Part III Differential Geometry Example Sheet 2

(1) (i) Not true – e.g. $\alpha = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ on $\mathbb{R}^4$ has $\alpha \wedge \alpha = 2 dx_1 \wedge \cdots \wedge dx_4$.

(ii) <u>Lemma</u> For $V = \mathbb{R}^n$, $\Lambda^{p-1} V^* \xrightarrow[\wedge\alpha]{} \Lambda^p V^* \xrightarrow[\wedge\alpha]{} \Lambda^{p+1} V^*$ is exact at $\Lambda^p V^*$.

<u>Pf</u> α a 1-form $\Rightarrow \alpha \wedge \alpha = 0 \Rightarrow$ image $\subseteq$ kernel.
Image of $\wedge \alpha$ in $\Lambda^p V^*$ has dim $\binom{n-1}{p-1}$ (can take basis for V^* consisting of $\alpha, \phi_2, \dots, \phi_n$) & image of $\wedge\alpha$ in $\Lambda^{p+1} V^*$ has dim $\binom{n-1}{p}$. So kernel of this latter map has dim $\binom{n}{p} - \binom{n-1}{p} = \binom{n-1}{p-1}$ & so image = kernel. $\square$

Now connect this argument works locally – may assume that $\alpha, \phi_2, \dots, \phi_n$ now a local frame over U.
Then $\alpha \wedge \phi_{i_1} \wedge \dots \wedge \phi_{i_{p-1}}$ $(i_1 < \dots < i_{p-1})$ is a frame for the subbundle $\alpha \wedge \Lambda^{p-1} T_M^* \subset \Lambda^p T_M^*$. Above Lemma $\Rightarrow \beta$ a smooth section of this subbundle

ie. $\beta = \sum f_{i_1 \dots i_{p-1}} \alpha \wedge \phi_{i_1} \wedge \dots \wedge \phi_{i_{p-1}}$ for some smooth functions $f_{\underline{i}}$

ie. $\beta = \alpha \wedge \gamma$ for smooth $(p-1)$-form $\gamma = \sum f_{\underline{i}} \phi_{i_1} \wedge \dots \wedge \phi_{i_{p-1}}$

So we can find a cover $\mathcal{U} = \{U_i\}_{i \in I}$ so that $\beta|_{U_i} = \alpha \wedge \gamma_i$ for some smooth $(p-1)$-form γ_i on U_i.
Take a partition of unity $\{\rho_j\}_{j \in J}$ subordinate to the cover $\mathcal{U}$ & set $\gamma = \sum_{j \in J} \rho_j \gamma_{i(j)}$.

Then $\alpha \wedge \gamma = \sum \rho_j \beta = \beta$ as required.

Q2/ $\exists$ standard orientation on $\mathbb{R}^{n+1}$ — this induces a standard orientation on $M = S^n$;
if $v_1, \dots, v_n \in T_pM$, say $\underline{p} = \vec{OP}$, then
$v_1, \dots, v_n$ are in standard orientation in $T_pM \overset{\text{def}}{\iff}$
$\underline{p}, v_1, \dots, v_n$ are in standard orientation in $T_p\mathbb{R}^{n+1} = \mathbb{R}^{n+1}$

Now $\mathbb{R}P^n = S^n/\{\pm 1\}$, quotient by antipodal map α where $\alpha(\underline{x}) = -\underline{x}$.
The induced map $\alpha_* : T_pS^n \longrightarrow T_{\alpha p}S^n$ sends
$v_1, \dots, v_n$ to $-v_1, \dots, -v_n$. This is orientation preserving $\iff$ $-\underline{p}, -v_1, \dots, -v_n$ are in same orientation as $\underline{p}, v_1, \dots, v_n$ $\iff$ $n+1$ even $\iff$ n odd.

So if μ an orientation on $S^n/\{\pm 1\}$, it induces a nowhere vanishing n-form on S^n with $\alpha^*(\mu) = \mu$
$\Rightarrow$ α orientation preserving $\Rightarrow$ n odd.

Conversely, if n odd, then can cover S^n by small open sets & their antipodal images, with charts satisfying the cond$^=$ that Jacobian matrices of coord transformations all have $\det > 0$.

<u>Alter</u> Use n-form
$$\omega = \sum_{i=1}^{n+1} (-1)^{i-1} x_i \, dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_{n+1}$$ & note
that $dr \wedge \omega = 2r\, dx_1 \wedge \dots \wedge dx_{n+1}$ $\quad (r^2 = \sum x_i^2)$

(3) STP identity for $\omega = f\,dg$.

So $d\omega = df \wedge dg$

$$d\omega(X,Y) = (Xf)(Yg) - (Xg)(Yf)$$

$$X\,\omega(Y) = X(f\,Y(g)) = (Xf)(Yg) + f(XY)g$$

$$Y\,\omega(X) = Y(f\,X(g)) = (Yf)(Xg) + f(YX)g$$

$$\omega[X,Y] = f(XY)(g) - f(YX)(g)$$ Hence result.

* Generalization: $d\omega(X_1, \ldots, X_{p+1})$

$$= \sum_{i=1}^{p+1} (-1)^{i+1} X_i(\,\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1})$$

$$+ \sum_{1 \le i < j \le p+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1}) \qquad (*)$$

$$= \Sigma_1 + \Sigma_2 \quad \text{say}$$

CLAIM RHS is a tensor: if X_{i_0} replaced by fX_{i_0}, then Σ_1 becomes

$$f\Sigma_1 + \sum_{i \neq i_0} (-1)^{i+1} (X_i f)\,\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1})$$

& using $[fX, Y] = f[X,Y] - (Yf)X$
$[X, fY] = f[X,Y] - (Xf)Y$

Σ_2 then becomes $f\Sigma_2 + \sum_{i<i_0} (-1)^{i+i_0} (X_i f)\,\omega(X_{i_0}, \ldots, \hat{i}, \ldots, \hat{i_0} \ldots)$

$$+ \sum_{i_0 < j} (-1)^{i_0+j} (X_j f)\,\omega(X_{i_0}, \ldots, \hat{i_0}, \ldots, i, \ldots)$$

& so $\Sigma_1 + \Sigma_2 \longmapsto f\Sigma_1 + f\Sigma_2$

By inspection RHS is alternating & hence a $(p+1)$-form. $\therefore$ STP identity for local v fields $\partial/\partial x_{i_1}, \ldots, \partial/\partial x_{i_{p+1}}$ & for $\omega = f\, dx_1 \wedge \ldots \wedge dx_p$.

Then RHS $= \Sigma_1 + 0 = 0$ unless $(i_1, \ldots, i_{p+1}) = (1, 2, \ldots, p, k)$ for some $k > p$, in which case we get $(-1)^p\, \partial f/\partial x_k$.

But $d\omega = \sum_{k>p} \partial f/\partial x_k \, dx_k \wedge dx_1 \wedge \ldots \wedge dx_p$ and evaluated on $\partial/\partial x_{i_1}, \ldots, \partial/\partial x_{i_{p+1}}$ agrees with α.

Q4 If we set $\tan\theta = y/x$ on $\mathbb{R}^2 \setminus \{(0,y)\} = U_1$

then
$$\underset{\frac{x^2+y^2}{x^2}}{\underset{\parallel}{\sec^2\theta}}\, d\theta = \frac{x\,dy - y\,dx}{x^2}$$

$$\Rightarrow d\theta = \frac{x\,dy - y\,dx}{x^2+y^2} = \omega$$

Setting $\cot\theta = x/y$ on $\mathbb{R}^2 \setminus x$ axis $= U_2$

we similarly deduce that $d\theta = \omega$

Embed $S^1 \hookrightarrow \mathbb{R}^2 \setminus \{0\}$ in obvious way. If ω exact on $\mathbb{R}^2 \setminus \{0\}$, the pullback exact on S^1

$\Rightarrow \int_{S^1} \omega = 0$ by Stokes.

But $\int_{S^1} \omega = \int_0^{2\pi} d\theta = 2\pi$ $\#$.

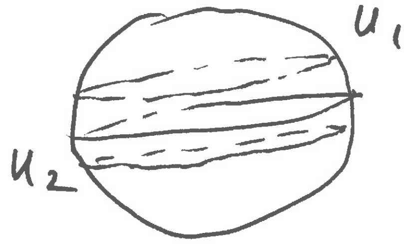
We can however write any 1-form on S^1 as $f(\theta)\, d\theta$; set $g(\theta) = \int_0^{\theta} f\, d\theta$

If $g(2\pi) = \lambda 2\pi$, we set $h(\theta) = g - \lambda\theta$, a well-defined function on S^1.

Then $f\, d\theta = \lambda\, d\theta + dh \Rightarrow$

$H^1_{DR}(S^1) = \mathbb{R}\langle d\theta \rangle \cong \mathbb{R}$. $\square$

Q5 (i) Cover S^2 by two charts each diffeo to open disc under stereographic projection (from respectively south/north poles), say U_1, U_2, where $U_1 \cap U_2 \cong S^1 \times (-\varepsilon, \varepsilon)$



A closed 1-form ω on S^2 yields closed 1-forms ω_i on each U_i & hence $\omega_i = dh_i$, $h_i \in C^\infty(U_i)$ by Poincaré.

Then $d(h_1 - h_2) = 0$ on $U_1 \cap U_2 \Rightarrow$

$h_1 - h_2 = \text{const} \underset{\text{wlog}}{=} 0$

$\therefore \exists$ function h on S^2 s.t. $h|_{U_i} = h_i$

$\Rightarrow \quad \omega = dh \quad \Rightarrow \quad H^1_{DR}(S^2) = 0$

Note Exactly same argument yields $H^1_{DR}(S^n) = 0 \ \forall n > 1$ whilst Q4 $\Rightarrow H^1_{DR}(S^1) = \mathbb{R}$. Since S^n connected for $n > 0$, have $H^0_{DR}(S^n) = \mathbb{R}$ for $n > 0$
& $= \mathbb{R}^2$ for $n = 0$.

(ii)* Similarly decompose S^n as $U_1 \cup U_2$ with $U_1 \cap U_2 \cong S^{n-1} \times (-\varepsilon, \varepsilon)$. We quote the generalised Poincaré Lemma: for any mfld M,

$H^p(M \times \mathbb{R}, \mathbb{R}) = H^p(M, \mathbb{R}) \quad \forall p$

($\Rightarrow$ standard Poincaré by induction - Bott & Tu p35)

Given a closed p-form ω on S^n $(p > 1)$, set $\omega = d\eta_i$ on U_i $\quad \therefore d(\eta_1 - \eta_2) = 0$ on $U_1 \cap U_2$.
$\therefore$ we can set $\tau = \eta_1 - \eta_2$ closed $(p-1)$-form on $U_1 \cap U_2$

If $\omega' = \omega + d\eta$ for some η, above construction yields same $(p-1)$-form τ (defined up to an exact form arising from choices of η_1, η_2. Hence we have a well-def HM $H^p(S^n, \mathbb{R}) \to H^{p-1}(U_1 \cap U_2, \mathbb{R})$

Conversely, given a closed $(p-1)$-form τ on $U_1 \cap U_2$, choose partition of unity $\{\rho_1, \rho_2\}$ wrt $\{U_1, U_2\}$

s.t. $\mathrm{supp}(\rho_i) \subset U_i$, $\rho_1 + \rho_2 = 1$

Define η_i on U_i by $\eta_1 = \rho_2 \tau$, $\eta_2 = -\rho_1 \tau$

$\Rightarrow$ $\eta_1 - \eta_2 = \tau$ on $U_1 \cap U_2$.

Set $\omega = d\eta_1$ on U_1 & $\omega = d\eta_2$ on U_2, &

so ω is a closed p-form on S^2.

If moreover $\tau' = \tau + d\gamma$, then $\eta_1' = \eta_1 + \rho_2 d\gamma$

$\eta_2' = \eta_2 - \rho_1 d\gamma$

$\Rightarrow \omega' = \omega + d\rho_2 \wedge d\gamma$ on U_1, $\omega' = \omega - d\rho_1 \wedge d\gamma$ on U_2
$= \omega - d\rho_1 \wedge d\gamma$

$\Rightarrow \omega' = \omega + d(d\rho_1 \wedge \gamma)$ (noting that $\mathrm{supp}(d\rho_1) \subset U_1 \cap U_2$)

$\Rightarrow \omega_1' \sim \omega_2'$. So we have a well-defined inverse

map $H^{p-1}(U_1 \cap U_2, \mathbb{R}) \longrightarrow H^p(S^n, \mathbb{R})$ i.e.

$\exists$ IM $H^p(S^n, \mathbb{R}) \xrightarrow{\sim} H^{p-1}(S^{n-1}, \mathbb{R})$ (generalized Poincaré)

$\therefore$ $H^n(S^n, \mathbb{R}) \cong \mathbb{R}$ by induction, $n > 0$

$H^p(S^n, \mathbb{R}) = 0$ by induction for $1 < p < n$

Q6 Identifying $T_{\underline{x}} S^{2n+1}$ with

$\{\underline{v} \in \mathbb{R}^{2n+2}$ s.t. $\underline{x} \cdot \underline{v} = 0\}$, the required v field

is given by taking tangent vector

$(-x_2, x_1, -x_4, x_3, \dots, -x_{2n+2}, x_{2n+1})$ at $\underline{x}$

(corresponding, if we identify S^{2n+1} as given by $|z_1|^2 + \dots + |z_{n+1}|^2 = 1$ in $\mathbb{C}^{n+1}$, to mult^n by i, ie if ϕ_t is the corresponding local flow on S^{2n+1}, then $\phi(z_1, \dots, z_{n+1}) = e^{it}(z_1, \dots, z_{n+1})$)

Q7 $\omega^1, \dots, \omega^d$ form the dual coframe for T^*G, dual to the frame $X_1, \dots, X_d$ for TG, and hence form a basis of $T^*_g G$ at all g and are smooth global forms.

Now $(L_g^* \omega^i)_a (X_j) = \omega^i_{ga}((L_g)_* X_j)$

$= \omega^i_{ga}(X_j) = \delta^i_j = \omega^i_a(X_j) \quad \forall i$

$\Rightarrow L_g^* \omega^i = \omega^i \quad \forall i.$

Recall now that $(L_g)_*[X, Y] = [(L_g)_* X, (L_g)_* Y]$

If therefore $[X_i|_e, X_j|_e] = \sum_k C^k_{ij} X_k|_e$,

then $[X_i|_g, X_j|_g] = \sum_k C^k_{ij} X_k|_g$

where $C_{ji} = -C_{ij}$

So identity in Q3 $\Rightarrow$

$$d\omega^k(X_p, X_q) = \cancel{X_p\,\omega^k(X_q)} - \cancel{X_q\,\omega^k(X_p)} - \omega^k([X_p, X_q])$$

$$= -\omega^k\Big(\sum_l c^l_{pq} X_l\Big) = -c^k_{pq}.$$

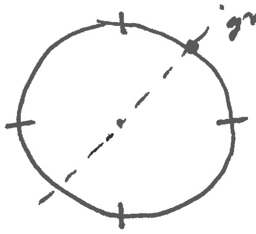
& $$-\tfrac{1}{2}\sum_{i,j} c^k_{ij}\,\omega^i\wedge\omega^j\,(X_p, X_q)$$

$$= -\tfrac{1}{2}\sum_{i,j}(\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})\,c^k_{ij}$$

$$= -\tfrac{1}{2}c^k_{pq} + \tfrac{1}{2}c^k_{qp} = -c^k_{pq}$$

Hence $$d\omega^k = -\tfrac{1}{2}\sum_{i,j} c^k_{ij}\,\omega^i\wedge\omega^j$$

Q8 $\quad S^1 = \mathbb{R}P^1$



$U_0 = \mathbb{R}$, coord x, $(1:x)\mapsto x$

$U_1 = \mathbb{R}$, coord y, $(y:1)\mapsto y$

Choose trivialization of bundle given by (cf Lectures)

$$\Phi_0 : (w, wx) \longmapsto \big((1:x),\, w\sqrt{1+x^2}\big)$$

$$\Phi_1 : (vy, v) \longmapsto \big((y:1),\, v\sqrt{1+y^2}\big)$$

So $\Phi_1\circ\Phi_0^{-1}((1:x), t)$ $\qquad (y = 1/x)$

$= \Phi_1\left(t/\sqrt{1+x^2},\ tx/\sqrt{1+x^2} \right) \in E_{(1:x)}$

$= \Phi_1\left(t|y|/\sqrt{y^2+1},\ t\,\dfrac{|y|/y}{\sqrt{y^2+1}} \right) \in E_{(y:1)}$

$= \left((y:1),\ \frac{|y|}{y}\, t \right)$ so transition fn

given by $y/|y| \in \{\pm 1\}$

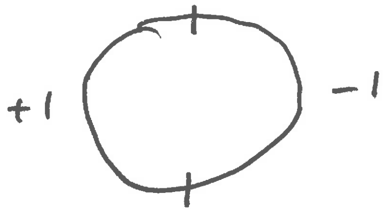
Identifying $\mathbb{RP}^1$ with S^1

$$\theta \longmapsto 2\theta$$

$$\begin{tikzcd} S^1 \arrow[rr] \arrow[dr] & & S^1 \\ & \mathbb{RP}^1 \arrow[ur, "\sim"'] & \end{tikzcd}$$

we have transition functions given

as shown



$\rightsquigarrow$ infinite Möbius strip

Transition function of $E \otimes E$ is just $1 = (+1)^2 = (-1)^2$

$\Rightarrow$ $E \otimes E$ corresponds to glueing over S^1.

Q9 $\Phi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{R}^k$ — standard i.p.

on $\mathbb{R}^k$ $\rightsquigarrow$ trivial metric on $U_\alpha \times \mathbb{R}^2$

$\rightsquigarrow$ metric on $\pi^{-1}(U_\alpha)$, say $\langle\ ,\ \rangle_\alpha$

Now take partition of unity subordinate to cover $\{U_\alpha\}$,

say $\{\rho_i\}_{i \in I}$ with $\operatorname{supp}(\rho_i) \subset U_{\alpha(i)}$

Define a global metric $<\ ,\ >$ on E by

$$< \ , \ > = \sum_i \rho_i < \ , \ >_{\alpha(i)}$$

Wrt trivialization $\Phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^k$ this gives a metric on $\begin{array}{c} U_\alpha \times \mathbb{R}^k \\ \downarrow \\ U_\alpha \end{array}$, given by a

smooth map $g_\alpha : U_\alpha \to \mathrm{Sym}_+(k, \mathbb{R})$

For each α, choose an o-n frame for $\pi^{-1}(U_\alpha)$, obtained by Gram-Schmidt o-n & a new trivialization $\Psi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^k$ determined by this new frame. The transition functions therefore take o-n bases to o-n bases & hence are given by functions $h_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow O(k, \mathbb{R})$

The map $E \longrightarrow E^*$ given by

$e \mapsto \langle e, - \rangle \in E^*$, easily checked to be an IM of bundles.

Q10 Define mult^n by $[E] \otimes [F] = [E \otimes F]$

Identity of group is just trivial bundle $\mathbb{1}_M$.

Why is collection of such classes a set!? Answer

A bundle is determined up to IM by trans fns wrt some trivializing cover.

Inverse of $[E]$ is just $[E^*]$, the class of the dual bundle; if E has transition fns $f_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathbb{R}$ wrt a trivializing cover $\{U_\alpha\}$ then E^* has transition functions $f_{\alpha\beta}^{-1}$ & $E \otimes E^*$ has transition functions $f_{\alpha\beta} \times f_{\alpha\beta}^{-1} = 1$ i.e. it's IM to the trivial line bundle 1_M.

Finally, given a class $\neq$ identity, its represented by a line bundle $E \not\cong 1_M$. But Q9 $\Rightarrow E \cong E^*$ & hence $E \otimes E \cong E \otimes E^* \cong 1_M$ i.e. $[E]^2 = id$.

Q11 Recall that we define $L_X(Y)$ by

$$L_X(Y)(P) = \lim_{h \to 0} \frac{\phi_h^*(Y)(P) - Y(P)}{h}$$

($\phi_t : U \to M$ local flow corr to X)

$$= \lim_{h \to 0} \frac{(\phi_{-h})_* Y_{\phi_h(P)} - Y_P}{h}$$

i.e.
$$L_X(Y) = \lim_{h \to 0} \left\{ \frac{(\phi_{-h})_* Y - Y}{h} \right\}$$

CLAIM $\quad L_X(Y) = [X, Y]$

Pf For f smooth fn in nbhd of P, compute $[L_X(Y)]_P(f)$: for h small,

$$[(\phi_{-h})_* Y_{\phi_h(P)}](f) - Y_P(f)$$

$$= [Y_{\phi_h(P)}](f \circ \phi_{-h}) - [Y_{\phi_h(P)}](f)$$

$$+ [Y_{\phi_h(P)}](f) - Y_P(f)$$

where $\lim_{h \to 0} \frac{1}{h} \{ [Y_{\phi_h(P)}](f \circ \phi_{-h}) - [Y_{\phi_h(P)}](f) \}$

$$= - \lim_{h \to 0} \left[Y_{\phi_h(P)} \left(\frac{f \circ \phi_{-h} - f}{-h} \right) \right]$$

$$= -Y[X(f)]_P \quad \text{since} \quad X(f) = L_X(f)$$

$$= \lim_{k \to 0} \frac{f \circ \phi_k - f}{k}$$

& $\lim_{h \to 0} \frac{1}{h} \{ [Y(f)]_{\phi_h(P)} - [Y(f)]_P \}$

$$= L_X [Y(f)]_P = X[Y(f)]_P$$

So $L_X Y = [X, Y]$.

Q12** For two forms,

$(\omega_1 \wedge \omega_2)(X_1, \dots, X_{p+q}) =$

$$\frac{1}{p!\,q!} \sum_{\pi \in S_{p+q}} \varepsilon(\pi)\, \omega_1(X_{\pi(1)}, \dots, X_{\pi(p)})\, \omega_2(X_{\pi(p+1)}, \dots, X_{\pi(p+q)})$$

$$= \sum_{p,q \text{ shuffles}} \varepsilon(\pi)\, \omega_1(\quad)\, \omega_2(\quad).$$

i.e. $\pi(1) < \pi(2) < \dots < \pi(p)$
$\pi(p+1) < \pi(p+2) < \dots < \pi(p+q)$

Claim $\quad i(X_1)(\omega_1 \wedge \omega_2) = (i(X_1)\omega_1) \wedge \omega_2 + (-1)^p \omega_1 \wedge (i(X)\omega_2)$

$LHS(X_2, \dots, X_{p+q}) =$

$$\sum_{\substack{p,q \text{ shuffles} \\ \pi(1)=1}} \varepsilon(\pi)\, \omega_1(\quad)\, \omega_2(\quad) + \sum_{\substack{p,q \text{ shuffles} \\ \pi(p+1)=1}} \varepsilon(\pi)\, \omega(\quad)\, \omega_2(\quad).$$

$$= (i(X_1)\omega_1 \wedge \omega_2)(X_2, \dots, X_{p+q}) + (-1)^p (\omega_1 \wedge i(X_1)\omega_2)(X_2, \dots, X_{p+q}).$$

For this question, only need result for $p = 1$, for which argument considerably shortened.

In lectures, we showed also that

(a) $L_X(\omega \wedge \omega') = L_X(\omega) \wedge \omega' + \omega \wedge L_X(\omega')$

(b) $L_X(d\omega) = d(L_X \omega)$.

In particular, (b) $\Rightarrow$ $L_X(dg) = d(L_X g) = d(X(g))$ for any smooth fn g.

Suppose ω is a p form – result clear for $p=0$ & so assume $p>0$ & write $\omega = dg \wedge \eta$, η a p-1 form.
Assume result holds for η.

$\therefore$ $L_X(\omega) = L_X(dg) \wedge \eta + dg \wedge L_X \eta$

$= d(X(g)) \wedge \eta + \underline{dg \wedge i(X) d\eta} + dg \wedge d\, i(X)\eta$.

Now $i(X)\, d\omega = -i(X)\, dg \wedge d\eta = -(X(g))\, d\eta + \underline{dg \wedge i(X) d\eta}$

& $d(i(X)\omega) = d(X(g)\eta) + dg \wedge d\, i(X)\eta$.

$0 = L_X \omega = d\, i(X)\omega + \cancel{i(X) d\omega}$

$\Rightarrow$ $i(X)\omega = dH$ since $H^1_{DR}(M) = 0$

Have $dH \neq 0$ at $P \in \{H = 0\}$ $\Rightarrow$
tgt space is codim 1 subspace of $T_{M,P}$ given by $dH = 0$
& level set is locally a codim 1 submfld of M
(since $H: M \to \mathbb{R}$ has level set as fibre & $(dH)_P$ surj)
Note that if U = open nbhd of fibre cont P, then $v \in T_{U,P}$ acts on $f \in \mathcal{O}_P(M)$ by $f \longmapsto v(f|_U)$. Then for all $g \in \mathcal{O}_{H(P)}(\mathbb{R})$, $(dH)(v)(g) = v(g \circ H) = v(g(H(P))) = 0$
$\Rightarrow$ $T_{U,P} = T_{M,P} \cap \{dH = 0\}$.