Part III Differential Geometry Example Sheet 3

Q1 Given connections D_1, D_2 on E_1, E_2

(a) $\exists$ connection $D = D_1 \oplus D_2$ on $E_1 \oplus E_2$ defined by

$$D(s_1, s_2) = (D_1 s_1, D_2 s_2) \in \Omega^1(E_1 \oplus E_2)$$
$$= (D_1 s_1, 0) + (0, D_2 s_2)$$

(b) $\exists$ connection D on $E_1 \otimes E_2$ defined locally by

$$D(s_1 \otimes s_2) = D_1 s_1 \otimes s_2 + s_1 \otimes D_2 s_2,$$

well-defined since $D(f s_1 \otimes s_2) = D(s_1 \otimes f s_2)$

$$= f D(s_1 \otimes s_2) + df\, s_1 \otimes s_2$$

& then D clearly a connection

(c) The formula $X(s^*, s) = (D^*_X s^*, s) + (s^*, D_X s)$

clearly determines $D^*_X s^*$ $\quad \forall X$ & s^*.

We need to check it's a connection

(i) $$(D^*_{fX} s^*, s) = f X(s^*, s) - f(s^*, D_X s)$$
$$= f(D^*_X s^*, s) = (f D^*_X s^*, s)$$

(ii) $$(D^*_X(f s^*), s) = X(f(s^*, s)) - f(s^*, D_X s)$$
$$= f(D^*_X s^*, s) + X(f)(s^*, s)$$
$$= (X(f) s^* + f D^*_X s^*, s)$$

Set $De_j = \sum \theta_{kj} e_k$, $D_X e_j = \sum \theta_{kj}(X) e_k$

$D^* \varepsilon_i = \sum \theta^*_{li} \varepsilon_l$, $D_X \varepsilon_i = \sum \theta^*_{li}(X) \varepsilon_l$

$$\therefore \quad 0 = X(\varepsilon_i, e_j) = (D^*_X \varepsilon_i, e_j) + (\varepsilon_i, D_X e_j)$$

$$= \sum_l \theta^*_{li}(X)(\varepsilon_l, e_j) + \sum_k \theta_{kj}(X)(\varepsilon_i, e_k)$$

$$= \theta^*_{ji}(X) + \theta_{ij}(X) \Rightarrow \theta^*_\varepsilon = -\text{transpose of } \theta_e .$$

Q2 First comment that $\mathrm{Hom}(E_1, E_2) = E_1^* \otimes E_2$ & so given D_1, D_2, we have a connection $D = D_1^* \otimes D_2$ on $E_1^* \otimes E_2$. When $E_2 = \mathbb{1}$ & D_2 the trivial connection given by $D_2 f = 0$ ($f(x) = (x, 1)$), then $D(\varepsilon_i \otimes e) = D^* \varepsilon_i \otimes f$ — $\varepsilon_1, \dots, \varepsilon_n$ (co)-frame for E_1^*)

Coordinate free description (cf Q1):

Given $\theta \in \Gamma(\mathrm{Hom}(E_1, E_2))$ & $s \in \Gamma(E_1)$, then

$$D_X(\theta(s)) = (\tilde{D}_X \theta)(s) + \theta(D_X s)$$

dropping suffices 1 & 2 from D_1 & D_2

— easy check this is a connection on $\mathrm{Hom}(E_1, E_2)$ (cf Q5).

Mechanical check that these two definitions agree.

Given a local frame $e_1, \dots, e_n$ for E_1, dual coframe $\varepsilon_1, \dots, \varepsilon_n$ for E_1^*, frame $f_1, \dots, f_m$ for E_2 (with dual coframe $\phi_1, \dots, \phi_n$ for E_2^*), have frame $f_i \otimes \varepsilon_j$ for $\mathrm{Hom}(E_1, E_2)$ – any elt $\theta \in \Gamma(\mathrm{Hom}(E_1, E_2))$ may be written as $\sum h_{ij} f_i \otimes \varepsilon_j$. For X any v field, we calculate $\hat{D}_X(f_i \otimes \varepsilon_j)$ in the two ways:

<u>Coord free description</u>:

$$(\tilde{D}_X \theta)(e_k) + \theta(D_X e_k) = D_X(\theta(e_k))$$

When $\theta \leftrightarrow f_i \otimes \varepsilon_j$, get

$$\tilde{D}_X(f_i \otimes \varepsilon_j)(e_k) = \delta_{jk} D_X(f_i) - (f_i \otimes \varepsilon_j)(D_X e_k)$$

$$= \delta_{jk} D_X(f_i) - \varepsilon_j(D_X e_k) f_i$$

<u>Other description</u> $\quad \tilde{D}_X(f_i \otimes \varepsilon_j)(e_k)$

$$= (D_X f_i \otimes \varepsilon_j)(e_k) + (f_i \otimes D_X^* \varepsilon_j)(e_k)$$

$$= \delta_{jk} D_X f_i + (D_X^* \varepsilon_j)(e_k) f_i$$

where $(D_X^* \varepsilon_j)(e_k) = \cancel{X(\delta_{jk})} - \varepsilon_j(D_X e_k)$.

<u>Q3</u> Check this for elts $\sigma = \omega \otimes s \in \Omega^p(E)$;

$$\underset{\overset{\shortparallel}{\sigma}}{d^E(\omega \otimes s)} = d\omega \otimes s + (-1)^{\deg \omega} \omega \wedge d^E s$$

$$\therefore\ d^E \circ d^E(\omega \otimes s) = d^E(d\omega \otimes s) + (-1)^{\deg \omega} d^E(\omega \wedge d^E s)$$

$$= \cancel{(-1)^{\deg\omega+1} d\omega \wedge d^E s} + \cancel{(-1)^{\deg\omega} d\omega \wedge d^E s} + \omega \wedge d^E(d^E s)$$

$$= \omega \wedge (R \wedge s) = R \wedge \sigma \in \Omega^{p+2}(E)$$

$(R \in \Omega^2(\operatorname{End} E))$.

Q4 Wlog can take $\mu = \alpha \otimes s$ locally

$$\therefore\ d^E \mu = d\alpha \otimes s - \alpha \wedge Ds$$

$$\therefore\ d^E\mu(X,Y) = \{X\alpha(Y) \otimes s - Y\alpha(X) \otimes s - \alpha([X,Y]) \otimes s\}$$

$$- \alpha(X) D_Y s + \alpha(Y) D_X s$$

where $\{\ \} = d\alpha(X,Y) \otimes s$ (using Ex Sht 2, Q3)

$$= D_X(\mu(Y)) - D_Y(\mu(X)) - \mu([X,Y])$$

Now $R\sigma = d^E(D\sigma)$; apply above with $\mu = D\sigma$:

So $(R\sigma)(X,Y) = D_X(D_Y(\sigma)) - D_Y(D_X(\sigma)) - D_{[X,Y]}\sigma$

$\Rightarrow$ claim

Q5 $\tilde{D}$ defined by $(\tilde{D}_X \theta)(\sigma) = D_X(\theta(\sigma)) - \theta(D_X \sigma)$

Easy check this is a connection on End E

[$(\tilde{D}_X \theta)(f s) = X(f)\theta(s) + f D_X(\theta(s)) - X(f)\theta(s) - f\theta(D_X s)$

$= f \tilde{D}_X(\theta)(s) \Rightarrow \tilde{D}_X \theta \in \mathrm{Hom}(E|_U, E|_U)$ for any $\theta \in \mathrm{Hom}(E|_U, E|_U)$.

Moreover $\tilde{D}$ is linear in X &

$(\tilde{D}_X f\theta)(s) = D_X(f\theta(s)) - f\theta(D_X s)$

$= X(f)\theta(s) + f\tilde{D}_X \theta(s)$

$\Rightarrow \tilde{D}_X(f\theta) = X(f)\theta + f\tilde{D}_X \theta$ $\therefore \tilde{D}$ a connection]

Denote corresponding covariant derivative by

$d^{\mathrm{Hom}} : \Omega^p(\mathrm{End}\, E) \to \Omega^{p+1}(\mathrm{End}\, E)$

Locally $d^{\mathrm{Hom}}\left(\sum_k \omega_k \wedge F_k\right)$ $\left[\sum_k \omega_k \wedge F_k = \sum \omega_k F_k\right]$

$= \sum_k \left(d\omega_k \wedge F_k + (-1)^p \omega_k \wedge \tilde{D}(F_k)\right)$

For $\theta \in \mathrm{End}\, E$, $\sigma \in \Gamma(E)$, $\nabla(\theta\sigma) = (\tilde{\nabla}\theta)\sigma + \theta(\nabla\sigma)$

Extend this: for $F \in \Omega^p(\mathrm{End}\, E)$, $\tau \in \Omega^q(E)$,

get $d^E(F \wedge \tau) = (d^{\mathrm{Hom}} F) \wedge \tau + (-1)^p F \wedge d^E \tau$.

(Proof: Write $F = \sum_k F_k \wedge \omega_k = \sum \omega_k \wedge F_k$

as above, $F_k \in \Gamma(U, \mathrm{End}\, E)$, and

$\mu = \sum s_j \wedge \eta_j$ over U, $s_j \in \Gamma(U, E)$

& expand both sides — a boring check).

The curvature of D on E represented by $R \in \Omega^2(\mathrm{End}\, E)$

CLAIM $\quad d^{\mathrm{Hom}} R = 0$

Proof $\quad d^E(R \wedge \sigma) = (d^{\mathrm{Hom}} R) \wedge \sigma + R \wedge d^E \sigma$

for any $\sigma \in \Gamma(E)$

$\quad\quad \Vert$

$$d^E(d^E d^E \sigma) \underset{Q3}{=} R \wedge d^E \sigma \Rightarrow d^{\mathrm{Hom}} R = 0.$$

Choose a local trivialization $e_1, \ldots, e_r$ for E over U

$\therefore\ D(e_k) = \sum \theta_{lk} e_l$, $D^2(e_k) = \sum \Theta_{lk} e_l$

If $\varepsilon_1, \ldots, \varepsilon_r$ is dual coframe, then

$$R = \sum_{p,q} \Theta_{qp}\, \varepsilon_p \otimes e_q$$

$$d^{\mathrm{Hom}} R = \sum d\Theta_{qp}\, \varepsilon_p \otimes e_q + \sum \Theta_{qp} \wedge \tilde{D}(\varepsilon_p \otimes e_q)$$

$$\therefore (d^{\mathrm{Hom}} R)(e_i) = \sum_q d\Theta_{qi}\, e_q + \sum_{p,q} \Theta_{qp}\, \tilde{D}(\varepsilon_p \otimes e_q)(e_i)$$

$$= \sum_q d\Theta_{qi}\, e_q + \sum_{p,q} \Theta_{qp} \Big(\delta_{ip} \sum_k \theta_{kq} e_k - \theta_{pi} e_q\Big)$$

$$= \sum_q \Big(d\Theta_{qi} + \sum_k \Theta_{ki}\, \theta_{kq} - \sum_p \Theta_{qp}\, \theta_{pi} \Big) e_q$$

So this is zero for all i $\iff$

$$d \Theta_{qi} = \sum_k \left(\Theta_{qk} \theta_{ki} - \theta_{qk} \Theta_{ki} \right) \quad \forall i, q$$

$$\iff \quad d\Theta = \Theta \wedge \theta - \theta \wedge \Theta$$

i.e. form of Bianchi from lectures.

Q6** Given a smooth curve $\gamma : [a,b] \to M$ & a connection ∇ on v bundle E over M, we can define parallel transport in exactly the same way as we did for Koszul connections: Given local coords $x_1, \dots, x_n$ on M & local frame $e_1, \dots, e_r$ for E, we set $\nabla_i := \nabla_{\partial/\partial x_i}$ & $\nabla_i e_p = \sum \Gamma^q_{ip} e_q$.

A section $V(t)$ of E along γ is then locally of the form $V(t) = \sum v_j(t) e_i(\gamma(t))$. We define

$$\frac{DV}{dt} = \sum_{j=1}^{r} \left\{ \frac{dv_j}{dt} + \sum_{k=1}^{r} \sum_{i=1}^{n} \frac{d\gamma_i}{dt} \Gamma^j_{ik}(\gamma(t)) v_k(t) \right\} e_j(\gamma(t))$$

Given now $V_0 \in E_{\gamma(0)}$, we can solve for section V along t for which $V(0) = V_0$, $\dfrac{DV}{dt} = 0$

$\therefore$ Get parallel translation map

$$\tau_t : E_{\gamma(0)} \longrightarrow E_{\gamma(t)}.$$

For the case of E a vector bundle over a hypercube $H = I^n$ & ∇ a flat connection on E, we proceed as follows:

choose any basis $e_1, \to e_r$ for $E_{\underline{0}}$, $\underline{0} \in I^n = (-1,1)^n$

Parallel translate to get a frame $e_1(x_1), \dots, e_r(x_1)$ for $E|_{x_2 = \dots = x_n = 0}$ over $-1 < x_1 < 1$

For given $x_1 = a_1$, parallel translate to get frame $e_1(a_1, x_2), \dots, e_r(a_1, x_2)$ for $E|_{x_1 = a_1, x_3 = \dots x_n = 0}$

Since these depend smoothly on initial cond$^{=}$s $e_1(a_1, 0), \dots, e_r(a_1, 0)$, we obtain a smooth frame for $E|_{x_3 = \dots = x_n = 0}$. Continue by induction to get a smooth frame $e_1(x_1, \dots, x_n), \dots, e_r(x_1, \dots, x_n)$ for E over H.

<u>CLAIM</u> $\nabla_i e_p = 0 \quad \forall i, p$

<u>Proof by induction</u> $i = 1$: clearly $\nabla_1 e_p = 0 \ \forall p$ along the curve $x_2 = \dots = x_n = 0$, since we obtained $e_p(x_1)$ by parallel transport.

Curvature of ∇ is zero $\iff$ $\nabla_i \nabla_j = \nabla_j \nabla_i \quad \forall i, j$

Hence at points $(a_1, a_2, 0, \dots, 0)$ for instance

$\nabla_2(\nabla_1 e_p) = \nabla_1 \nabla_2 e_p = 0 \Rightarrow \nabla_1 e_p$ also

parallel along curve $x_1 = a_1, x_3 = \dots = x_n = 0$

But $\nabla_1 e_p = 0$ at $x_2 = 0 \Rightarrow \nabla_1 e_p = 0 \quad \forall x_2$

By induction, deduce $\nabla_1 e_p = 0$ at all $\underline{x} \in I^n$.

Similarly, induction shows that if $\nabla_i e_p = 0$

for $i \leq r$, then it's zero for $i = r+1$

Hence $\nabla_i e_p = 0 \quad \forall i, p$ & so ∇ is the

trivial connection on the trivial bundle E wrt

frame $e_1, \dots, e_r$ constructed.

Q7 (i) [To prove the result stated, we usually

choose a Riemannian metric on M and take the

geodesic distance metric ρ as defined in Ex Sht 4.

One then shows that for $0 < \varepsilon << 1$, the geodesic

ball $B(P, \varepsilon)$ is geodesically convex, i.e. for any

$Q_1, Q_2 \in B(P, \varepsilon)$, $\exists!$ minimum length geodesic

joining Q_1 to Q_2 and this geodesic segment $\subset B(P, \varepsilon)$;

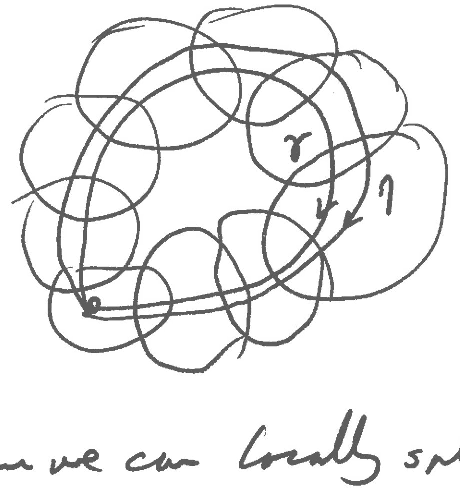
for the case $n=2$, this is discussed in Ch 8 of my book Curved Spaces. The intersection of any two such sets is also geodesically convex & hence connected]

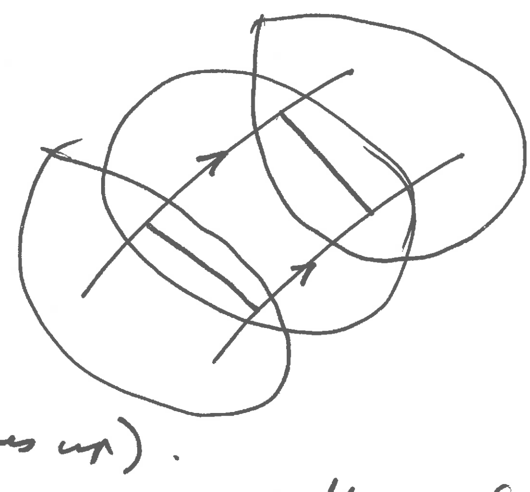
<u>Easy Lemma</u> Suppose U has local coords $x_1, \dots, x_n$ & $E = U \times \mathbb{R}^r$ is the trivial bundle, with standard frame $e_1, \dots, e_r$ & (flat) connection which is trivial wrt frame. Suppose $s_i = \sum_{j=1}^{r} f_{ji} e_j$ is another frame for E, $f_i \in C^\infty(U)$, then the connection is trivial wrt frame $s_1, \dots, s_r \iff$
$\partial f_{ji}/\partial x_k = 0 \quad \forall i,j \ \forall k$ (change of frame formula)
$\iff f_{ji}$ are locally constant.

Now we just choose an appropriate trivialising cover $\{U_\alpha\}$ for E s.t. $U_\alpha \cap U_\beta$ connected $\forall \alpha, \beta$. Previous question $\Rightarrow$ wlog can assume that for each U_α we have a frame wrt which the connection is trivial. Easy Lemma $\Rightarrow$ the transition functions locally constant $\Rightarrow$ they are constant by connectedness of intersections $U_\alpha \cap U_\beta$.

(ii) Choose base point $P_0 \in B$ & basis $e_1, \dots, e_r$ for E_{P_0}. For any point $Q \in B$, choose a smooth curve γ joining P to Q and parallel translate frame along γ to a basis of $E_{\gamma(1)} = E_Q$. When $B = (-\varepsilon, \varepsilon)^n$ as in previous question, this is locally indep of γ & we obtain the frame $e_1, \dots, e_r$ for E which trivializes the connection.

If B simply connected, claim that for any closed smooth curve γ starting and finishing at P_0, parallel translation is just the identity. This is a standard homotopy argument — may deduce from compactness & local result, that for any curve η suff close to γ (smooth, closed, starting & ending at P_0), parallel translation along γ & η yield same frame of E_{P_0}.





(since we can locally split curves up).

Using compactness again, same then follows for any smooth closed curve η (base point P_0) which is homotopic to γ. If B simply connected, we get well-def global frame for E over B & local result $\Rightarrow$ connection trivial wrt this frame

Q8 Given a parallel frame $V_1(t), \ldots, V_n(t)$

along γ, i.e. $\tau_t V_i(0) = V_i(t)$, the

$\tau^*_{-t} : T^*_{\gamma(0)} M \longrightarrow T^*_{\gamma(t)} M.$ is parallel transport map.

Thus if $\phi_1, \ldots, \phi_n$ is dual basis for $T^*_{\gamma(0)} M$,

we obtain a parallel coframe $\phi_i(t) = \tau^*_{-t}(\phi_i(0))$
$\in T^*_{\gamma(t)} M$

(since $\nabla_{\dot\gamma(t)} \phi_i = 0$).

But $\phi_i(t)(V_j(t)) = \phi_i(0)(\tau_{-t}(V_j(t)))$

$= \phi_i(0)(V_j(0)) = \delta_{ij}$

i.e. $\phi_1, \ldots, \phi_n$
is the dual coframe along γ.

Now for v fields X, Y & $P \in M$, choose γ s.t.
$\dot\gamma(0) = X_P$. Have parallel frame $V_1, \ldots, V_n$ along γ
coframe $\phi_1, \ldots, \phi_n$

Write $Y(\gamma(t)) = \sum g_j(t) V_j(t)$ & $\omega(\gamma(t)) = \sum f_i(t) \phi_i(t)$.

& so $\omega(Y)(\gamma(t)) = \sum f_i g_j \delta_{ij} = \sum_i f_i g_i$

$\therefore \nabla_{X_P}(\omega(Y)) = \sum_i (f_i g_i)'(0) = \sum_i \left(f_i'(0) g_i(0) + f_i(0) g_i'(0)\right)$

But $\nabla_{X_P} \omega = \sum f_i'(0) \phi_i(0)$, $\nabla_{X_P} Y = \sum g_j'(0) V_j(0)$

& hence $\nabla_{X_P}(\omega(Y)) = (\nabla_{X_P}\omega)(Y) + \omega(\nabla_{X_P} Y)$.

From the formula $\nabla_X(\omega(Y)) = (\nabla_X \omega)(Y) + \omega(\nabla_X Y)$,

it follows as in Q5 that ∇_X is well-defined (independent of choice of curves γ with $\dot\gamma(0) = X_P$) & is a connection.

Q9 <u>Lemma</u> Suppose D is an operator taking smooth fns $\mathcal{F}_M \to \mathcal{F}_M$ & smooth v fields $\mathcal{H}_M \to \mathcal{H}_M$ which are both linear / $\mathbb{R}$ & s.t.

$$D(fY) = f.DY + Df.Y. \qquad (+)$$

Then D has a unique extⁿ to an operator D taking tensor fields $T^k_l \to T^k_l$ s.t.

(1) D linear over $\mathbb{R}$

(2) $D(A \otimes B) = DA \otimes B + A \otimes DB$

(3) For any contraction $C : T^k_l \to T^{k-1}_{l-1}$, have $DC = CD$

<u>Pf</u> For a 1-form ω, we want for any v field Y we want $D(\omega \otimes Y) = D\omega \otimes Y + \omega \otimes D(Y)$.

Using (3), then have $D(\omega(Y)) = (D\omega)(Y) + \omega(D(Y))$

Easy check, as before that this satisfies basic relⁿ (+).

Thus D determined also on 1-forms & so (2) determines D on any tensor. Easy check by induction that D satisfies (2) (linearity $\Rightarrow$ need only check for decomposable tensors) & also that (3) continues to hold, by calculation above for 1-forms. Hence $\exists!$ extension to an operator D on T^k_l satisfying required condⁿs. $\square$

Two cases of interest here:

(a) $Df = Xf$ for X a v field

$DY = L_X Y = [X, Y]$

$\exists!$ ext$^{=}$ to L_X on tensors.

(b) $Df = Xf$, $DY = \nabla_X Y$, the case of interest to us. $\exists!$ ext$^{=}$ to covariant deriv ∇_X on T^k_l.

[Above argument $\Rightarrow$ induced connection on forms is just that of the previous question]

To check we have a connection ∇ on T^k_l, need to check that $\nabla_{fX+gY} A = f\nabla_X A + g\nabla_Y A$ (follows from use of (2)). So ∇ is a connection & ∇_X is the unique ext$^{=}$ of the covariant derivative satisfying (1) - (3).

If now we define ∇_X by means of parallel transport, we checked in lectures that we do recover ∇_X on $\mathfrak{I}$ on Ⓗ$_M$. The cond$^{=}$s (1) - (3) are then easily checked to hold for this extension to tensors, and so it must coincide with the unique extension defined above, and we say that this extension did yield a connection on T^k_l.

Q10* ∇ on TM induces connection $\overset{\nabla}{}$ on End T & hence a covariant exterior derivative $d^{\mathrm{End}} : \Omega^2(\mathrm{End}\, T) \to \Omega^3(\mathrm{End}\, T)$.

Have curvature term $R \in \Gamma(\Omega^2(\mathrm{End}\, T))$ & Q5 $\Rightarrow d^{\mathrm{End}}(R) = 0$

However ∇ induces a connection $\tilde{\nabla}$ on v bundle $(\Lambda^2 T^* M) \otimes \mathrm{End}\, T$ & so we have a diagram

thought of as anti-symmetric tensors

$$\begin{tikzcd}
\Omega^2(\mathrm{End}\, T) \arrow[rr, "\tilde{\nabla}"] \arrow[dr, "d^{\mathrm{End}\, T}"'] & & \Omega^1 \otimes \Omega^2(\mathrm{End}\, T) \arrow[dl, "\wedge"] \\
& \Omega^3(\mathrm{End}\, T) &
\end{tikzcd}$$

For any tensor $\mu \in \Omega^1 \otimes \Omega^1(\mathrm{End}\, T)$, the image $\tilde{\mu}$ of μ under $\wedge$ satisfies

$$\tilde{\mu}(X, Y, Z) = \mu(X; Y, Z) + \mu(Y; Z, X) + \mu(Z; X, Y)$$

Suppose now ∇ is symmetric on T & so $\exists$ coord system $x_1, \ldots, x_n$ in nbhd of any given point P s.t.

$$\nabla(\partial/\partial x_j) = 0 \quad \underline{\text{at } P} \quad \forall j.$$

So $d^{\mathrm{End}}(\omega \otimes \theta) = d\omega \otimes \theta = \omega \wedge \nabla \theta$

$= d\omega \otimes \theta$ $\underline{\text{at } P}$

since easy check verifies that $\nabla \theta = 0$ $\underline{\text{at } P}$

So $d^{\mathrm{End}\,T}(\omega \otimes \theta)\left(\partial/\partial x_r, \partial/\partial x_s, \partial/\partial x_t\right)_P$

$$= d\omega\left(\partial/\partial x_r, \partial/\partial x_s, \partial/\partial x_t\right)_P \otimes \theta_P$$

$$= \left(\partial/\partial x_r\left(\omega\left(\partial/\partial x_s, \partial/\partial x_t\right)\right) + \partial/\partial x_s(\quad) + \partial/\partial x_t(\quad)\right)_P \otimes \theta(P)$$

(cf Ex Sht 2, Q3).

Now $\tilde{\nabla}(\omega \otimes \theta) = \nabla\omega \otimes \theta + \omega \otimes \nabla\theta = \nabla\omega \otimes \theta$ $\underline{\underline{\text{at } P}}$

So $\tilde{\nabla}(\omega \otimes \theta)\left(\partial/\partial x_i ; \partial/\partial x_j, \partial/\partial x_k\right)_P$

$$= \nabla_{\partial/\partial x_i}(\omega)\left(\partial/\partial x_j, \partial/\partial x_k\right) \otimes \theta(P) \qquad \underline{\underline{\text{at } P}}$$

$$= \partial/\partial x_i\left(\omega\left(\partial/\partial x_j, \partial/\partial x_k\right)\right) \otimes \theta(P) \qquad \text{at } P.$$

since $\nabla_i(\omega)\left(\partial/\partial x_j, \partial/\partial x_k\right) = \partial/\partial x_i\left(\omega\left(\partial/\partial x_j, \partial/\partial x_k\right)\right)$

$$- \underbrace{\omega\left(\nabla_i \partial/\partial x_j, \partial/\partial x_k\right)}_{= 0 \text{ at } P} - \underbrace{\omega\left(\partial/\partial x_j, \nabla_i \partial/\partial x_k\right)}_{= 0 \text{ at } P.}$$

Thus $d^{\mathrm{End}\,T} = \wedge \circ \tilde{\nabla}$ under torsion free assumption

Now $(\wedge \circ \tilde{\nabla})(R)\left(\partial/\partial x_i, \partial/\partial x_j, \partial/\partial x_k\right)$

$$= (\nabla_i R)\left(\partial/\partial x_j, \partial/\partial x_k\right) + (\nabla_j R)\left(\partial/\partial x_k, \partial/\partial x_i\right) + (\nabla_k R)\left(\partial/\partial x_i, \partial/\partial x_j\right)$$

$\underline{\text{at } P}$

$$= \left(d^{\mathrm{End}\,T} R\right)\left(\partial/\partial x_i, \partial/\partial x_j, \partial/\partial x_k\right)$$

$= 0$ as required for 2nd Bianchi