$. So $\exists N$ such that $x_n \in B(x, \epsilon) \subseteq U$ $\forall n \geqslant N$. ∎

This means that the convergence of x_n may be rephrased solely in terms of open subsets in (X, d).

Caveat. Consider $X := C[0, 1]$, and the space (X, d_1). Let $g_n \in X$ be as follows:



Then $g_n(0) = 1\ \forall n$, but $g_n \to 0$ as $n \to \infty$. Is this not counter-intuitive?

Definition 1.9. ***(ϵ-δ definition of continuity).*** A function $f : (X, \rho_1) \to (Y, \rho_2)$ is

(i) **continuous** at $x \in X$ if $\forall \epsilon > 0, \exists \delta > 0$ such that $\rho_1(x', x) < \delta \Rightarrow \rho_2(f(x'), f(x)) < \epsilon$;

(ii) **uniformly continuous** on X if $\forall \epsilon > 0, \exists \delta > 0$ such that $\rho_1(x_1, x_2) < \delta \Rightarrow \rho_2(f(x_1), f(x_2)) < \epsilon$.

Note that item (i) of the above definition may be rephrased as: $f : (X, \rho_1) \to (Y, \rho_2)$ is continuous at $x \in X$ iff $\forall \epsilon > 0, \exists \delta > 0$ such that $f(B(x, \delta)) \subseteq B(f(x), \epsilon)$, or equivalently, $B(x, \delta) \subseteq f^{-1}(B(f(x), \epsilon)) = \{x' \in X : f(x') \in B(f(x), \epsilon)\}$.

Lemma 1.10. If $f : (X, \rho_1) \to (Y, \rho_2)$ is continuous and $x_n \to x$ in (X, ρ_1), then $f(x_n) \to f(x)$ in (Y, ρ_2).

Proof. $\forall \epsilon > 0, \exists \delta > 0$ such that $\rho_1(x', x) < \delta \Rightarrow \rho_2(f(x'), f(x)) < \epsilon$. As $x_n \to x$, $\exists N$ such that $n \geqslant N \Rightarrow \rho_1(x_n, x) < \delta$. So $n \geqslant N \Rightarrow \rho_2(f(x_n), f(x)) < \epsilon$. Therefore $f(x_n) \to f(x)$. ▮

Example. Consider the identity map id : $(C[0, 1], d_\infty) \to (C[0, 1], d_1)$. Since $d_\infty(f, g) < \epsilon \Leftrightarrow \sup_{x \in [0,1]} |f(x) - g(x)| < \epsilon \Rightarrow d_1(f, g) < \epsilon$, we see that id is continuous.

We can use the functions $f_n \in C[0, 1]$ in the proof of Proposition 1.5 on page 5 to show that the identity in the other direction, id : $(C[0, 1], d_1) \to (C[0, 1], d_\infty)$, is not continuous, again by noting that $d_1(f_n, 0) \to 0$ but $d_\infty(f_n, 0) \to \infty$ as $n \to \infty$.

We wish to express the continuity of a map purely in terms of open sets.

Proposition 1.11. A map $f : (X, \rho_1) \to (Y, \rho_2)$ is continuous $\Leftrightarrow$

(i) $\forall$ open subsets $U \subseteq Y$, $f^{-1}(U)$ are open in X; or equivalently

(ii) $\forall$ closed subsets $F \subseteq Y$, $f^{-1}(F)$ are closed in X.

Proof. (i) ($\Leftarrow$): For given $\epsilon > 0$ and $x \in X$, take $U = B(f(x), \epsilon)$. Then $f^{-1}(U)$ is open $\Rightarrow$ $\exists \delta > 0$ such that $B(x, \delta) \subseteq f^{-1}(U)$. This means $\rho_1(x', x) < \delta \Rightarrow \rho_2(f(x'), f(x)) < \epsilon$.

($\Rightarrow$): For an open subset $U \in Y$ and any $x \in f^{-1}(U)$, we can choose an open ball $B(f(x), \epsilon) \subseteq U$. Since f is continuous at x, $\exists \delta > 0$ such that $B(x, \delta) \subseteq f^{-1}(B(f(x), \epsilon)) \subseteq f^{-1}(U)$. This is true for all such x, therefore $f^{-1}(U)$ is open in X.

(ii) ($\Rightarrow$): F is closed in $Y \Leftrightarrow Y \setminus F$ is open in Y. By (i), this implies $f^{-1}(Y \setminus F)$, which is equal to $X \setminus f^{-1}(F)$, is open in $X \Leftrightarrow f^{-1}(F)$ is closed in X.

($\Leftarrow$): Suppose U is open in Y. Then $Y \setminus U$ is closed in Y so by the hypothesis, $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$ is closed in $X \Leftrightarrow f^{-1}(U)$ is open in X. By (i), f is continuous. ∎

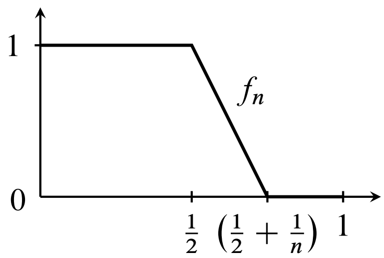
1.4 Completeness

A metric space (X, d) is called **complete** if, for all sequences of the form $x_1, x_2, \dots$ in X satisfying "$\forall \epsilon > 0, \exists N$ such that $d(x_m, x_n) < \epsilon\ \forall m, n \geqslant N$", we have $x_n \to x$ for some $x \in X$.

$(\mathbb{R}, d_1)$ is complete. This is known as Cauchy's principle of convergence. However, $(\mathbb{Q}, d_1)$ and $((0, 1) \subseteq \mathbb{R}, d_{\text{Eucl}})$ are both not complete.

Example. Let $X := C[0, 1]$. We show that (X, d_1) is not complete.

For $n > 1$, take $f_n \in X$ as follows:



Then $d_1(f_m, f_n) \leqslant \frac{1}{N}$ for $m, n \geqslant N$, so f_n form a Cauchy sequence. Suppose $f_n \to f \in X$ as $n \to \infty$, i.e. $\int_0^1 |f_n - f| \to 0$. By the triangle inequality, $\int_0^1 |f_n - f| \geqslant \int_0^{1/2} (|f - 1| - |f_n - 1|) + \int_{1/2}^1 (|f| - |f_n|) \to \int_0^{1/2} |f - 1| + \int_{1/2}^1 |f|$, giving $\int_0^{1/2} |f - 1| = \int_{1/2}^1 |f| = 0$.

Assuming f is continuous, this means

$$f(x) = \begin{cases} 1 & \text{if } x < \frac{1}{2}, \\ 0 & \text{if } x > \frac{1}{2}, \end{cases}$$

which is discontinuous at $x = \frac{1}{2}$. Contradiction.

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§2 TOPOLOGICAL SPACES

2.1 Introduction

Consider the open subsets of a metric space with properties as described in Lemma 1.7 on page 7. These properties may be abstracted out for a definition of a topological space:

Definition 2.1. A **topological space** (X, τ) consists of a set X and a set (the "**topology**") τ of subsets of X (hence $\tau \subseteq \mathcal{P}(X)$, the power set of X), where by definition we call the elements of τ the "open" subsets, satisfying:

(i) $X, \emptyset \in \tau$;
(ii) If $U_i \in \tau \; \forall i \in I$, then $\bigcup_{i \in I} U_i \in \tau$;
(iii) If $U_1, U_2 \in \tau$, then $U_1 \cap U_2 \in \tau$.

By induction, we also get the closure property (iii) for finite intersections.

A subset $Y \subseteq X$ in a topological space (X, τ) is called closed if $X \setminus Y$ is open. Therefore we can describe a topology on a set X by specifying the closed sets in X which satisfy

(i) $X, \emptyset$ are closed;
(ii) If F_i are closed $\forall i \in I$, then so too is $\bigcap_{i \in I} F_i$;
(iii) If F_1, F_2 are closed, then so too is $F_1 \cup F_2$.

This description of a topology is sometimes more natural: Consider the non-metric topologies in examples (ii) and (iii) below.

Lemma 1.7 on page 7 implies that every metric space (X, d) gives rise to a topology. This is called the **metric topology**.

Two metrics ρ_1, ρ_2 on X are said to be **topologically equivalent** if their associated topologies are the same.

Exercise. Show that ρ_1, ρ_2 are Lipschitz equivalent $\Rightarrow$ ρ_1, ρ_2 are topologically equivalent.

Example. The discrete metric on a set X gives rise to the **discrete topology** in which every subset in X is open, i.e. $\tau = \mathscr{P}(X)$.

Examples of non-metric topologies

(i) Let X be a set with at least two elements, and $\tau := \{X, \emptyset\}$, the **indiscrete topology**.

(ii) Let X be any infinite set, and $\tau := \{\emptyset\} \cup \{Y \subseteq X : X \setminus Y \text{ is finite}\}$, the **co-finite topology**. If $X = \mathbb{R}$ or $\mathbb{C}$ then this is known as the **Zariski topology**, where open sets are "complements of zeros of polynomials". The Zariski topologies on $\mathbb{R}^n$ or $\mathbb{C}^n$ are very important in algebraic geometry.

(iii) Let X be any uncountable set e.g. $\mathbb{R}$ or $\mathbb{C}$, and $\tau := \{\emptyset\} \cup \{Y \subseteq X : X \setminus Y \text{ is countable}\}$, the **co-countable topology**.

(iv) For the set $X = \{a, b\}$, there are exactly 4 distinct topologies: $\tau := \big\{\emptyset, \{a\}, \{b\}, \{a, b\}\big\}$ i.e. the discrete / metric topology, $\tau := \big\{\emptyset, \{a, b\}\big\}$ i.e. the indiscrete topology, $\tau := \big\{\emptyset, \{a\}, \{a, b\}\big\}$, or $\tau := \big\{\emptyset, \{b\}, \{a, b\}\big\}$.

Example. The **half-open interval topology** τ on $\mathbb{R}$ consists of arbitrary unions of half-open intervals $[a, b)$ where $a < b$, $a, b \in \mathbb{R}$. Clearly $\mathbb{R}, \emptyset \in \tau$ and τ is closed under unions. Suppose $U_1, U_2 \in \tau$. We show that $\forall P \in U_1 \cap U_2$, $\exists\, [a, b)$ such that $P \in [a, b) \subseteq U_1 \cap U_2$ and hence $U_1 \cap U_2 \in \tau$.

Since $P \in U_1$ and $P \in U_2$, we have $P \in [a_1, b_1) \subseteq U_1$ and $P \in [a_2, b_2) \subseteq U_2$ for some $a_1 < b_1$, $a_2 < b_2$. Let $a = \max\{a_1, a_2\}$ and $b = \min\{b_1, b_2\}$. Then $P \in [a, b) \subseteq U_1 \cap U_2$, as required.

More definitions

Definition 2.2. Let (X, τ_1) and (Y, τ_2) be topological spaces. Let $P \in (X, \tau_1)$.

(i) An **open neighbourhood (nbhd)** of P is an open set $U \subseteq X$ with $P \in U$. (cf. Definition 1.8 on page 7.)

(ii) A sequence of points x_n **converges** to **limit** x if, for any open neighbourhood $U \ni x$, $\exists N$ such that $x_n \in U$ $\forall n \geqslant N$. (cf. Section 1.3 on page 7.)

(iii) A map $f : (X, \tau_1) \to (Y, \tau_2)$ is **continuous** if, for any open set $U \subseteq Y$, $f^{-1}(U)$ is open in X. (cf. Proposition 1.11(i) on page 8.)

The proof of Proposition 1.11(ii) on page 8 shows that f is continuous $\Leftrightarrow$ $\forall$ closed set $F \subseteq Y$, $f^{-1}(F)$ is closed in X.

Example. The identity map $\mathrm{id} : (\mathbb{R}, \tau_{\mathrm{Eucl}}) \to (\mathbb{R}$, co-finite topology) is continuous, because closed sets in the co-finite topology, namely, the finite sets and $\mathbb{R}$, are closed in the Euclidean topology.

The identity map $\mathrm{id} : (\mathbb{R}, \tau_{\mathrm{Eucl}}) \to (\mathbb{R}$, co-countable topology) is not continuous, because $\mathbb{Q} \subseteq \mathbb{R}$ is closed in the co-countable topology, but not so in the Euclidean topology.

Definition 2.3. A map $f : (X, \tau_1) \to (Y, \tau_2)$ is a **homeomorphism** if:

(i) f is bijective;

(ii) Both f and f^{-1} are continuous.

In this case, the open subsets of X correspond precisely to the open subsets of Y under the bijection f. This can be used to define an equivalence relation between topological spaces.

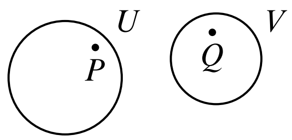
A property on topological spaces is called a **topological property** if (X, τ_1) has the property, and (X, τ_1) is homeomorphic to (Y, τ_2) implies that (Y, τ_2) has the same property.

Example. Consider the topological spaces $(\mathbb{R}, \tau_{\mathrm{Eucl}})$ and $\big((-1,1), \tau_{\mathrm{Eucl}}\big)$. Let $f : \mathbb{R} \to (-1,1)$, defined by $f(x) = x/(1+|x|)$. f is bijective with inverse $g : (-1,1) \to \mathbb{R}$ given by $g(y) = y/(1-|y|)$. Both f and g are continuous, and so f and g are homeomorphisms.

Note that $(\mathbb{R}, d_{\mathrm{Eucl}})$ is complete, whilst $\big((-1,1), d_{\mathrm{Eucl}}\big)$ is not. Therefore this example shows that 'completeness' is a property of the metric on a metric space, and not just a topological property.

However, it also shows that using the homeomorphism, we can construct a complete metric on $(-1,1)$ coming from d_{Eucl} on $\mathbb{R}$ which is topologically equivalent to d_{Eucl} on $(-1,1)$.

Definition 2.4. A topological space (X, τ) is called **Hausdorff** if $\forall P, Q \in X$ with $P \neq Q$, $\exists$ disjoint open sets $U \ni P$ and $V \ni Q$, i.e. we can separate points by open sets. This is a topological property.



Example. $\mathbb{R}$ with the indiscrete, co-finite or co-countable topology is not Hausdorff, because any two non-empty open sets intersect non-trivially (see the Examples (i), (ii) and (iii) on page 10). But any metric space is clearly Hausdorff. So these topologies are non-metric.

Example. The half-open interval topology on $\mathbb{R}$ is Hausdorff. If $a, b \in \mathbb{R}$ with $a < b$, then take $[a, b) \ni a$, $[b, b+1) \ni b$ will do. Example Sheet 2, Question 18 shows that this topology is non-metric.

Definition 2.5. In a topological space X, for any $A \subseteq X$, we say that $x_0 \in X$ is an **accumulation point** or **limit point** of A if any open neighbourhood U of x_0 satisfies $U \cap A \neq \emptyset$.

Lemma 2.6. $A \subseteq X$ is closed $\Leftrightarrow$ A contains all of its accumulation points, i.e. if $x_0 \in X$ is an accumulation point of A then $x_0 \in A$.

Proof. ($\Rightarrow$): Suppose A is closed and $x_0 \in X \setminus A$. Then take $U := X \setminus A$, an open neighbourhood of x_0. Now $U \cap A = \emptyset$, so x_0 is not an accumulation point.

($\Leftarrow$): Suppose A is not closed, then $X \setminus A$ is not open. $\exists x_0 \in X \setminus A$ such that no open neighbourhood U of x_0 is contained in $X \setminus A$, i.e. any open neighbourhood U of x_0 satisfies $U \cap A \neq \emptyset$. So x_0 is an accumulation point of A but not in A. ∎

Remark. Suppose we have a convergent sequence $x_n \to x \in X$ with $x_n \in A\ \forall n$. Then for any open neighbourhood U of x, $\exists N$ such that $x_n \in U\ \forall n \geqslant N$. So x is an accumulation point of A. Therefore if A is closed, we must have $x \in A$.

A topological space (X, τ) has a countable base of open neighbourhoods (or is **first countable**) if, $\forall P \in X$, $\exists$ open neighbourhoods $N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots$ of P with the property that, $\forall$ open neighbourhood U of P, $\exists m$ such that $N_i \subseteq U\ \forall i \geqslant m$.

Example. Any metric space is first countable: Given $P \in X$, take the open balls $B\left(P, \frac{1}{n}\right)$ for $n \in \mathbb{N}$.

Lemma 2.7. If (X, τ) is first countable, and $A \subseteq X$ has the property that "$\forall$ convergent sequences $x_n \to x \in X$ with $x_n \in A\ \forall n$, the limit x is in A", then A is closed.

Proof. By Lemma 2.6, it suffices to prove that any accumulation point x of A is the limit of some sequence x_n with $x_n \in A\ \forall n$. Let $N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots$ be a base of open neighbourhoods for x. Then $\forall i$, since x is an accumulation point, $\exists x_i \in A \cap N_i$, giving the sequence (x_i). $\forall$ open neighbourhood U of x, by first countability $\exists m$ such that $N_i \subseteq U\ \forall i \geqslant m$. So $x_i \in U$ $\forall i \geqslant m$, i.e. $x_n \to x$. ∎

Remark. Example Sheet 1, Question 19 (revised version on the DPMMS website) shows that we do need the first countability condition here.

2.2 Interiors and closures

Given any $A \subseteq X$, we define the **interior** $\mathrm{Int}(A)$ or A^0 of A to be the union of all open subsets contained in A. $\mathrm{Int}(A)$ is the *largest* open subset contained in A: If U is open and $U \subseteq A$, then $U \subseteq \mathrm{Int}(A)$.

Define the **closure** $\mathrm{Cl}(A)$ or $\overline{A}$ of A to be the intersection of all closed sets containing A. $\mathrm{Cl}(A)$ is the *smallest* closed set containing A: If F is closed and $A \subseteq F$, then $\mathrm{Cl}(A) \subseteq F$.

> *Example.* Consider $\mathbb{R}$ with the Euclidean topology. $\mathrm{Int}(\mathbb{Q}) = \emptyset$ (because any non-empty open subset contains irrationals) and $\mathrm{Cl}(\mathbb{Q}) = \mathbb{R}$. Also, $\mathrm{Int}([0, 1]) = (0, 1)$ and $\mathrm{Cl}((0, 1)) = [0, 1]$.

For any set, we have $\mathrm{Int}(A) \subseteq A \subseteq \mathrm{Cl}(A)$. The **boundary** or **frontier** of A is defined to be $\partial A := \mathrm{Cl}(A) \setminus \mathrm{Int}(A)$.

A set $A \subseteq X$ is called **dense** if $\mathrm{Cl}(A) = X$.

If $A \subseteq B$ are subsets of X, then $\mathrm{Int}(A) \subseteq \mathrm{Int}(B)$ and $\mathrm{Cl}(A) \subseteq \mathrm{Cl}(B)$.

Proposition 2.8. Let $A \subseteq X$. Then:

(i) $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))) = \mathrm{Int}(\mathrm{Cl}(A))$;
(ii) $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))) = \mathrm{Cl}(\mathrm{Int}(A))$.

Proof. (i): Since $\mathrm{Int}(\mathrm{Cl}(A))$ is open and $\mathrm{Int}(\mathrm{Cl}(A)) \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$, taking interiors of both sides we have $\mathrm{Int}(\mathrm{Cl}(A)) \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))))$. Since $\mathrm{Cl}(A)$ is closed and $\mathrm{Int}(\mathrm{Cl}(A)) \subseteq \mathrm{Cl}(A)$, taking closures then interiors of both sides we have $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))) \subseteq \mathrm{Int}(\mathrm{Cl}(A))$.

(ii): Similar argument. See Example Sheet 1, Question 11. ∎

This shows that if we start from an arbitrary set $A \subseteq X$ and take successive interiors and closures, we may obtain at most 7 distinct sets, namely, A, $\mathrm{Int}(A)$, $\mathrm{Cl}(A)$, $\mathrm{Cl}(\mathrm{Int}(A))$, $\mathrm{Int}(\mathrm{Cl}(A))$, $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ and $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$. There is such an A where all 7 of these are distinct: See Example Sheet 1, Question 11.

2.3 Base of open subsets for a topology

Given a topological space (X, τ), we say that a collection $\mathcal{B} := \{U_i\}_{i \in I}$ of "basic" open sets form a **base** or **basis** for the topology if any open set is the union of elements of $\mathcal{B}$.

Question: When does an arbitrary collection $\mathcal{B} := \{U_i\}_{i \in I}$ of subsets of a set X form the base for some topology on X? *Answer*: If, $\forall i, j$, $U_i \cap U_j$ is the union of some U_k's in $\mathcal{B}$. If so, we can define a topology by specifying that an open set is just a union of some U_i's in $\mathcal{B}$ (and also include X, if necessary).

A topological space (X, τ) is called **second countable** if it has a countable base of open sets. (X, τ) is second countable $\Rightarrow$ (X, τ) is first countable, because $\forall P \in X$ and open neighbourhood U of P, U is the union of some basic open sets and so $\exists U_i \subseteq U$ with $P \in U_i$ $\forall i \in \mathbb{N}$.

2.4 Subspaces, quotients and products

Let (X, τ) be a topological space, and $Y \subseteq X$. The **subspace topology** on Y is $\tau|_Y := \{U \cap Y : U \in \tau\}$. Consider the inclusion map $i : Y \hookrightarrow X$. It is continuous (because $i^{-1}(U) = U \cap Y$), and the subspace topology is the *smallest* topology on which i is continuous.

Proposition 2.9.

(i) Let $\mathscr{B}$ be a base for the topology τ. Then $\mathscr{B}|_Y := \{U \cap Y : U \in \mathscr{B}\}$ is a base for the subspace topology;

(ii) Let (X, ρ) be a metric space and ρ_1 be the restriction of the metric to Y. Then the subspace topology on Y induced from the ρ-metric topology on X is the same as the ρ_1-metric topology on Y.

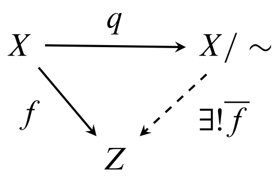
Proof. (i): Any open set $U \subseteq X$ is $U = \bigcup_\alpha U_\alpha$ where $U_\alpha \in \mathscr{B}$, so $U \cap Y = \bigcup_\alpha (U_\alpha \cap Y)$.

(ii): A base for the metric topology on X is given by open balls $B_\rho(x, \delta)$ for a fixed $x \in X$. $\forall y \in B_\rho(x, \delta) \cap Y, \exists \delta' > 0$ such that $B_\rho(y, \delta') \subseteq B_\rho(x, \delta)$, then $B_{\rho_1}(y, \delta') = B_\rho(y, \delta') \cap Y \subseteq B_\rho(x, \delta) \cap Y$. Therefore $B_\rho(x, \delta) \cap Y$, a base for the subspace topology, is the union of open ρ_1-balls. ∎

Example. $(0, 1] \subseteq \mathbb{R}$ with the Euclidean topology has a base for the subspace topology consisting of open intervals (a, b) and half-open intervals $(a, 1]$, where $0 < a < b < 1$.

Let (X, τ) be a topological space and $\sim$ be an equivalence relation on X. Denote the quotient set by $Y := X/\sim$ and the quotient map by $q : X \to Y$ where $x \mapsto [x]$. The **quotient topology** on Y is then given by $\{U \subseteq Y : q^{-1}(U) \in \tau\}$, i.e. the subsets U of the quotient set Y, for which the union of the equivalence classes in X corresponding to points of U is an open subset of X.

Remark. q is continuous, and the quotient topology is the largest topology on Y for which this is so. If $f : X \to Z$ is any continuous map between topological spaces such that $x \sim y \Rightarrow f(x) = f(y)$, then there is a unique factorisation and $\overline{f}$, defined by $\overline{f}([x]) = f(x)$, is continuous. $(q^{-1}(\overline{f}^{-1}(U)) = f^{-1}(U)$ is open in $X \Rightarrow \overline{f}$ is continuous.)



Examples.

(i) Define $\sim$ on $\mathbb{R}$ by $x \sim y$ iff $x - y \in \mathbb{Z}$. The map $\phi : \mathbb{R}/\sim \to \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, where $[x] \mapsto e^{2\pi i x}$, is well-defined and a homeomorphism. (See Example Sheet 1, Question 15).

(ii) Define the 2-D torus T to be $\mathbb{R}^2/\sim$ where $(x_1, y_1) \sim (x_2, y_2)$ iff $x_1 - x_2 \in \mathbb{Z}$ and $y_1 - y_2 \in \mathbb{Z}$. The topology in fact comes from a metric on T (See Example Sheet 1, Question 18) and hence well-behaved.

In general, one can get rather nasty (non-Hausdorff, for instance) topologies from an arbitrary equivalence relation.

If $A \subseteq X$ we can define $\sim$ on X by $x \sim y$ iff $x = y$ or $x, y \in A$. The quotient space is sometimes written as X/A, in which A is scrunched down to a point. Usually closed A is taken.

Example. Let D to be the closed unit disc in $\mathbb{C}$, with boundary C, the unit circle. Then D/C is homeomorphic to S^2, the 2-sphere. (See Example Sheet 2, Question 13.)

Proposition 2.10. Suppose (X, τ) is Hausdorff and $A \subseteq X$ is closed, and $\forall x \in X \setminus A$, $\exists$ open sets U, V with $U \cap V = \emptyset$, $A \subseteq U$, $x \in V$, then X/A is Hausdorff.

Proof. Given any two points $\overline{x} \neq \overline{y}$ in X/A, either one of the following holds:

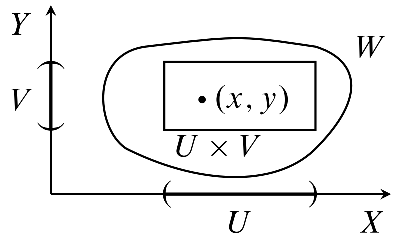
(i) Neither $\overline{x}$ or $\overline{y}$ correspond to A: then $\exists! x, y \in X$ corresponding to $\overline{x}, \overline{y}$. Now $\exists U_x \supseteq A$ and $V_x \ni x$ such that $U_x \cap V_x = \emptyset$. $\exists U_y \supseteq A$ and $V_y \ni y$ such that $U_y \cap V_y = \emptyset$. Since X is Hausdorff, wlog we assume $V_x \cap V_y = \emptyset$. The corresponding open sets $q(V_x), q(V_y)$ in X/A separate $\overline{x}$ and $\overline{y}$.

(ii) $\overline{x} = q(x)$ where $x \in X \setminus A$ and $\overline{y}$ corresponds to A. Then $\exists$ open sets $U \supseteq A$ and $V \ni x$ such that $U \cap V = \emptyset$. The corresponding open sets $q(U), q(V) \in X/A$ separate $\overline{x}$ and $\overline{y}$. ∎

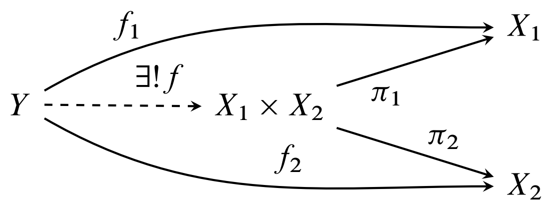
Given topological spaces (X, τ) and (Y, σ), we define the **product topology** $\tau \times \sigma$ on $X \times Y$ by $W \subseteq X \times Y$ is open iff $\forall (x, y) \in W$, $\exists$ open sets $U \subseteq X$ and $V \subseteq Y$ such that $x \in U$, $y \in V$ and $U \times V \subseteq W$. A base of open sets for $(X \times Y, \tau \times \sigma)$ consists of sets $U \times V$, where U is open in X and V is open in Y. Note that $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$, so this does define a base for the topology.

Similarly if $(X_i, \tau_i)_{i=1}^n$ are topological spaces, the product topology on $X_1 \times \cdots \times X_n$ has a base of open sets of the form $U_1 \times \cdots \times U_n$, where U_i is open in X_i.

Example. Consider $\mathbb{R}$ with the Euclidean topology. The product topology for $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is just the metric topology, where open rectangles $I_1 \times I_2$, with I_1 and I_2 being open intervals in $\mathbb{R}$, form its base and also a base for the Euclidean topology on $\mathbb{R}^2$.



Lemma 2.11. Given topological spaces (X_1, τ_1) and (X_2, τ_2), the projection maps $\pi_i : (X_1 \times X_2, \tau_1 \times \tau_2) \to (X_i, \tau_i)$ are continuous. Given a topological space (Y, τ) and continuous maps $f_i : Y \to X_i$, there is a unique factorisation, and f (in the diagram) is continuous.



Proof. For the first part, use $\pi_1^{-1}(U) = U \times X_2$ and $\pi_2^{-1}(V) = X_1 \times V$. For the second part, we have $f(y) = (f_1(y), f_2(y))$. $\forall$ basic open set $U \times V \subseteq X_1 \times X_2$, where U is open in X_1 and V is open in X_2, $f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$ is open in Y. Since any any open set W in $X_1 \times X_2$ is the union of basic open sets, $f^{-1}(W)$ is the union of their inverses, and so $f^{-1}(W)$ is open in Y. Therefore f is continuous. ∎

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§3 CONNECTEDNESS

3.1 Various results

Definition 3.1. A topological space X is **disconnected** if $\exists$ non-empty open subsets U, V such that $U \cap V = \emptyset$ and $U \cup V = X$. We say U, V disconnect X. Otherwise X is **connected**.

From this definition, we have X is connected iff $\forall$ open sets U, V where $U \cap V = \emptyset$ and $U \cup V = X$, either $U = \emptyset$ (whence $V = X$) or $V = \emptyset$ (whence $U = X$). Connectedness is a topological property.

If $Y \subseteq X$, then Y is disconnected in the subspace topology iff $\exists$ open sets $U, V \subseteq X$ such that $U \cap Y \neq \emptyset$, $V \cap Y \neq \emptyset$, $U \cap V \cap Y = \emptyset$ and $Y \subseteq U \cup V$. We say that U, V disconnect Y.

Proposition 3.2. Let X be a topological space. Then the following are equivalent:

(i) X is connected;
(ii) The only subsets of X which are both open and closed are $\emptyset, X$;
(iii) Every continuous function $f : X \to \mathbb{Z}$ is constant.

Proof. (i) $\Leftrightarrow$ (ii): Trivial. For $U \subseteq X$, consider $U \cup (X \setminus U) = X$.

Not (iii) $\Rightarrow$ Not (i): Suppose $\exists$ non-constant $f : X \to \mathbb{Z}$ which is continuous, then $\exists m, n \in f(X)$ with $m < n$. $f^{-1}(\{k : k \leqslant m\})$ and $f^{-1}(\{k : k > m\})$ are non-empty, open sets disconnecting X.

Not (i) $\Rightarrow$ Not (iii): $\exists$ non-empty disjoint open sets U, V such that $U \cup V = X$. Let $f : X \to \mathbb{Z}$ where $f(x) = 0$ if $x \in U$, $f(x) = 1$ if $x \in V$. This is continuous but non-constant. (It is locally constant, however.) ∎

Proposition 3.3. Continuous images of a connected space are connected.

Proof. If $f : X \to Y$ is a continuous surjective map between topological spaces, and if U, V disconnect Y, then $f^{-1}(U), f^{-1}(V)$ disconnect X. So X is connected $\Rightarrow$ Y is connected. ∎

Connectedness in $\mathbb{R}$

A subset $I \subseteq \mathbb{R}$ is called an **interval** if given $x, z \in I$ where $x \leqslant z$, we have $y \in I$ $\forall y$ satisfying $x \leqslant y \leqslant z$. There are the cases $\inf I = a \in \mathbb{R}$ and $a \in I$, or $a \notin I$, or $\inf I = -\infty$. Similarly we have the cases $\sup I = b \in \mathbb{R}$ and $b \in I$, or $b \notin I$, or $\sup I = \infty$. So any interval I takes the form $[a, b]$, $[a, b)$, $(a, b]$, (a, b), $[a, \infty)$, (a, ∞), $(-\infty, b]$, $(-\infty, b)$ or $(-\infty, \infty)$.

Theorem 3.4. A subset of $\mathbb{R}$ is connected $\Leftrightarrow$ it is an interval.

Proof. ($\Rightarrow$): If $X \subseteq \mathbb{R}$ is not an interval, $\exists x, y, z$ where $x < y < z$, and $x, z \in X$ but $y \notin X$. Then $(-\infty, y)$ and (y, ∞) disconnect X.

($\Leftarrow$): Let I be an interval and U, V be open subsets of $\mathbb{R}$ disconnecting I. $\exists u \in U \cap I$ and $v \in V \cap I$ where, wlog, $u < v$. Since I is an interval, $[u, v] \subseteq I$. Let $s = \sup([u, v] \cap U)$.

If $s \in U$, then $s \neq v$ so $s < v$ and, since U is open, $\exists \delta > 0$ such that $(s - \delta, s + \delta) \subseteq U$. So $\exists s' \in [u, v] \cap U$ with $s' > s$. This contradicts s being an upper bound.

If $s \in V$, then $\exists \delta > 0$ such that $(s - \delta, s + \delta) \subseteq V$. In particular, $(s - \delta, s + \delta) \cap U = \emptyset$, so $[u, v] \cap U \subseteq [u, s - \delta]$. This contradicts s being the least upper bound. ∎

Corollary 3.5. ***(Intermediate value theorem).*** Let $a < b$ and $f : [a, b] \to \mathbb{R}$ be continuous. If $y \in [f(a), f(b)]$ (or $[f(b), f(a)]$) then $\exists x \in [a, b]$ such that $f(x) = y$.

Proof. Suppose not, then $f^{-1}\big((-\infty, y)\big)$ and $f^{-1}\big((y, \infty)\big)$ disconnect $[a, b]$. ∎

Connected subsets and subspaces

Proposition 3.6. Given connected subspaces $\{Y_\alpha\}_{\alpha \in A}$ of a topological space X, where $Y_\alpha \cap Y_\beta \neq \emptyset$ $\forall \alpha, \beta \in A$, the union $Y = \bigcup_\alpha Y_\alpha$ is connected.

Proof. Using Proposition 3.2(iii), it suffices to prove that any continuous function $f : Y \to \mathbb{Z}$ is constant. Using the same proposition, $f_\alpha := f|_{Y_\alpha}$ is constant, equal to n_α, say, on Y_α $\forall \alpha \in A$. Since $\forall \beta \neq \alpha$, $\exists z \in Y_\alpha \cap Y_\beta$, we have $n_\alpha = f(z) = n_\beta$. Therefore f is constant on Y. ∎

A **connected component** of a topological space X is a maximal connected subset $Y \subseteq X$, i.e. if $Z \subseteq X$ is connected and $Z \supseteq Y$ then $Z = Y$.

Each point $x \in X$ is contained in a unique connected component of X, namely, $\bigcup\{Z \subseteq X : Z$ is connected and $x \in Z\}$. The fact that this is connected is a result of Proposition 3.6.

Example. Let $X := \{0\} \cup \left\{\frac{1}{n} : n = 1, 2, \ldots\right\} \subseteq \mathbb{R}$, with the subspace topology. The connected component containing $\frac{1}{n}$ is $\left\{\frac{1}{n}\right\}$, both open and closed in X. The connected component containing 0 is $\{0\}$ which is closed but not open in X.

Proposition 3.7. If Y is a connected subset of a topological space X, then the closure $\overline{Y}$ is connected.

Proof. Suppose $f : \overline{Y} \to \mathbb{Z}$ is continuous. Proposition 3.2(iii) on page 16 $\Rightarrow$ $f|_Y$ is constant, equal to m, say. Given $x \in \overline{Y}$, let $n := f(x)$, then $f^{-1}(n)$ is an open neighbourhood of x in $\overline{Y}$, i.e. of the form $U \cap \overline{Y}$ where U is open in X. Since $U \cap \overline{Y} \neq \emptyset$, (considering $\overline{Y} = X \setminus \operatorname{Int}(X \setminus Y)$) it contains a point $z \in Y$ at which $f(z) = m$, so $n = m$, i.e. f is constant on $\overline{Y}$. ∎

This shows that connected components of a topological space must be closed (because they are *maximal* connected subsets), but are not necessarily open, as the above example shows.

We call a topological space X **totally disconnected** if its connected components are just the single points, or equivalently, its only connected subsets are single points. Any discrete topological space is totally disconnected, as is the above example.

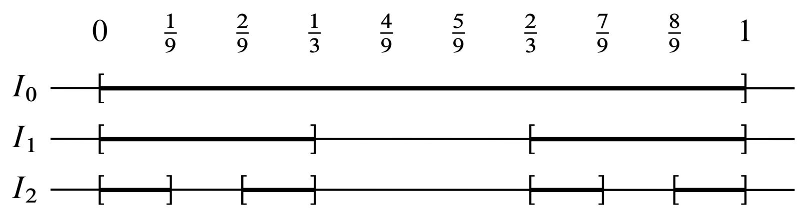
Lemma. If X is a topological space, and $\forall x, y \in X$ where $x \neq y$, X may be disconnected by $U, V \subseteq X$ where $x \in U$ and $y \in V$, then X is totally disconnected.

Proof. For any $Y \subseteq X$ with points x, y where $x \neq y$, take U, V as given. Then U, V disconnect Y. ∎

The set of irrationals $\mathbb{R} \setminus \mathbb{Q}$ is totally disconnected: Use the above lemma and the fact that between any two distinct irrationals there is a rational.

Cantor set

Consider starting with $I_0 := [0, 1]$ and removing $\left(\frac{1}{3}, \frac{2}{3}\right)$ to get $I_1 := \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. Now remove the middle third from both $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$ to get I_2, and so on. The **Cantor set** is $C := \bigcap_{n \geqslant 0} I_n$.



We can understand this in terms of ternary (i.e. base 3) expansions. I_1 consists of numbers with ternary expansion $0.a_1a_2a_3\ldots$ where $a_1 = 0$ or 2. (Wlog we impose $\frac{1}{3} = 0.022\ldots$.) I_2 consists of numbers where $a_1 = 0$ or 2, and $a_2 = 0$ or 2, and so on for I_3 etc. So C consists of numbers with a ternary expansion where each $a_i = 0$ or 2.

Suppose now we are given two distinct points x, y in C. For some n, the ternary expansions will differ first in the n-th place. $C \subseteq I_n$, and I_n consists of 2^n disjoint closed intervals, one of which contains x and another contains y.

So we can disconnect I_n by open $U, V \subseteq [0, 1]$ where $x \in U \cap I_n$ and $y \in V \cap I_n$. This implies U, V disconnect C where $x \in U \cap C$ and $y \in U \cap C$. By the previous lemma, C is totally disconnected. Also, both C and its complement are uncountable: Consider the ternary expansions.

3.2 Path-connectedness

Let X be a topological space and $x, y \in X$. A **continuous path** from x to y is a continuous function $\phi : [a, b] \to X$ such that $\phi(a) = x$, $\phi(b) = y$. (You may take $a = 0$, $b = 1$ if you prefer.) We say X is **path-connected** if $\forall x, y \in X$, $\exists$ a path from x to y. This is a topological property.

Proposition 3.3*. Continuous images of a path-connected space are path-connected. (cf. Proposition 3.3 on page 17.)

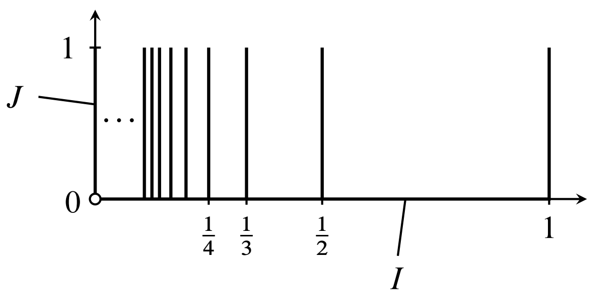
Proof. Suppose $f : X \to Y$ is a continuous surjection. Given $y_1, y_2 \in Y$, pick $x_i \in f^{-1}(y_i)$ for $i = 1, 2$, and a path $\gamma : [a, b] \to X$ from x_1 to x_2. Then $f \circ \gamma : [a, b] \to Y$ is a path from y_1 to y_2. ∎

Proposition 3.8. Path-connected $\Rightarrow$ Connected.

Proof. Let X be a path-connected topological space, and suppose $U, V \subseteq X$ disconnect X. Choose some $u \in U$, $v \in V$. Then $\exists$ a continuous function $\phi : [a, b] \to X$ with $\phi(a) = u$ and $\phi(b) = v$. Now $\phi^{-1}(U), \phi^{-1}(V) \subseteq [a, b]$ disconnect $[a, b]$, contradiction. ∎

But connected $\not\Rightarrow$ path-connected:

Example. In $\mathbb{R}^2$, let $I := \{(x, 0) : 0 < x \leqslant 1\}$ and $J := \{(0, y) : 0 < y \leqslant 1\}$. For $n = 1, 2, \ldots$, let $L_n := \left\{\left(\frac{1}{n}, y\right) : 0 \leqslant y \leqslant 1\right\}$. Set $X = I \cup J \cup \left(\bigcup_{n \geqslant 1} L_n\right)$, with the subspace topology.



X is not path-connected: For any continuous path $\gamma(t) = \big(\gamma_1(t), \gamma_2(t)\big)$ from $(1, 0)$ to $(0, 1)$, $\exists s$ such that $\gamma_1(s) = 0$ and $\gamma_1(t) > 0\ \forall t < s$, so we must also have $\gamma_2(s) = 0$. This means γ passes through $(0, 0) \notin X$.

X is connected: Suppose $f : X \to \mathbb{Z}$ is a continuous function. f is constant on J and on $Y := I \cup \left(\bigcup_{n \geqslant 1} L_n\right)$. However, the points $\left(\frac{1}{n}, \frac{1}{2}\right) \in Y$ have limit $\left(0, \frac{1}{2}\right) \in J$ as $n \to \infty$, hence continuity of $f \Rightarrow$ the two constants agree, i.e. f is constant on X.

Given a topological space X, we can define an equivalence relation $\sim$ on X by $x \sim y$ iff $\exists$ a path from x to y in X. Reflexivity of $\sim$ is trivial. Symmetry of $\sim$: If $\phi : [a,b] \to X$ is a path from x to y, then $\psi(t) = \phi(-t)$ gives a path $\psi : [-b,-a] \to X$ from y to x. Transitivity of $\sim$: If $\phi : [a,b] \to X$ and $\psi : [c,d] \to X$ are paths from x to y and y to z respectively, then $\chi : [a, b+d-c] \to X$ where

$$\chi(t) := \begin{cases} \phi(t) & \text{if } t \in [a,b], \\ \psi(t+c-b) & \text{if } t \in [b, b+d-c], \end{cases}$$

is a path from x to z. The equivalence classes under $\sim$ are called the **path-connected components** of X.

Theorem 3.9. Let X be an open subset of the Euclidean space $\mathbb{R}^n$, then X is connected $\Leftrightarrow$ X is path-connected.

Proof. ($\Leftarrow$): Proved as Proposition 3.8. ($\Rightarrow$): Let $x \in X$ and U be the equivalence class of x under $\sim$ defined as above.

U is open in X: Let $y \in U$, whence $x \sim y$. Since X is open, $\exists \delta > 0$ such that $B(y,\delta) \subseteq X$. Then $\forall z \in B(y,\delta)$, we have $y \sim z$ (take the straight line segment), so transitivity of $\sim \Rightarrow x \sim z$, i.e. $B(y,\delta) \subseteq U$.

Similarly, $X \setminus U$ is open in X. Now X is connected and $U \neq \emptyset$, so we have $X \setminus U = \emptyset \Leftrightarrow U = X \Leftrightarrow x \sim y \; \forall y \in X$. ∎

3.3 Products of connected spaces

Let X, Y be topological spaces. Consider $X \times Y$ with the product topology.

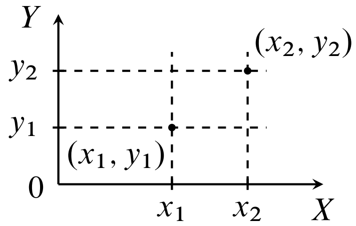
Proposition 3.10. If X, Y are path-connected, then so too is $X \times Y$.

Proof. Let (x_1, y_2) and (x_2, y_2) be in $X \times Y$. $\exists$ paths $\gamma_1 : [0,1] \to X$ and $\gamma_2 : [0,1] \to Y$ from x_1 to x_2 and y_1 to y_2, respectively. Define $\gamma : [0,1] \to X \times Y$ by $\gamma(t) := \big(\gamma_1(t), \gamma_2(t)\big)$. Any base for the topology on $X \times Y$ consists of open sets $U \times V$ where U is open in X and V is open in Y. Now $\gamma^{-1}(U \times V) = \gamma_1^{-1}(U) \cap \gamma_2^{-1}(V)$ is open. So γ is continuous and gives a path from (x_1, y_1) to (x_2, y_2). ∎

Proposition 3.11. If X, Y are connected, then so too is $X \times Y$.

First we make some general comments about the product topology $X \times Y$. $\forall y \in Y$, $X \times \{y\}$ with the subspace topology is homeomorphic to X via the projection map $\pi_1 : X \times \{y\} \to X$. Similarly, $\forall x \in X$, $\{x\} \times Y$ is homeomorphic to Y. So X, Y are connected $\Rightarrow$ $X \times \{y\}$ and $\{x\} \times Y$ are connected.

Proof of 3.11. Let $f : X \times Y \to \mathbb{Z}$ be continuous. By connectedness, f is constant on each of $\{x\} \times Y$ and $X \times \{y\}$. Take two points (x_1, y_1) and (x_2, y_2) in $X \times Y$, then $f(x_1, y_1) = f(x_1, y_2) = f(x_2, y_2)$ (See the following diagram), hence f constant on $X \times Y$, and $X \times Y$ is connected. ∎



A similar argument proves Proposition 3.10: $\exists$ paths from (x_1, y_2) to (x_1, y_2) and from (x_1, y_2) to (x_2, y_2) in $X \times Y$.

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§4 COMPACTNESS

4.1 Various results

Definition 4.1. Let X be a topological space and $Y \subseteq X$. An **open cover** of Y is a collection $\{U_\gamma : \gamma \in \Gamma\}$ of open subsets such that $Y \subseteq \bigcup_{\gamma \in \Gamma} U_\gamma$.

Such an open cover of Y provides a base of open sets $\{U_\gamma \cap Y : \gamma \in \Gamma\}$ for Y with the subspace topology, and conversely. A **subcover** of an open cover $\mathcal{U} = \{U_\gamma : \gamma \in \Gamma\}$ of Y is a subcollection $\mathcal{V} \subseteq \mathcal{U}$ which is still an open cover of Y.

Example. The intervals $I_n = (-n, n)$ where $n = 1, 2, \ldots$ form an open cover of $\mathbb{R}$, and I_{n^2} is a proper sub-cover. The intervals $J_n = (n-1, n+1)$ where $n \in \mathbb{Z}$ form an open cover of $\mathbb{R}$ with no proper subcover.

Definition 4.2. A topological space X is **compact** if every open cover has a finite subcover.

By the above example, $\mathbb{R}$ is not compact. Any finite topological space is compact, as is any set with the indiscrete or co-finite topology.

By the way we defined it, compactness is a topological property.

Lemma 4.3. Let X be a topological space. Then $Y \subseteq X$ with the subspace topology is compact $\Leftrightarrow$ every open cover $\{U_\gamma\}$ of Y has a finite subcover.

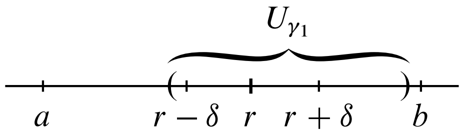
Proof. ($\Rightarrow$): Let $\{U_\gamma : \gamma \in \Gamma\}$ be an open cover of Y, then $Y = \bigcup_{\gamma \in \Gamma}\{U_\gamma \cap Y\}$, and the $U_\gamma \cap Y$'s are open in Y. Since Y is compact, $\exists \gamma_1, \ldots, \gamma_n \in \Gamma$ such that $Y = \bigcup_{i=1}^n U_{\gamma_i} \cap Y \Rightarrow$ $\{U_{\gamma_i} : i = 1, \ldots, n\}$ covers Y.

($\Leftarrow$): Suppose $Y = \bigcup_{\gamma \in \Gamma} V_\gamma$ where the V_γ's are open in Y. Write $V_\gamma = U_\gamma \cap Y$ where the U_γ's are open in X and form an open cover of Y. So $\exists \gamma_1, \ldots, \gamma_n \in \Gamma$ such that $Y \subseteq \bigcup_{i=1}^n U_{\gamma_i}$, hence $Y = \bigcup_{i=1}^n V_{\gamma_i}$. ∎

The open interval $(0, 1) \subseteq \mathbb{R}$ is not compact: Consider the open cover consisting of intervals $\left(\frac{1}{n}, 1 - \frac{1}{n}\right)$ for $n = 3, 4, \ldots$.

Theorem 4.4. ***(Heine-Borel).*** The closed interval $[a, b] \subseteq \mathbb{R}$ is compact.

Proof. Let $[a, b] \subseteq \bigcup_{\gamma \in \Gamma} U_\gamma$, where the U_γ's are open in $\mathbb{R}$. Let $K := \{x \in [a, b] : [a, x]$ is covered by finitely many U_γ's$\}$. $K \neq \emptyset$ since $a \in K$. K is bounded above by b. Let $r := \sup K$, then $r \in [a, b]$, so $r \in U_{\gamma_1}$ for some γ_1. Since U_{γ_1} is open, $\exists \delta > 0$ such that $[r - \delta, r + \delta] \subseteq U_{\gamma_1}$.



By definition of $\sup K$, $\exists c \in [r - \delta, r]$ such that $[a, c]$ is covered by finitely many of the U_γ's. Including U_{γ_1}, we have $[a, r + \delta] \cap [a, b]$ is covered by finitely many of the U_γ's. This contradicts r being an upper bound unless $r = b$, in which case the above argument says that $[a, b]$ is covered by finitely many of the U_γ's. So $[a, b]$ is compact. ∎

Proposition 4.5. Continuous images of a compact set are compact.

Proof. Suppose $f : X \to Y$ is a continuous map between topological spaces, and $K \subseteq X$ is compact. Suppose $f(K) \subseteq \bigcup_{\gamma \in \Gamma} U_\gamma$, where U_γ is open in Y. Then $K \subseteq \bigcup_{\gamma \in \Gamma} f^{-1}(U_\gamma)$, and since f is continuous, each $f^{-1}(U_\gamma)$ is open in X. Since K is compact, $\exists \gamma_1, \ldots, \gamma_n \in \Gamma$ such that $K \subseteq \bigcup_{i=1}^n f^{-1}(U_{\gamma_i})$. So $f(K) \subseteq \bigcup_{i=1}^n U_{\gamma_i}$. ∎

Proposition 4.6. Closed subsets of a compact topological space are compact.

Proof. Let X be a compact topological space, and $K \subseteq X$ be closed. If $K = \emptyset$ then this is trivial, so assume not. Suppose $K \subseteq \bigcup_{\gamma \in \Gamma} U_\gamma$, where the U_γ's are open in X. Then $X = (X \setminus K) \cup \left(\bigcup_{\gamma \in \Gamma} U_\gamma\right)$, where $X \setminus K$ is also open. Since X is compact, $\exists \gamma_1, \ldots, \gamma_n \in \Gamma$ such that $X = (X \setminus K) \cup \left(\bigcup_{i=1}^n U_{\gamma_i}\right)$. So $K \subseteq \bigcup_{i=1}^n U_{\gamma_i}$. ∎

Proposition 4.7. Compact subsets of a Hausdorff space are closed.

Proof. Suppose X is a Hausdorff space, and $K \subseteq X$ is compact. If $K = X$ then this is trivial, so assume not. We show that $X \setminus K$ is open.

Let $x \in X \setminus K$. Since X is Hausdorff, $\forall y \in K$, $\exists$ disjoint open sets $U_y \ni x$ and $V_y \ni y$. Now $\{V_y : y \in K\}$ is an open cover of K. Since K is compact, $\exists y_1, \ldots, y_n \in K$ such that $K \subseteq \bigcup_{i=1}^n V_{y_i}$. Then $U := \bigcap_{i=1}^n U_{y_i}$ is an open neighbourhood of x with $U \cap K = \emptyset$, so $U \subseteq X \setminus K$, as required. ∎

Corollary 4.8. A subset $X \subseteq \mathbb{R}$ is compact $\Leftrightarrow$ X is closed and bounded.

Proof. ($\Rightarrow$): Since $\mathbb{R}$ is Hausdorff, Proposition 4.7 $\Rightarrow$ X is closed. Suppose X is not bounded, then the intervals $(-n, n)$ where $n = 1, 2, \ldots$ form an open cover of X with no finite subcover. Contradiction.

($\Leftarrow$): $\exists M > 0$ such that $X \subseteq [-M, M]$. Since $\mathbb{R} \setminus X$ is open, we have $(\mathbb{R} \setminus X) \cap [-M, M]$ is open in $[-M, M]$, i.e. X is closed in $[-M, M]$. By Heine-Borel (Theorem 4.4), $[-M, M]$ is compact. By Proposition 4.6, X is compact. ∎

Together with Theorem 3.4 on page 17, this shows that the only connected compact subsets of $\mathbb{R}$ are the closed intervals. See also the remark after Theorem 4.11 on page 24.

Example. Consider the Cantor set $C = \bigcap_{n \geqslant 0} I_n$ as defined on page 18. Each I_n is the disjoint union of 2^n closed intervals, so I_n is closed. So C is closed and bounded $\Rightarrow C$ is compact.

Corollary 4.9. Let $f : X \to Y$ be a continuous bijection from a compact space X to a Hausdorff space Y. Then f is a homeomorphism.

Proof. Write $g := f^{-1}$. Let $F \subseteq X$ be closed. X is compact, so Proposition 4.6 $\Rightarrow F$ is compact. f is continuous, so Proposition 4.5 $\Rightarrow g^{-1}(F) = f(F)$ is compact. Y is Hausdorff, so Proposition 4.7 $\Rightarrow g^{-1}(F)$ is closed in Y. So g is continuous. ∎

This result is particularly useful in identifying quotient spaces. E.g. If we define $\sim$ on $\mathbb{R}$ by $x \sim y$ iff $x - y \in \mathbb{Z}$, and $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ (the unit circle with the subspace topology coming from $\mathbb{C}$), then it is clear that the map $f : \mathbb{R} \to \mathbb{T}$, given by $x \mapsto e^{2\pi i x}$, is continuous. It induces a bijection $\overline{f} : \mathbb{R}/\!\sim \to \mathbb{T}$, which, by definition of quotient topology, is also continuous. (See the remark about quotient topologies on page 14.)

The restricted quotient map $[0, 1] \to \mathbb{R}/\!\sim$ is a continuous surjection. $[0, 1]$ is compact, so Proposition 4.5 $\Rightarrow \mathbb{R}/\!\sim$ is compact. $\mathbb{T}$ is Hausdorff, so Corollary 4.9 $\Rightarrow \overline{f}$ is a homeomorphism.

Similarly, we can show that the 2-D torus $\mathbb{R}^2/\mathbb{Z}^2$, defined on page 14, is homeomorphic both to the product space $S^1 \times S^1$ (where S^1 is the unit circle) and to the embedded torus $X \subseteq \mathbb{R}^3$ consisting of points $\big((2 + \cos\phi)\cos\theta, (2 + \cos\phi)\sin\theta, \sin\phi\big)$, where $\theta, \phi \in [0, 2\pi)$.

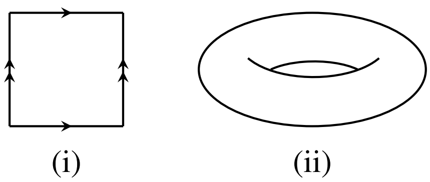


Figure 2: (i) The 2-D torus $\mathbb{R}^2/\mathbb{Z}^2$; (ii) The embedded torus $X \subseteq \mathbb{R}^3$.

In Analysis, you learnt that continuous real-valued functions on $[a, b] \subseteq \mathbb{R}$ are bounded and attain their bounds. What is being used here is the compactness of $[a, b]$. This statement is still true for continuous real-valued functions on the Cantor set, for instance.

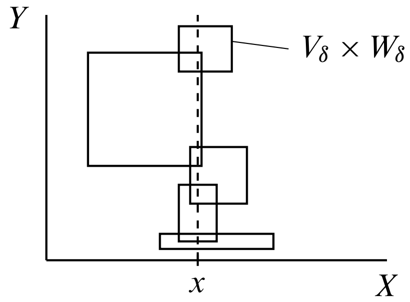
Proposition 4.10. Continuous real-valued functions on a compact space X is bounded and attain their bounds.

Proof. Let X be a compact space, and $f : X \to \mathbb{R}$ be continuous. Proposition 4.5 $\Rightarrow f(X)$ is compact. Corollary 4.8 $\Rightarrow f(X)$ is closed and bounded. Now $f(X)$ is closed $\Leftrightarrow f(X)$ contains all its accumulation points. $\sup\{f(X)\}, \inf\{f(X)\}$ (exist because $f(X)$ is bounded) are accumulation points for $f(X)$, so $\sup\{f(X)\}, \inf\{f(X)\} \in f(X)$, hence the result. ∎

Theorem 4.11. The product of two compact spaces is compact.

Proof. Let X, Y be compact spaces and $X \times Y = \bigcup_{\gamma \in \Gamma} U_\gamma$, where the U_γ's are open in $X \times Y$. Each U_γ is the union of open sets of the form $V \times W$, where V is open in X and W is open in Y. So $X \times Y = \bigcup_{\delta \in \Delta}(V_\delta \times W_\delta)$, where each V_δ is open in X, each W_δ is open in Y, and each $V_\delta \times W_\delta \subseteq U_\gamma$ for some γ.

Let $x \in X$. Then $\{x\} \times Y \subseteq \bigcup_{\delta \in \Delta : x \in V_\delta} V_\delta \times W_\delta$. In particular, $Y = \bigcup_{\delta \in \Delta : x \in V_\delta} W_\delta$. Since Y is compact, $\exists \delta_1, \ldots, \delta_m \in \Delta$, where $x \in V_{\delta_i}$ for each i, such that $Y = \bigcup_{i=1}^m W_{\delta_i}$.



Now let $V_x := \bigcap_{i=1}^m V_{\delta_i}$, an open neighbourhood of x satisfying $V_x \times Y \subseteq \bigcup_{i=1}^m V_{\delta_i} \times W_{\delta_i}$. Considering all $x \in X$, we see the V_x's form an open cover of X. Since X is compact, $\exists x_1, \ldots, x_n \in X$ such that $X = \bigcup_{j=1}^n V_{x_j}$. So $X \times Y = \bigcup_{j=1}^n V_{x_j} \times Y$.

Each $V_{x_j} \times Y$ has a finite cover by the $V_\delta \times W_\delta$'s, so the same is true for $X \times Y$. Using $V_\delta \times W_\delta \subseteq U_\gamma$, we see that $X \times Y$ has a finite cover by the U_γ's, hence the result. ∎

Remark. Given topological spaces X, Y, Z, the product $X \times Y \times Z$ is homeomorphic to $X \times (Y \times Z)$ (the open sets correspond). By induction, the above theorem implies the product of finitely many compact spaces is compact.

As a consequence, and with the use of Corollary 4.8 on page 22, $[-M, M]^n$ is a compact subset of $\mathbb{R}^n$. Now a set $X \subseteq \mathbb{R}^n$ is bounded means $\exists M > 0$ such that $X \subseteq [-M, M]^n$. So the proof of the same corollary may be extended to show $X \subseteq \mathbb{R}^n$ is compact $\Leftrightarrow$ X is closed and bounded.

Proposition 4.12. Let X be a compact metric space, Y be any metric space, and $f : X \to Y$ be a continuous map. Then f is uniformly continuous, i.e. $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x, y \in X$, $d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon$.

Proof. Since f is continuous, $\forall x \in X, \exists \delta_x > 0$ such that $d_X(x, x') < 2\delta_x \Rightarrow d_Y(f(x), f(x')) < \frac{\epsilon}{2}$. Let $U_x := \{x' \in X : d_X(x, x') < \delta_x\}$. Considering all $x \in X$, we see the U_x's form an open cover of X. Since X is compact, $\exists x_1, \ldots, x_n \in X$ such that $X = \bigcup_{i=1}^n U_{x_i}$.

Let $\delta := \min\{\delta_{x_i}\}$. Suppose $d_X(y, z) < \delta$ for some $y, z \in X$. Since the U_{x_i} form an open cover of X, $\exists\, i \in \{1, \ldots, n\}$ such that $d(y, x_i) < \delta_{x_i}$. Since $d_X(y, z) < \delta \leqslant \delta_{x_i}$, we have $d_X(z, x_i) < 2\delta_{x_i}$ by the triangle inequality. Therefore $d_Y(f(y), f(z)) \leqslant d_Y(f(y), f(x_i)) + d_Y(f(x_i), f(z)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. ∎

4.2 Sequential compactness

Definition 4.13. A topological space X is **sequentially compact** if every sequence in X has a convergent subsequence.

For a general topological space, compactness and sequential compactness do *not* imply each other. However:

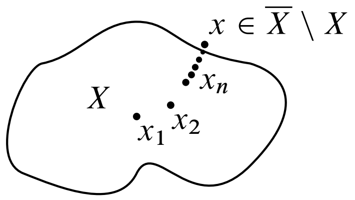
Proposition 4.14. Compact metric spaces are sequentially compact.

Proof. Let (X, d) be a metric space and (x_n) be a sequence in X with no convergent subsequence. Then (x_n) has infinitely many distinct members. $\forall x \in X, \exists \delta > 0$ such that $d(x, x_n) < \delta$ for finitely many n only. (If not, then $\exists x \in X$ such that $\forall m \in \mathbb{N}$, $d(x, x_n) < \frac{1}{m}$ for infinitely many n, and so $\exists$ a subsequence of (x_n) tending to x.)

We let $U_x := \{y \in X : d(x, y) < \delta\}$. Each U_x contains finitely many x_n's only, so $\{U_x : x \in X\}$ is an open cover of X with no finite subcover. X is not compact. ∎

This result implies the Bolzano-Weierstrass theorem: Any closed bounded subset of $\mathbb{R}^n$ is sequentially compact.

Example. Let $X \subseteq \mathbb{R}^n$ be sequentially compact. Then (1) X is bounded: Otherwise $\exists (x_n)$ such that $d(x, x_n) > n$, with no convergent subsequence; (2) X is closed: otherwise $\exists (x_n)$ such that $x_n \to x \notin X$.



In fact, we have the following more general result:

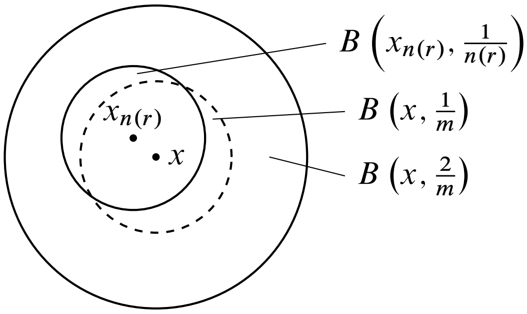
Theorem 4.15. Let (X, d) be a sequentially compact metric space. Then:

(i) $\forall \epsilon > 0, \exists x_1, \ldots, x_n \in X$ such that $X = \bigcup_{i=1}^{n} B(x_i, \epsilon)$;
(ii) $\forall$open cover $\mathcal{U}$ of X, $\exists \epsilon > 0$ such that $\forall x \in X$, $B(x, \epsilon) \subseteq U$ for some $U \in \mathcal{U}$;
(iii) (X, d) is compact.

Proof. (i): Suppose not. Then by induction we can construct a sequence (x_n) in X such that $d(x_m, x_n) \geqslant \epsilon \ \forall m \neq n$. Its subsequences are not Cauchy, hence divergent. Contradiction.

(ii): Suppose not. Then $\exists$ an open cover $\mathcal{U}$ of X such that $\forall n, \exists x_n \in X$ such that $B\left(x_n, \frac{1}{n}\right) \nsubseteq U$, $\forall U \in \mathcal{U}$. But (x_n) has a subsequence $(x_{n(r)})$ tending to $x \in X$. So let $x \in U_0$ for some $U_0 \in \mathcal{U}$. Since U_0 is open, $\exists m > 0$ such that $B\left(x, \frac{2}{m}\right) \subseteq U_0$.

Now $\exists N$ such that $x_{n(r)} \in B\left(x, \frac{1}{m}\right) \ \forall r \geqslant N$. Additionally if $n(r) > m$, and $y \in B\left(x_{n(r)}, \frac{1}{n(r)}\right)$, then $d(x, y) \leqslant d(x, x_{n(r)}) + d(x_{n(r)}, y) < \frac{2}{m}$. So for such $n(r)$, $B\left(x_{n(r)}, \frac{1}{n(r)}\right) \subseteq B\left(x, \frac{2}{m}\right) \subseteq U_0$. Contradiction. (See the diagram.)



(iii): Let $\mathcal{U}$ be an open cover of X. Choose an $\epsilon > 0$ as provided by (ii). For this ϵ, using (i) $\exists x_1, \ldots, x_n \in X$ such that $X = \bigcup_{i=1}^n B(x_i, \epsilon)$. For each i, $B(x_i, \epsilon) \subseteq U_i$ for some $U_i \in \mathcal{U}$ because of (ii). So $X = \bigcup_{i=1}^n U_i$. X is compact. ∎

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