Part III Complex Manifolds, Lent 2013 pmhw@dpmms.cam.ac.uk

Example Sheet 2

(1) Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space X, and $\mathcal{U} = \{U_\alpha\}$ an open cover of X. Suppose furthermore that $H^q(U_\alpha, \mathcal{F}) = 0$ for all $q > 0$ and all α. Give a counterexample to the assertion that $H^p(\mathcal{U}, \mathcal{F}) \cong H^p(X, \mathcal{F})$ for all $p \geq 0$.

(2) Let $M = \mathbf{P}^1(\mathbf{C})$ and P, Q distinct points of M. Let $\mathcal{O}_M(-P-Q)$ denote the sheaf of holomorphic functions vanishing at both P and Q. Show that there is a short exact sequence of sheaves

$$0 \to \mathcal{O}_M(-P-Q) \to \mathcal{O}_M \to \mathbf{C}_P \oplus \mathbf{C}_Q \to 0,$$

where the sheaf on the right should be carefully defined. Deduce that the map on sections $\Gamma(M, \mathcal{O}_M) \to \Gamma(M, \mathbf{C}_P \oplus \mathbf{C}_Q)$ is not surjective and that $H^1(\mathcal{O}_M(-P-Q)) \neq 0$.

(3) If $\iota : Y \hookrightarrow X$ is the inclusion of a closed subspace Y into a paracompact space X and $\mathcal{F}$ is a sheaf of abelian groups on Y, prove that $H^j(Y, \mathcal{F}) \cong H^j(X, \iota_* \mathcal{F})$ for all $j \geq 0$. Give an example to show that this statement fails for Y an open subset.

(4) For M a Riemann surface and $P \in M$, let $\mathcal{M}_P$ denote the germs of meromorphic functions at P — these are essentially determined by local Laurent series. The image of a germ $f \in \mathcal{M}_P$ in $\mathcal{M}/\mathcal{O}_{\mathcal{M},\mathcal{P}}$, will be called the *principal part* of f at P. Suppose now $P_1, \ldots, P_n \in M$ and we specify certain principle parts at the P_i. If $H^1(M, \mathcal{O}_M) = 0$, show that there is a global meromorphic function h with the specified principle parts at the P_i and holomorphic elsewhere. Show that this is not true without the assumption on cohomology.

(5) A slightly stronger version of the $\bar{\partial}$-Poincaré lemma (proved in Griffiths and Harris) states that for any polydisc $\Delta \subset \mathbf{C}^n$, the Dolbeault cohomology $H^{p,q}_{\bar{\partial}}(\Delta) = 0$ for all $q > 0$. Assuming this (for $n = 1$), prove that $H^1(\mathbf{P}^1(\mathbf{C}), \mathcal{O}_{\mathbf{P}^1}) = 0$. Calculate the dimensions $h^0(\mathbf{P}^1(\mathbf{C}), \mathcal{O}_{\mathbf{P}^1}(d))$ for all integers d (where h^i denotes the complex dimension of the cohomology group H^i). Using long exact sequences of cohomology, calculate $h^1(\mathbf{P}^1(\mathbf{C}), \mathcal{O}_{\mathbf{P}^1}(d))$ for all integers d, and verify the formula that

$$h^1(\mathbf{P}^1(\mathbf{C}), \mathcal{O}_{\mathbf{P}^1}(d)) = h^0(\mathbf{P}^1(\mathbf{C}), \mathcal{O}_{\mathbf{P}^1}(-2-d))$$

for all integers d. (This is just a very special case of Serre duality, which for a compact Riemann surface M and holomorphic vector bundle E on M, states that $h^1(M, \mathcal{O}_M(E)) = h^0(M, \mathcal{O}_M(K_M \otimes E^*))$.)

(6) Show that any holomorphic line bundle on a disc $\Delta \subset \mathbf{C}$ is trivial. Deduce that any holomorphic line bundle on $\mathbf{P}^1(\mathbf{C})$ is of the form $[H]^{\otimes n}$ for some integer n.

(7) Let M be a complex manifold, and let J denote the endomorphism of its real tangent bundle (corresponding to multiplication by i on the holomorphic tangent bundle T'_M). Given a Riemanian metric on M (considered as a real manifold), find a necessary and sufficient condition for it to come from a hermitian metric on M.

(8) Let X be a smooth manifold and E a complex vector bundle on X. For ψ a complex 1-form on X, we consider $d\psi$ as an alternating 2-form via the natural identification.

For complex vector fields X, Y, show that

$$2\, d\psi(X,Y) = X\psi(Y) - Y\psi(X) - \psi([X,Y]).$$

Suppose that $D : \mathcal{A}(E) \to \mathcal{A}^1(E)$ is a connection on E, with $R : \mathcal{A}(E) \to \mathcal{A}^2(E)$ the curvature of D. With the 2-form part of $R \in \mathcal{A}^2(\mathrm{Hom}(E,E))$ considered as an alternating form, show that

$$2\, R(X,Y) = [D_X, D_Y] - D_{[X,Y]}.$$

(9) Suppose that L_1, L_2 are complex line bundles on a smooth manifold X, with connections D_1, D_2 respectively. Show that $D = D_1 + D_2$ defines a connection on $L_1 \oplus L_2 = E$. Deduce that $c_1(E) = c_1(L_1) + c_1(L_2)$ in $H^2_{DR}(M, \mathbf{C})$, and $c_2(E) = c_1(L_1)c_1(L_2)$ in $H^4_{DR}(M, \mathbf{C})$.

(10) Suppose E is a rank r complex vector bundle on a manifold and D is a hermitian connection on E. For $1 \le m \le r$, show that the induced connection $D^{(m)}$ on $\Lambda^m E$ is a hermitian connection (with respect to the induced hermitian metric).

(11) Let D be a connection on a complex vector bundle E. We define the dual connection D^* on E^* by specifying that for local sections σ of E^* and s of E, we have the identity

$$(D^*\sigma)(s) = d(\sigma(s)) - \sigma(Ds).$$

Check that D^* is a connection.

Given a hermitian metric on E, we define a dual metric on E^* by specifying that the dual frame to any unitary frame is unitary. If D is a hermitian connection on the hermitian vector bundle E, show that D^* is a hermitian connection on E^*.

Suppose now E is a hermitian holomorphic vector bundle over a complex manifold and that D is the Chern connection on E; show that D^* is the Chern connection on E^*.

(12) If E is a holomorphic vector bundle on a complex manifold M, and $F \subset E$ is a holomorphic subbundle, then a hermitian metric on E induces one on F and we have a direct sum decomposition of complex smooth bundles $E = F \oplus F^{\perp}$. If D_E is the Chern connection on E, show that the composite (in obvious notation) $\pi_F \circ D_E$ is the Chern connection on F.