The canonical class on a smooth curve

Given a non-zero rational differential ω on an irreducible smooth curve V and $P \in V$, choose a local parameter $t \in m_{V,P}$. Writing $\omega = f dt$, we define $v_P(\omega) = v_P(f)$.

Lemma 5.5. Our definition of $v_P(\omega)$ does not depend on the choice of local parameter at P, and $v_P(\omega) = 0$ except at finitely many points P of V.

The two statements of (5.5) are proved in the two lemmata below.

Lemma 1. *(i) The numbers $v_P(dh)$ for $h \in \mathcal{O}_{V,P}$ are bounded below.*
(ii) $v_P(dh) \geq 0$ for all $h \in \mathcal{O}_{V,P}$.
(iii) $v_P(dt') = 0$ for any local parameter t' at P.

Proof. (i) Wlog we can assume $V \subset \mathbf{A}^n$ affine. An element of $\mathcal{O}_{V,P}$ has the form $h = f(x_1, \ldots, x_n)/g(x_1, \ldots, x_n)$, where $g(P) \neq 0$ and $x_i \in \mathcal{O}_{V,P}$ is the ith coordinate function on V. Therefore

$$dh = (gdf - fdg)/g^2 = \sum \alpha_i \, dx_i \quad \text{for suitable } \alpha_i \in \mathcal{O}_{V,P} .$$

Thus $v_P(dh)$ is bounded below by $\min\{v_P(dx_i) \,:\, i = 1, \ldots, n\}$.

(ii) Let $m \geq 0$ be the minimum integer such that $v_P(dh) \geq -m$ for all $h \in \mathcal{O}_{V,P}$; such an m exists because of (i). We show that $m = 0$.

Suppose we have $h \in \mathcal{O}_{V,P}$ with $v_P(dh) = -m < 0$. Observe that $dh = d(h - h(P)) = d(th_1)$ for some $h_1 \in \mathcal{O}_{V,P}$. Thus $dh = h_1 dt + t dh_1$, and since $v_P(dh_1) \geq -m$, we deduce that $v_P(dh) > -m$, contrary to assumption. The claim therefore follows.

(iii) Write $t' = ut$ with u a unit in $\mathcal{O}_{V,P}$. Therefore

$$dt' = udt + tdu = (u + th)dt$$

for some $h \in \mathcal{O}_{V,P}$ with $du = hdt$. By (ii), we know that $v_P(h) = v_P(du) \geq 0$, and hence that $v_P(dt') = v_P(u + th) = 0$. QED

In particular, we deduce from (iii) that $v_P(\omega)$ does not depend on the choice of local parameter t, since for any other local parameter t', the rational differential dt' is a multiple of dt by a unit in $\mathcal{O}_{V,P}$. We observe that ω is regular at P iff $v_P(\omega) \geq 0$.

Lemma 2. *If V a smooth irreducible curve and ω a non-zero rational differential, then $v_P(\omega) = 0$ for all but finitely many points P on V.*

Proof. Reduce to the affine case and consider the differential dx_1 for x_1 an affine coordinate function on the curve. Sufficient then to prove the result for dx_1. Clearly dx_1 has only finitely many poles (using Lemma 1), and we show that it has only finitely many zeros by considering the finite extension of fields $k(V)/k(x_1)$. Each coordinate function x_i satisfies an irreducible polynomial equation $f_i(x_1, x_i) = 0$ in $k(V)$, for which $\partial f_i/\partial x_i$ defines a non-zero function on V. More precisely, there are only finitely many points P with $\partial f_i/\partial x_i(P) = 0$. This is true for all i, and so can reduce down to considering points P with $\partial f_i/\partial x_i(P) \neq 0$ for all $i > 1$. For such points P, we must have $v_P(dx_1) = 0$ — to see this, observe that $\partial f_i/\partial x_1 \; dx_1 + \partial f_i/\partial x_i \; dx_i = 0$ in $\Omega^1_{k(V)/k}$ for $i > 1$. Thus if $v_P(dx_1) > 0$, we would have $v_P(dx_i) > 0$ for all i, contradicting the fact that P is a smooth point, since one of the functions $x_i - x_i(P)$ must then be a local parameter at P, and hence in particular $v_P(dx_i) = 0$ for some i. QED

We can now define the divisor (ω) of ω in the obvious way: $(\omega) = \sum_{P \in V} v_P(\omega)P$; such a divisor is called a *canonical divisor*, usually denoted K_V. Any other non-zero rational differential ω' is of the form $\omega' = h\omega$ for some $h \in k(V)^*$, and so $(\omega') = (h) + (\omega)$, i.e. we have a uniquely defined divisor class on V, also denoted K_V, the *canonical class* on V.

Proposition 5.6. *Let $K_V = (\omega)$, then $\Omega^1_V \cong \mathcal{O}_V(K_V)$.*

Proof. For any open $U \subset V$, $\omega' \in \Gamma(U, \Omega^1_V) \iff v_P(\omega') \geq 0$ for all $P \in U$ $\iff \omega' = f\,\omega$ and $(K_V + (f))|_U \geq 0 \iff f \in \Gamma(U, \mathcal{O}_V(K_V))$.
Thus sending ω' to f determines the required isomorphism of sheaves.

Thus, if we write $h^i(V, D)$ for $h^i(V, \mathcal{O}_V(D))$, Serre duality implies that $h^1(V, D) = h^0(V, K_V - D)$, for V a smooth projective curve. We define the *genus* $g(V)$ of V by

$$g(V) = h^1(V, \mathcal{O}_V) = h^0(V, K_V),$$

which is also the dimension of the space global regular forms on V by (5.6).

Theorem 5.7. (Riemann–Roch Theorem)
If V is a smooth projective curve and D a divisor on V, then

$$h^0(V, D) - h^0(V, K_V - D) = 1 - g(V) \; + \deg(D).$$

In particular, taking $D = K_V$, we have $\deg(K_V) = 2g(V) - 2$.