Example Sheet 4

1. Let M be a smooth manifold, equipped with a Riemannian metric, and let $\gamma : [a, b] \to M$ denote any smooth curve on M; we define length$(\gamma) := \int_a^b \langle \dot\gamma(t), \dot\gamma(t)\rangle^{1/2} dt$. This in turn gives rise to a distance function ρ, where $\rho(P, Q)$ is the infimum of lengths of smooth curves from P to Q. Prove that (M, ρ) is a metric space.

2. Let M be an embedded submanifold of a manifold N. Show that there is a vector bundle $TN|_M$ on M, containing TM as a sub-bundle, and that any Koszul connection ∇ on N induces a linear connection (which we also denote as ∇) on $TN|_M$.
   Suppose now that g is a Riemannian metric on N; show that g induces a Riemannian metric on M, and also determines an orthogonal projection map $\pi : TN|_M \to TM$ of bundles on M. If now ∇ denotes the Levi-Civita connection on N, identify the Levi-Civita connection on M in terms of ∇ and π.

3. Suppose that g is a Riemannian metric on a smooth manifold M, and r is a strictly positive real number. Show that the effect of scaling the Riemannian metric by r^2 (and hence distances by r) is to leave the Levi–Civita connection unchanged but to scale the sectional curvatures by $1/r^2$.

4. Consider the embedded submanifold $S^{n-1} \subset \mathbb{R}^n$, the unit sphere, and let the symbol dS^2 denote the expression for the induced Riemannian metric on the unit sphere S^{n-1} with respect to suitable local coordinates on the sphere. Show that the Euclidean metric on $\mathbb{R}^n \setminus \{0\}$ can be expressed as $g = dr^2 + r^2 dS^2$, where $r = |x|$, $x \in \mathbb{R}^n$. [You might like to consider the dimensions $n = 2$ or 3 first, using polar coordinates.]

5. With the setup as in Question 2, and ∇ denoting the Levi–Civita connection on the Riemannian manifold N, for local vector fields V, W on M we set $II(V, W)$ to be the normal component of $\nabla_V W$ with respect to the metric. Show that II is symmetric in V and W, and hence deduce that it induces a symmetric bilinear form on the tangent bundle of M with values in the normal bundle (called the *second fundamental form*). If R denotes the curvature tensor on N and $\bar R$ the curvature tensor on M, for tangent vectors v, w, x, y to M at P, prove Gauss's formula that

$$\langle \bar R_{vw} x, y\rangle = \langle R_{vw} x, y\rangle + \langle II(v, y), II(w, x)\rangle - \langle II(v, x), II(w, y)\rangle.$$

   [Hint: Use Q4 from Example Sheet 3.]

6. Suppose now in the previous question that N is the Euclidean space $\mathbf{R}^n$ and that M is a smooth hypersurface defined by a smooth function $f = 0$. Show that there is a globally defined field of unit normal vectors $\mathbf{N}$ defined on M, thereby defining a smooth map $\mathbf{N} : M \to S^{n-1}$ (the *Gauss map*). Show that the tangent space to S^{n-1}

at $\mathbf{N}(P)$ has a natural identification with T_PM, and hence that the derivative $d_P\mathbf{N}$ may be regarded as an endomorphism of T_PM. Prove that

$$II(v,w) = -\langle d_P\mathbf{N}(v), w\rangle\,\mathbf{N} \quad \text{for all } v,w \in T_PM.$$

Deduce that the sectional curvatures of the embedded hypersphere $S^{n-1} \subset \mathbf{R}^n$ of radius $r > 0$ are all $1/r^2$.

7*. An almost complex structure on a manifold M is an endomorphism J of its tangent bundle TM such that $J^2 = -1$. If M has an almost complex structure, show that it is even dimensional.

We say that a connection ∇ on TM is *compatible* with J if $\nabla(J) = 0$, where ∇ here denotes also the induced connection on $\mathrm{End}(TM)$. Given a metric g and a compatible almost complex structure J on M, that is $g(JX,JY) = g(X,Y)$ for all X,Y, show that there are metric connections on M which are compatible with J, and that there is a distinguished choice for such a connection.

If M has dimension $2r$ and it is equipped with a metric and a compatible almost complex structure J, show that parallel transport around closed curves (defined by any such metric connection compatible with J) is represented by elements of $U(r)$.

8. Show that the Riemann curvature (R_{ijkl}) of (M,g) defines a symmetric bilinear form the fibres of $\wedge^2 TM$. Show that if $\dim M = 3$ then the Riemann curvature $R(g)$ is determined at each point of M by the Ricci curvature $\mathrm{Ric}(g)$.
[Hint: the assignment of $\mathrm{Ric}(g)$ to $R(g)$ is a linear map, at each point of M. A special feature of the dimension 3 is that the spaces of 1-forms and 2-forms on $\mathbb{R}^3$ have the same dimension.]

9. Suppose we have two Riemannian manifolds (M,g) and (N,h); show that there is a natural product metric $g+h$ on $M\times N$. If X is a vector field on M and Y one on N, we may regard both of these as vector field on the product. If ∇ denotes the Levi-Civita connection on $M\times N$, show that $\nabla_X Y = 0$. Conclude that $R(X,Y,X,Y) = 0$. [This means that product metrics have many sectional curvatures which are zero.]

10. Let $q(\mathbf{x},\mathbf{y}) = -x_0y_0 + \sum_{i=1}^n x_iy_i$, a Lorentzian bilinear form on $\mathbb{R}^{n+1}$, and set $\mathbf{H}(r)$ to be the upper branch of the hypersurface $q(\mathbf{x},\mathbf{y}) = -r^2$ (i.e. the part with $x_0 > 0$). Show that $\mathbf{H}(r)$ is a smooth manifold, and that the restriction of q to the tangent spaces of $\mathbf{H}(r)$ (considered as codimension one subspaces of $\mathbb{R}^{n+1}$) defines a Riemannian metric on $\mathbf{H}(r)$. Show that the sectional curvatures of $\mathbf{H}(r)$ are all $-1/r^2$. [The Riemannian manifold $\mathbf{H}(r)$ is called *hyperbolic space* of dimension n.]

11. Suppose M is a connected Riemannian manifold of dimension at least three, and that the Ricci curvatures are constant at each point. Show that the metric is Einstein.