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Example Sheet 2

1. (*i*) Is $\alpha \wedge \alpha = 0$ true for every differential form α of positive degree?
   (*ii*) Let α be a nowhere-zero 1-form. Prove that for a p-form β ($p \geq 1$), one has $\alpha \wedge \beta = 0$ if and only if $\beta = \alpha \wedge \gamma$ for some $(p-1)$-form γ. [You might like to do it on $\mathbb{R}^n$ first. Partitions of unity are useful in the general case.]

2. Prove that $\mathbb{R}P^n$ is orientable if and only if n is odd.
   [Hint: consider the $2:1$ map $S^n \to \mathbb{R}P^n$ and a suitable choice of orientation n-form on S^n.]

3. Prove the identity $d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$, for a 1-form ω and vector fields X, Y. *Can you generalize this result to the case when ω is a p-form?

4. Show that
   $$d\omega = 0, \qquad \text{where} \quad \omega = \frac{-ydx + xdy}{x^2+y^2},$$
   but ω cannot be written as df for any smooth function f on $\mathbb{R}^2 \setminus \{0\}$.
   [Hint: consider an appropriate embedding of S^1 in $\mathbb{R}^2$ and integrate the pull-back of ω over S^1.]
   Hence or otherwise deduce that the de Rham cohomology of the circle is $H^1(S^1) = \mathbb{R}$.

5. (*i*) Show that every closed 1-form on S^2 is exact.
   (*ii*) *Construct isomorphisms of de Rham cohomology $H^k(S^n) \cong H^{k-1}(S^{n-1})$, for all $k, n > 1$. Calculate the de Rham cohomology $H^k(S^n)$ for every k, n.
   [You may assume a generalised version of the Poincaré Lemma, namely that for M any smooth manifold, $H^k(M \times \mathbb{R}) \cong H^k(M)$ for all k.]

6. Construct a nowhere-vanishing (smooth) vector field on S^{2n+1} for any n.

7. Let G be a matrix Lie group and X_i, $i = 1, \dots, d = \dim G$, a system of linearly independent left-invariant vector fields on G induced by a basis of $T_I G$. Show that the condition that $\omega^i(X_j) = \delta^i_j$ identically on G defines a system of linearly independent smooth 1-forms ω^i on G. Show further that the 1-forms ω^i are *left-invariant* in the sense that
   $$L_g^*(\omega^i) = \omega^i, \qquad \text{for every } g \in G.$$
   Let C^k_{ij} be a set of real constants determined by $[X_i, X_j] = \sum_k C^k_{ij} X_k$. Deduce from the identity of Question 3 the formula
   $$d\omega^k = -\frac{1}{2} \sum_{i,j} C^k_{ij} \omega^i \wedge \omega^j.$$

8. Modify the construction of Hopf bundle, given in the lectures, replacing $\mathbb{C}$ everywhere by $\mathbb{R}$ to obtain a rank one real vector bundle E over S^1. The total space of this $\mathbb{R}$-analogue of Hopf (vector) bundle is thus a surface (2-dimensional manifold). Can you identify this surface? What is the surface corresponding to $E \otimes E$?

9. Show that every (real) vector bundle can be given a positive definite inner product, varying smoothly with the fibres, i.e. given in each local trivialization (U_α, Φ_α) by a smooth map $g_\alpha : x \in U_\alpha \to g_\alpha(x) \in \mathrm{Sym}_+(k, \mathbb{R})$. Here $k = \operatorname{rank} E$ and $\mathrm{Sym}_+(k, \mathbb{R})$ denotes the set of all real positive-definite $k \times k$ symmetric matrices.
[Hint: you might like to use a partition of unity.]
   Deduce that any vector bundle admits an $O(n)$-structure. Deduce also that any (real) vector bundle is (non-canonically) isomorphic to its dual.

10. Show that the isomorphism classes of line bundles over a manifold M may be given the structure of a (multiplicative) group, where the group operation, inverses and identity should be specified, in which all elements (not equal to the identity) have order 2.

11. Given a vector field X on a manifold M, we let L_X denote the Lie derivative acting on vector fields. If Y is another vector field on M, prove that $L_X(Y) = [X, Y]$.

12** Given a form ω of degree $r > 0$ and a vector field X on a manifold M, we define $i(X)\omega$, the *interior product* of X with ω, to be the $(r-1)$-form given by

$$(i(X)\omega)(X_1, \ldots, X_{r-1}) = \omega(X, X_1, \ldots, X_{r-1}).$$

If L_X denotes the Lie derivative acting on forms, prove the formula

$$L_X\omega = i(X)d\omega + di(X)\omega.$$

   If ω is a closed 2-form with $L_X\omega = 0$ on a manifold M with $H^1_{DR}(M) = 0$, deduce that $i(X)\omega = dH$ for some smooth function H on M. If $i(X)\omega$ is non-zero at a point P, show that the level set of H through P is locally near P a codimension one submanifold of M, and that its tangent space at P is the codimension one subspace of T_PM defined by $\{v \in T_PM \,:\, (i(X)\omega)(v) = 0\}$.