**Part III Complex Manifolds, Lent 2013** pmhw@dpmms.cam.ac.uk

**Example Sheet 3**

(1) Let $\Delta$ denote the $\bar{\partial}$-Laplacian for compactly supported smooth functions $f$ on $\mathbf{C}^n$ with the standard metric (and taking the holomorphic bundle to be trivial). Show that $\Delta(f) = -\frac{1}{2}\sum_j(\partial^2 f/\partial x_j^2 + \partial^2 f/\partial y_j^2)$.

(2) Let $M$ be a complex torus. After choosing a basis for the complex holomorphic 1-forms, show that $H^{1,1}(M) \cap H^2(M, \mathbf{R})$ is isomorphic to the space $V$ of hermitian $n \times n$ matrices. Under this identification, what does the $n$-fold product map $\alpha \mapsto \alpha^n \in H^{2n}(M, \mathbf{R}) = \mathbf{R}$ on $H^{1,1}(M) \cap H^2(M, \mathbf{R})$ correspond to? Show that the classes of real $(1,1)$-forms corresponding to Kähler metrics on $M$ form a certain open cone in $V$, and that the Kähler classes $\kappa$ with $\kappa^n = 1$ may be identified as the quotient space $SL(n, \mathbf{C})/SU(n)$.

(3) Calculate the dimensions of the cohomology groups $H^i(\mathbf{P}^n(\mathbf{C}), \mathcal{O}_{\mathbf{P}^n}(r))$ for all values of $n$, $i$ and $r$.

(4) Let $M$ be the Hopf manifold $\mathbf{C}^d \setminus \{\mathbf{0}\}/\sim$, where $(w_1, \ldots, w_d) \sim (z_1, \ldots, z_d)$ iff $w_j = 2^s z_j$ for all $j$ (for some fixed $s \in \mathbf{Z}$). For $d > 1$, show that $h^0(M, \Omega^r_M) = 0$ for all $1 \le r \le d$ (you may assume that any holomorphic function on $\mathbf{C}^d \setminus \{\mathbf{0}\}$ extends to one on $\mathbf{C}^d$ if $d > 1$).

(5) Let $W = \mathbf{C}^2 \setminus \{\mathbf{0}\}$, and $\alpha$ a complex number with $0 < |\alpha| < 1$. If $G = \langle g \rangle$ acts on $\mathbf{C} \times W$ by

$$g(t, z_1, z_2) = (t, \alpha z_1 + t z_2, \alpha z_2),$$

show that the quotient $M = (\mathbf{C} \times W)/G$ is a complex manifold, with an obvious map to $\mathbf{C}$; the fibres $M_t$ for $t \in \mathbf{C}$ form what is called an *analytic family.*

Show, if $t \neq 0$, that $M_t \cong M_1$. *On the other hand, prove $M_0 \not\cong M_1$ (much harder!). Is $M$ a Kähler manifold?

(6) Let $E$ be a holomorphic line bundle on a compact complex manifold $M$, and suppose that $L$ is a positive holomorphic line bundle. For all $i > 0$, show that $H^i(M, E \otimes L^{\otimes k}) = 0$ for all $k >> 0$. Assuming the Kodaira embedding theorem, namely that for all $k >> 0$, $L^{\otimes k}$ corresponds to a hyperplane bundle for some projective embedding of $M$, and Bertini's theorem that a general hyperplane section of a complex projective manifold is a complex submanifold, show that $h^0(M, E \otimes L^{\otimes r})$ is a polynomial in $r$ for all $r >> 0$.

(7) Let $M$ be a compact Kähler manifold. Using the exponential short exact sequence, show that any integral class in $H^{1,1}(M)$ is the Chern class of some line bundle on $M$.

(8) Let $V$ be a codimension one complex submanifold of a complex manifold $M$ of dimension $n$, and let $1 \leq p \leq n$; show that there exist short exact sequences of sheaves

$$0 \to \Omega^p_M(-V) \to \Omega^p_M \to \Omega^p_M|_V \to 0 \quad \text{on } M,$$

$$0 \to \Omega^{p-1}_V(-V) \to \Omega^p_M|_V \to \Omega^p_V \to 0 \quad \text{on } V,$$

defining carefully the sheaves involved.

(9) Suppose in the previous question that the line bundle $[V]$ is positive on a compact complex manifold $M$; deduce that $h^{p,q}(M) = h^{p,q}(V)$ for $p+q < n-1$, and $h^{p,q}(M) \leq h^{p,q}(V)$ for $p+q = n-1$. Deduce the Lefschetz Hyperplane theorem for the betti numbers, that $b_i(M) = b_i(V)$ for $i < n-1$ and $b_i(M) \leq b_i(V)$ for $i = n-1$. Give an example of a smooth surface $V \subset \mathbf{P}^3(\mathbf{C})$ for whch $b_2(\mathbf{P}^3) < b_2(V)$.

(10) Let $M$ be a compact Kähler manifold of dimension $n$, with Kähler form $\omega$. We define the operator $L$ on $r$-forms by $L(\eta) := \omega \wedge \eta$, and let $\Lambda = L^* = (-1)^r * L *$ be its formal adjoint. An $r$-form $\eta$ is called *primitive* if $\Lambda(\eta) = 0$. Show that any *closed* primitive $(p,p)$-form on $M$ is harmonic.

Given a non-zero $(1,1)$-form $\alpha$ on $M$, show that there exists a unique element $\theta \in A^{1,0}(T'_M)$ whose contraction with $\omega \in A^{1,1}(M) = A^{0,1}(T'^*_M)$ satisfies $i(\theta)(\omega) = \alpha$.

Suppose now that $\alpha$ is real and primitive; show that $i(\theta)(\alpha) = -\frac{1}{2}\Lambda(\alpha \wedge \alpha)$. (You may find the formula for $\Lambda$ at the top of page 114 of Griffiths and Harris useful for this part.) Furthermore, for any real $(1,1)$-form $\beta$, prove that

$$i(\theta)(\beta) \wedge \omega^{n-1} = -(n-1)\alpha \wedge \beta \wedge \omega^{n-2}.$$

When $\alpha$ is also closed, deduce that its cohomology class satisfies $[\alpha]^2 \cup [\omega]^{n-2} < 0$. In this case, show that $i(\theta)(\alpha)$ is a closed form if and only if the form $\alpha \wedge \alpha$ is harmonic.