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Example Sheet 1

1. (*i*) Do the charts $\varphi_1(x) = x$ and $\varphi_2(x) = x^3$ ($x \in \mathbb{R}$) belong to the same C^∞ differentiable structure on $\mathbb{R}$?
   (*ii*) Let R_j, $j = 1, 2$, be the manifold defined by using the chart φ_j on the topological space $\mathbb{R}$. Are R_1 and R_2 diffeomorphic?

2. Let X be the metric space defined as follows: Let $P_1, \ldots, P_N$ be distinct points in the Euclidean plane with its standard metric d, and define a distance function (not a metric for $N > 1$) ρ^* on $\mathbb{R}^2$ by $\rho^*(P, Q) = \min\{d(P, Q), \min_{i,j}(d(P, P_i) + d(P_j, Q))\}$. Let X denote the quotient of $\mathbb{R}^2$ obtained by identifying the N points P_i to a single point $\bar{P}$ on X. Show that ρ^* induces a metric on X (the *London Underground metric*). Show that, for $N > 1$, the space X cannot be given the structure of a topological manifold.

3. (*i*) Prove that the product of smooth manifolds has the structure of a smooth manifold.
   (*ii*) Prove that n-dimensional real projective space $\mathbb{R}P^n = S^n/\{\pm 1\}$ has the structure of a smooth manifold of dimension n.
   (*iii*) Prove that complex projective space $\mathbb{C}P^n := (\mathbb{C}^{n+1} \setminus \{\mathbf{0}\})/\mathbb{C}^*$ has the structure of a smooth manifold of dimension $2n$.

4. (*i*) Prove that the complex projective line $\mathbb{C}P^1$ is diffeomorphic to the sphere S^2.
   (*ii*) Show that the natural map $(\mathbb{C}^2 \setminus \{\mathbf{0}\}) \to \mathbb{C}P^1$ induces a smooth map of manifolds $S^3 \to S^2$, the *Poincaré map*. Show that the map induces surjections on the tangent spaces, and that all the fibres are diffeomorphic to S^1.

5. Show that
   (*i*) $SU(2)$ is diffeomorphic to S^3;
   (*ii*) Give an explicit diffeomorphim between TS^1 and $S^1 \times \mathbb{R}$;
   (*iii*) if G is a Lie group, then TG is diffeomorphic to $G \times \mathbb{R}^d$, where $d = \dim G$;
   (*iv*) TS^3 is diffeomorphic to $S^3 \times \mathbb{R}^3$.

6. Given smooth function f on a neighbourhood of the origin in $\mathbf{R}^n$, show that on an appropriate open ball B around the origin, we can write

$$f(\mathbf{r}) = f(\mathbf{0}) + \sum_{i=1}^{n} r_i \frac{\partial f}{\partial r_i}(\mathbf{0}) + \sum_{i,j=1}^{n} r_i r_j \, g_{ij}(\mathbf{r}),$$

with the g_{ij} smooth functions on B.

7. Let $f : M \to N$ be a diffeomorphism of manifolds. If X, Y denote smooth vector fields on M, define the corresponding vector fields f_*X, f_*Y on N. Show that f_* respects the relevant Lie brackets, i.e. that $f_*([X,Y]_M) = [f_*X, f_*Y]_N$ as vector fields on N.

8. Consider the smooth vector fields $X = x\partial/\partial y - y\partial/\partial x$ and $Y = 2(x^2+y^2)\partial/\partial z + x\partial/\partial x + y\partial/\partial y$ on $\mathbb{R}^3 \setminus \{x = y = 0\}$. At any given point $P \in \mathbb{R}^3 \setminus \{x = y = 0\}$, show that they are killed by the linear form $d_P z - 2x d_P x - 2y d_P y$ on the tangent space at P. By finding integrable submanifolds, or otherwise, show that X and Y determine an involutive smooth distribution. Deduce that the involutive distribution extends to one on all of $\mathbb{R}^3$.

9. Suppose that M^m is an embedded submanifold of a smooth manifold N^n and $P \in M$; prove that there exists a coordinate neighbourhood U of P in N and coordinates $x_1, \ldots, x_n$ on U such that $U \cap M$ is given by $x_{m+1} = \ldots = x_n = 0$, with $x_1, \ldots, x_m$ restricting to coordinates on $U \cap M$.

10. For which values of $c \in \mathbb{R}$ is the zero locus in $\mathbb{R}^3$ of the polynomial
$$z^2 - (x^2+y^2)^2 + c$$
an embedded manifold in $\mathbb{R}^3$, and for which values is it an immersed manifold?

11. Prove that the map
$$\rho(x:y:z) = \frac{1}{x^2+y^2+z^2}(x^2, y^2, z^2, xy, yz, zx)$$
gives a well-defined *embedding* of $\mathbb{R}P^2$ into $\mathbb{R}^6$. Construct an embedding of $\mathbb{R}P^2$ in $\mathbb{R}^4$. [Hint: Compose ρ with a suitable map.]

12. By quoting an appropriate theorem from lectures, show that the following groups are Lie groups (in particular, smooth manifolds):
   (*i*) special linear group $SL(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) : \det A = 1\}$;
   (*ii*) unitary group $U(n) = \{A \in GL(n,\mathbb{C}) : AA^* = I\}$, where A^* denotes the conjugate transpose of A and I is the $n \times n$ identity matrix;
   (*iii*) special unitary group $SU(n) = \{A \in U(n) : \det A = 1\}$.

   In each case, find the corresponding Lie algebra.

13. Show that the map $A \to (A - A^t)/2$ represents, on some open neighbourhood of the identity, a chart on $SO(n)$ to an open neighbourhood of $\mathbf{0}$ in its Lie Algebra. Demonstrate a 2-1 group homomorphism from the unit quaternions onto $SO(3)$, and show that it induces a diffeomorphism from $\mathbb{R}P^3$ to $SO(3)$. Deduce that $T(\mathbb{R}P^3)$ is diffeomorphic to $\mathbb{R}P^3 \times \mathbb{R}^3$.