Example Sheet 1

(1) If $f = (f_1, \ldots, f_n) : \mathbf{C}^n \to \mathbf{C}^n$ is holomorphic, where $f_j = u_j + iv_j$ and $z_k = x_k + iy_k$, show that

$$\det \left(\frac{\partial(u_1, v_1, \ldots, u_n, v_n)}{\partial(x_1, y_1, \ldots, x_n, y_n)} \right) = \left| \det \left(\partial f_j / \partial z_k \right) \right|^2 .$$

Deduce that any complex manifold is orientable.

(2) If X is a smooth manifold, U a connected open subset of X and $P \in U$, show that the natural map $\mathcal{A}_X(U) \to \mathcal{A}_{X,P}$ is always surjective, but not in general injective. If M is a complex manifold, $P \in U$ as before, show that the map $\mathcal{O}_M(U) \to \mathcal{O}_{M,P}$ is always injective, but not in general surjective.

(3) For $\phi : \mathcal{F} \to \mathcal{G}$ a morphism of sheaves on a topological space X, show that

(a) ϕ is injective $\iff$ $\phi_P : \mathcal{F}_P \to \mathcal{G}_P$ is injective for all $P \in X$.

(b) ϕ is an isomorphism $\iff$ $\phi_P : \mathcal{F}_P \to \mathcal{G}_P$ is an isomorphism for all $P \in X$.

(4) If two holomorphic vector bundles of rank r on a complex manifold M have the same transition functions for trivializations with respect to some open cover $\{\mathcal{U}_\alpha\}$, show that they are isomorphic as holomorphic vector bundles.

(5) Suppose $\mathcal{U} = \{U_\alpha\}$ is an open cover of a topological space X, and that on each U_α we have a sheaf of abelian groups $\mathcal{F}_\alpha$. If on each $U_\alpha \cap U_\beta$ we have an isomorphism of sheaves

$$g_{\alpha\beta} : \mathcal{F}_\beta|_{U_\alpha \cap U_\beta} \to \mathcal{F}_\alpha|_{U_\alpha \cap U_\beta}$$

such that $g_{\alpha\alpha} = \mathrm{id}$ over U_α, and over each non-empty $U_\alpha \cap U_\beta \cap U_\gamma$ the compatibility relation $g_{\alpha\gamma} = g_{\alpha\beta} g_{\beta\gamma}$, show there exists a sheaf $\mathcal{F}$ on X with $\mathcal{F}|U_\alpha \cong \mathcal{F}_\alpha$ for all α.

(6) Let $\mathcal{O}_{\mathbf{C}^n,\mathbf{0}}$ denote the $\mathbf{C}$-algebra of germs of holomorphic functions at $\mathbf{0} \in \mathbf{C}^n$. If $w_1, \ldots, w_n$ are the holomorphic coordinate functions on $\mathbf{C}^n$, let $\partial/\partial w_i$ denote $\partial/\partial w_i|_{\mathbf{0}}$, i.e. the map $\mathcal{O}_{\mathbf{C}^n,\mathbf{0}} \to \mathbf{C}$ given by $f \mapsto (\partial f/\partial w_i)(\mathbf{0})$. Show that the $\partial/\partial w_i$ are complex derivations of $\mathcal{O}_{\mathbf{C}^n,\mathbf{0}}$ for $i = 1, \ldots, n$, and that they form a basis for $T'_{\mathbf{C}^n,\mathbf{0}}$ over $\mathbf{C}$.

(7) Let $M = \mathbf{C}^n/\Lambda$ be a complex torus and N a complex submanifold of dimension d. Prove that the space of global holomorphic r-forms on N has dimension at least $\binom{d}{r}$ for all $r \leq d$.

(8) Given a point $x \in \mathbf{P}^n(\mathbf{C})$, let $[x]$ denote the line through the origin in $\mathbf{C}^{n+1}$ representing x. Show that the set $\coprod_{x \in \mathbf{P}^n} [x]$ can be given the natural structure of a line bundle on $\mathbf{P}^n(\mathbf{C})$, the *tautological* line bundle L. Show that L is dual to the hyperplane bundle $[H]$.

(9) Let M be an n-dimensional complex manifold M. Suppose that a holomorphic vector bundle $E = L_1 \oplus \ldots \oplus L_{n-r}$ is a sum of holomorphic line bundles on M, and that V is an r-dimensional submanifold of M defined by the vanishing of some global holomorphic section of E; prove that $N_V \cong E|_V$. Find a formula for the canonical bundle K_V.

Suppose $V \subset \mathbf{P}^n(\mathbf{C})$ is a projective variety defined by the vanishing of homogeneous polynomials $F_1 = \ldots = F_{n-r} = 0$, with rank $(\partial F_i/\partial X_j) = n - r$ at all points of V, where $r > 0$. Prove that V is an r-dimensional complex submanifold of $\mathbf{P}^n(\mathbf{C})$ (you may assume the fact that V is connected). If $\deg F_i = d_i$ for $i = 1, \ldots, n-r$, such a V is called a *complete intersection* of type $(d_1, \ldots, d_{n-r})$ in $\mathbf{P}^n(\mathbf{C})$ — we can clearly assume here that $1 < d_1 \leq d_2 \leq \ldots \leq d_{n-r}$. Prove that the canonical bundle

$$K_V \cong [(d_1 + \ldots + d_{n-r} - n - 1)H]\,|_V.$$

Hence find all the types of 3-dimensional complete intersections with K_V trivial.

(10) Let M be a complex manifold, $V \subset M$ a codimension one complex submanifold and $\mathcal{E}$ a torsion-free $\mathcal{O}_M$-module. Let $\{U_\alpha\}$ be an open cover of M such that the ideal sheaf $\mathcal{I}(V)|_{U_\alpha} = f_\alpha \mathcal{O}_{U_\alpha} \subset \mathcal{O}_{U_\alpha}$ for all α. Considering the $\mathcal{O}_{U_\alpha}$-module $\mathcal{F}_\alpha = \mathcal{E}|_{U_\alpha}$ for each α, and defining isomorphisms of $\mathcal{O}_{U_\alpha \cap U_\beta}$-modules

$$g_{\alpha\beta} : \mathcal{F}_\beta|_{U_\alpha \cap U_\beta} \to \mathcal{F}_\alpha|_{U_\alpha \cap U_\beta}$$

by $g_{\alpha\beta}(\sigma) = \frac{f_\alpha}{f_\beta}\sigma$ for all sections of $\mathcal{F}_\beta|_{U_\alpha \cap U_\beta}$, deduce the existence of a corresponding $\mathcal{O}_M$-module, called the sheaf of *meromorphic sections of $\mathcal{E}$ with at most a simple pole along V* and denoted $\mathcal{E}(V)$ — locally its sections may be considered of the form m/f for m a local section of $\mathcal{E}$ and f a local generator for the ideal of V. If $\mathcal{E}$ is the sheaf of holomorphic sections of a holomorphic vector bundle E, show that $\mathcal{E}(V)$ is isomorphic to the sheaf of holomorphic sections of $E \otimes [V]$.

(11) With the notation as in the previous question, consider the subsheaf $\Omega^r_M(\log V)$ of $\Omega^r_M(V)$ consisting of meromorphic r-forms ω with locally both $f\omega$ and $fd\omega$ holomorphic. Show that $\Omega^r_M(\log V)$ is a well-defined locally free $\mathcal{O}_M$-module, and when $r = n$ that $\Omega^n_M(\log V) = \Omega^n_M(V)$.

Given $\omega \in \Gamma(U, \Omega^r_M(\log V))$, write $\omega = df/f \wedge \omega' + \omega''$, with $\omega' \in \Gamma(U, \Omega^{r-1}_M)$ and $\omega'' \in \Gamma(U, \Omega^r_M)$, and then take the restriction $\omega'|_V$;

defines a morphism of sheaves $\Omega^r_M(\log V) \to \Omega^{r-1}_V$ on M , where Ω^{r-1}_V here denotes the $\mathcal{O}_M$-module $\iota_* \Omega^{r-1}_V$, with ι denoting the inclusion map $V \subset M$. Deduce that there is an exact sequence of $\mathcal{O}_M$-modules

$$0 \to \Omega^r_M \to \Omega^r_M(\log V) \to \Omega^{r-1}_V \to 0.$$

[The map defined above is called the *Poincaré Residue map*, and generalizes the concept of residues from the case $n = 1$.]

(12) In the case when V is a hypersurface in M, by taking $r = n$ in the previous question, deduce the adjunction formula.