 pmhw@dpmms.cam.ac.uk

Example Sheet 3

1. Given vector bundles E_1, E_2 over a manifold M with connections D_1, respectively D_2, define corresponding connections $D_1 \oplus D_2$ on $E_1 \oplus E_2$ and $D_1 \otimes D_2$ on $E_1 \otimes E_2$. If a connection D is given on a vector bundle E, and E^* denotes the dual bundle with $(\, , \,)$ the natural pairing from $E^* \times E$ to the trivial bundle, show that there is a connection D^* (the *dual connection*) satisfying
$$X(s^*, s) = (D^*_X s^*, s) + (s^*, D_X s)$$
for all local sections s^* of E^*, s of E and vector fields X. If D has connection matrix $\theta = (\theta_{ij})$ with respect to some local frame $e_1, \ldots, e_r$, what is the connection matrix of D^* with respect to the associated coframe on E^* (on each fibre giving the dual basis to $e_1, \ldots, e_r$)?

2. Show that there is a globally defined connection $\tilde{D}$ on the bundle $\mathrm{Hom}(E_1, E_2)$, associated to the connections D_1 and D_2, and specializing to the dual connection in the case when E_2 is trivial of rank one (and D_2 is the trivial connection). Give a coordinate free description of this connection on $\mathrm{Hom}(E_1, E_2)$.

3. We saw in lectures how a connection D on a vector bundle E over M gave rise to covariant derivatives $d^E : \Omega^p(E) \to \Omega^{p+1}(E)$ for all $p \geq 0$, and that the curvature map $\mathcal{R} = d^E \circ d^E : \Omega(E) \to \Omega^2(E)$ was represented by the curvature tensor $R \in \Omega^2(\mathrm{End}(E))$ via $\sigma \mapsto R \wedge \sigma$. Show that for all $p \geq 0$, the map $d^E \circ d^E : \Omega^p(E) \to \Omega^{p+2}(E)$ is given by $\sigma \mapsto R \wedge \sigma$.

4. With the notation as in the previous question, and using Example Sheet 2, Question 3, show that for any element $\mu \in \Omega^1(E)$ and vector fields X, Y,
$$d^E(\mu)(X, Y) = D_X(\mu(Y)) - D_Y(\mu(X)) - \mu([X, Y]).$$
Deduce that, for any section σ of E,
$$(\mathcal{R}\sigma)(X, Y) = D_X D_Y \sigma - D_Y D_X \sigma - D_{[X,Y]}\sigma.$$

5. If D is a connection on a vector bundle E, let $\tilde{D}$ denote the connection on $\mathrm{End}(E)$ you defined in Question 2. Set d^{Hom} to denote the corresponding covariant derivative. With $R \in \Omega^2(\mathrm{End}(E))$ denoting the curvature of E, prove that $d^{\mathrm{Hom}}(R) = 0$. Verify that this is just the Bianchi identity as given in lectures, written in a coordinate free form.

6**. Prove the following *integrability theorem* for flat connections. If E is a vector bundle over the open hypercube $H = \{x \in \mathbb{R}^n : \max_i |x_i| < 1\}$ and D is a flat connection on E then there is a bundle isomorphism taking E to the trivial bundle over H and D to the trivial (product) connection.
[Hint: One can define parallel translation (along a curve) on the fibres of E in the

obvious way. This can be used to define a global frame on H. It is then a good idea to use induction on n. Killing the coefficients Γ^k_{ij} for any fixed i amounts to solving a linear ODE (with parameters).]

7. (i) Assuming the fact that for any smooth manifold we can find an arbitrarily fine open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ such that $U_\alpha \cap U_\beta$ is connected for all $\alpha, \beta \in A$, deduce from the previous question that if a vector bundle admits a flat connection, then there is a choice of local trivializations of the bundle so that the corresponding transition functions are *constant*, $\psi_{\beta\alpha}(x) \equiv h_{\beta\alpha}$, for all $x \in U_\beta \cap U_\alpha$.

   (ii) Show further that a flat connection on a vector bundle over a simply-connected base manifold, B say, determines an isomorphism of this bundle to the trivial bundle, i.e. a (global) trivialization over all of B. [Hint: Covariantly constant sections.]

8. Let M be a smooth manifold and ∇ denote a connection on TM. Suppose that $V_1, \ldots, V_n$ denote a parallel frame along a smooth curve γ, and that $\varphi_1, \ldots, \varphi_n$ denotes the dual coframe along γ. Show that the φ_i are parallel along γ (with respect to the induced connection ∇ on T^*M defined via parallel transport). Deduce, as in lectures, that for any 1-form ω and any vector fields X, Y,
$$\nabla_X(\omega(Y)) = (\nabla_X\omega)(Y) + \omega(\nabla_X Y).$$

9. Let ∇ denote a Koszul connection on a smooth manifold M. Prove that the extension of ∇ to any tensor bundle $T^k_l(M)$, defined as in lectures by means of parallel transport, does indeed define a connection on the tensor bundle. [Hint: Argue by induction.]

10*. Suppose that ∇ is a Koszul connection on a manifold M, and that $\tilde{\nabla}$ is the induced connection on the bundle $\wedge^2 T^*M \otimes \mathrm{End}(TM)$, and thereby determining a map on sections
$$\Omega^2(\mathrm{End}(TM)) \to \Gamma(T^*M \otimes \wedge^2 T^*M \otimes \mathrm{End}(TM)).$$
Show also that there is a natural map
$$\wedge : T^*M \otimes \wedge^2 T^*M \otimes \mathrm{End}(TM) \to \wedge^3 T^*M \otimes \mathrm{End}(TM),$$
and hence also a corresponding map on sections. Let $d^{\mathrm{End}(TM)} : \Omega^2(\mathrm{End}(TM)) \to \Omega^3(\mathrm{End}(TM))$ denote the covariant exterior derivative map determined by the induced connection on $\mathrm{End}(TM)$; if ∇ is a symmetric connection, prove that
$$d^{\mathrm{End}(TM)} = \wedge \circ \tilde{\nabla}$$
as maps on $\Omega^2(\mathrm{End}(TM))$.

Deduce the second Bianchi identity for ∇ from the result in Question 5.