Algebraic Geometry 2017 Ex Sht 4 Answers

(Ex Class 4 : Tue 23 Jan at 4pm in MR4)

1) For U open in Y, have HMs

$$\phi_* \mathcal{F}(U) \otimes_{\mathcal{O}_Y(U)} \mathcal{G}(U) = \mathcal{F}(\phi^{-1}U) \otimes_{\mathcal{O}_Y(U)} \mathcal{G}(U)$$

$$\cong \mathcal{F}(\phi^{-1}U) \otimes_{\mathcal{O}_X(\phi^{-1}U)} \mathcal{O}_X(\phi^{-1}U) \otimes_{\mathcal{O}_Y(U)} \mathcal{G}(U)$$

$$\xrightarrow{\theta_1} \mathcal{F}(\phi^{-1}U) \otimes_{\mathcal{O}_X(\phi^{-1}U)} (\phi_* \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{G})(U)$$

$$\cong \mathcal{F}(\phi^{-1}U) \otimes_{\mathcal{O}_X(\phi^{-1}U)} (\phi_* \phi^* \mathcal{G})(U) \quad \text{by Ex Sht 2, Q16}$$

$$= \mathcal{F}(\phi^{-1}U) \otimes_{\mathcal{O}_X(\phi^{-1}U)} (\phi^* \mathcal{G})(\phi^{-1}U)$$

$$\xrightarrow{\theta_2} (\mathcal{F} \otimes_{\mathcal{O}_X} \phi^* \mathcal{G})(\phi^{-1}U) = \phi_*(\mathcal{F} \otimes_{\mathcal{O}_X} \phi^* \mathcal{G})(U)$$

(where θ_1, θ_2 are IMs when U (& hence $\phi^{-1}U$) affine by Remark 2 on tensor products over affine varieties from Lectures). Thus the induced morphism of sheaves (univ property of $\otimes$)

$$\phi_* \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G} \longrightarrow \phi_* (\mathcal{F} \otimes_{\mathcal{O}_X} \phi^* \mathcal{G}) \text{ is}$$

an IM as claimed.

2) Wlog X affine, $A = k[X]$. Given $P \in X$,

$\mathcal{M}_P$ is a f.g. $\mathcal{O}_{X,P} = A_m$ - module $\Rightarrow$ it is free

(since $\mathcal{O}_{X,P}$ a DVR, it is a PID)

ie. $\exists$ IM $\quad \mathcal{O}_{X,P}^m \xrightarrow{\sim} \mathcal{M}_P$.

For an appropriate affine open set $U \ni P$, this yields $\mathcal{O}_X(U)^m \to \mathcal{M}(U)$ & $\mathcal{O}_U^m \xrightarrow[\theta]{} \mathcal{M}|_U$ inducing an IM on stalks at P. Wlog we can always assume $\mathcal{O}_U^m \twoheadrightarrow \mathcal{M}|_U$ surjective [if $\mathcal{C} = \mathrm{Coker}\,\theta$, then $\mathcal{C}$ is f.g. & has zero stalk at P $\overset{(3.3)}{\Rightarrow}$ $\mathcal{C}$ has zero stalks in some nbhd U' of P i.e. $\mathcal{C}|_{U'} = 0$. So wlog take $U = U'$ & so $\mathcal{O}_U^m \twoheadrightarrow \mathcal{M}|_U$].

For next part, we now use coherence. Let $A = k[U]$, then $\mathcal{M}|_U \cong \tilde{M}$ for some f.g. A-module M, & $A^m \twoheadrightarrow M$ surjective. Let $K = \ker(A^m \to M)$ & $\mathcal{K} = \tilde{K}$; then $\mathcal{K}$ f.g. with $\mathcal{K}_P = 0$ $\Rightarrow$ $\mathcal{K}|_{U'} = 0$ for some $U' \ni P$. Shrinking U again, then have $\mathcal{O}_U^m \xrightarrow{\sim} \mathcal{M}|_U$ as required.

3) We know $\Omega^1_{A/k} = (A\,dx + A\,dy)/(2y\,dy - 3x^2\,dx)$

Consider $\omega = 2x\,dy - 3y\,dx \in \Omega^1_{A/k}$, clearly non-zero in $\Omega^1_{A/k}$ (and in $\Omega^1_{A_{m_P}/k}$); note $A \cong k[t^2, t^3]$ & an elt of $\Omega^1_{A/k}$ written uniquely as $(a + bx + cx^2)dy + \alpha\,dx$, $\alpha \in A$.

However $y\,\omega = 2xy\,dy - 3y^2\,dx = x(2y\,dy - 3x^2\,dx)$

$$= 0 \text{ in } \Omega^1_{A/k}.$$

4) Wlog V affine, $k[V] = A$, $V \subseteq \mathbb{A}^N$, $m_P \lhd \mathcal{O}_{V,P}$. The derivation $d : A \to \Omega^1_{A/k}$ induces derivation $d : \mathcal{O}_{V,P} \to \Omega^1_{A/k} \otimes \mathcal{O}_{V,P} = \Omega^1_{V,P}$

and hence a linear map of $k = \mathcal{O}_{V,P}/m_P$ vector spaces

$$m_P/m_P^2 \cong m_P \otimes_{\mathcal{O}_{V,P}} \mathcal{O}_{V,P}/m_P \xrightarrow{d_P = d\otimes 1} \Omega^1_{V,P} \otimes \mathcal{O}_{V,P}/m_P$$

$$\cong \Omega^1_{V,P}/m_P\,\Omega^1_{V,P}$$

(d_P zero on m_P^2 since $d_P(fg) = f(P)d_Pg + g(P)d_Pf$),

where RHS $\cong \Omega^1_{A/k}/M_P\,\Omega^1_{A/k}$, where $M_P \lhd A$ corresponds to P. (since $\Omega^1_{V,P} = \Omega^1_{A/k} \otimes \mathcal{O}_{V,P}$)

Now $\Omega^1_{A/k}$ generated $/A$ by $dX_1, \dots, dX_N$ & so d_P is surjective. RTP d_P injective

Suppose $\exists\ g \in m_P$ s.t. $d_P g = 0$; wlog we can take $g \in A$, since $d_P(f/h) = d_P(f)/h(P) - \frac{\overset{=0}{f(P)}}{h(P)^2} d_P h$.

Choose $G \in k[X_1, \dots, X_N]$ representing g and so

$$d_P G := \sum_{i=1}^{N} \frac{\partial G}{\partial X_i}(P)\, dX_i = 0 \quad \text{in } \Omega^1_{A/k}/M_P\,\Omega^1_{A/k}.$$

Wlog assume $P = \underline{0}$; now $\exists\ F_1, \dots, F_m \in I(V)$ & $a_1, \dots, a_m \in k$ s.t. $d_P G = \sum_{i=1}^{m} a_i\, d_P F_i$ in $\Omega^1_{k[\underline{X}]/k}/\tilde{M}_P\,\Omega^1_{k[\underline{X}]/k}$ by characterization (†) for $\Omega^1_{A/k}$ in §5, where we set $\tilde{M}_P = \langle X_1, \dots, X_N\rangle \lhd k[\underline{X}]$.

So $d_P(G - \sum a_i F_i) = 0$ ie. $G - \sum a_i F_i$ has no linear terms & hence is in $\tilde{M}_P^2 = \langle X_1, \dots, X_N\rangle^2$.

$\therefore\ g \in m_P^2$ & so d_P injective

5) We know that $\Omega^1_{\mathbb{A}^n}(\mathbb{A}^n) = \Omega^1_{k[x_1,\dots,x_n]/k}$

is free over $k[\underline{X}]$ generated by $dx_1, \dots, dx_n$.

$\therefore$ $\Omega^n_{\mathbb{A}^n}(\mathbb{A}^n) = \Omega^n_{k[x_1,\dots,x_n]/k}$ is free of rank 1

over $k[\underline{X}]$ generated by $\omega_0 = dx_1 \wedge \dots \wedge dx_n$.

If $X_0, \dots, X_n$ are homogeneous coords on $\mathbb{P}^n$ & $\mathbb{A}^n$ given

by $X_0 \neq 0$, then $x_i = X_i/X_0$ for $i = 1, \dots, n$ affine coords

Consider now the affine piece U_i given by $X_i \neq 0$;

set $\omega_i = (-1)^i \, d(1/x_i) \wedge d(x_1/x_i) \wedge \dots \wedge \widehat{d\left(\frac{x_i}{x_i}\right)} \wedge \dots \wedge d\left(\frac{x_n}{x_i}\right)$

Then $\Omega^n_{\mathbb{P}^n}\big|_{U_i} = \omega_i \cdot \mathcal{O}_{U_i}$

Clear that $\omega_j = 1/x_j^{n+1} \, \omega_0 = \left(X_0/X_j\right)^{n+1} \omega_0$

$$= \left(X_i/X_j\right)^{n+1} \omega_i$$

& so $g_j \omega_j = g_i \omega_i \iff g_j = \left(X_j/X_i\right)^{n+1} g_i$

So transition functions $\psi_{ji} = \left(X_j/X_i\right)^{n+1} \Rightarrow$

$\Omega^n_{\mathbb{P}^n} = K_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$.

6) A homogeneous monomial of degree $d = \sum_0^n d_i$ is of

the form $X_0^{r_0} \dots X_n^{r_n}$ with at least one $r_j \geq d_j$

— hence claim.

$\check{C}^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = \Gamma(U_0 \cap \dots \cap U_n, \mathcal{O}_{\mathbb{P}^n})$

(notation as in Q5)

Any elt of RHS of form $\sigma = F / x_0^{d_0} \dots x_n^{d_n}$ for suitable $d_i \geq 0$ & F homogeneous of degree $d = \sum d_i$

By previous part, write $F = \sum (-1)^j x_j^{d_j} G_j$

where $\tau_j = x_j^{d_j} G_j / x_0^{d_0} \dots x_n^{d_n} \in \Gamma(U_0 \cap \dots \cap \widehat{U_j} \cap \dots \cap U_n)$

i.e. $\sigma = \sum (-1)^j \tau_j$ is a coboundary

$\therefore \quad H^n(\mathcal{U}, \mathcal{O}_{\mathbb{P}^n}) = 0$ for affine cover $\mathcal{U} = \{U_0, \dots, U_n\}$

$\Rightarrow \quad H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = 0 \quad \forall n > 0$

<u>Case $n = 1$</u> : Use s.e.s.

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1} \to k \to 0 .$$

les on cohomology $\Rightarrow \quad 0 \to k \to k \to H^1(\mathcal{O}_{\mathbb{P}^1}(-1)) \to 0$

$\Rightarrow H^1(\mathcal{O}_{\mathbb{P}^1}(-1)) = 0$. For $s \geq 0$, use

$$0 \to \mathcal{O}_{\mathbb{P}^1}(s-1) \to \mathcal{O}_{\mathbb{P}^1}(s) \to k \to 0 \quad \&$$

les on cohomology to deduce $H^1(\mathcal{O}_{\mathbb{P}^1}(s)) = 0 \quad \forall s \geq 0$

<u>Case $n > 1$</u> Use induction : <u>assume</u>

$H^{n-1}(\mathcal{O}_{\mathbb{P}^{n-1}}(-r)) = 0$ if $r < n$. Use s.e.s.

$$0 \to \mathcal{O}_{\mathbb{P}^n}(s-1) \to \mathcal{O}_{\mathbb{P}^n}(s) \to \mathcal{O}_{\mathbb{P}^{n-1}}(s) \to 0$$

If $s > -n$, have $H^n(\mathcal{O}_{\mathbb{P}^n}(s-1)) \cong H^n(\mathcal{O}_{\mathbb{P}^n}(s))$,

& since $H^n(\mathcal{O}_{\mathbb{P}^n}) = 0$, all these are zero.

Serre Duality $\Rightarrow \quad h^n(\mathcal{O}_{\mathbb{P}^n}(-r)) = h^0(\mathcal{O}_{\mathbb{P}^n}(-n-1+r)) = 0$

iff $r < n+1$.

7) Wlog $W \subset \mathbb{P}^2$ covered by affine pieces
$\mathbb{A}^2_0\ (x,y)$ & $\mathbb{A}^2_1\ (u,v)$, where $x = 1/u$, $y = v/u$
i.e. $(0:0:1) \notin W$.

Suppose $W_0 \subset \mathbb{A}^2_0$ is defined by $f(x,y) = 0$
Note that $f(x,y) = f(1/u, v/u) = u^{-d} g(u,v)$
where $W_1 \subset \mathbb{A}^2_1$ defined by $g(u,v) = 0$.
Now $\partial f/\partial x\, dx + \partial f/\partial y\, dy = 0$ in $\Omega^1_{k(W)/k}$ (*)

$$\Rightarrow \omega := \frac{dx}{\partial f/\partial y} = -\frac{dy}{\partial f/\partial x} \quad \text{in } \Omega^1_{k(W)/k}$$

For $P \in W_0$, one of dx, dy must be a local generator for $\Omega^1_{W,P}$ (one of $v_P(x - x(P)), v_P(y - y(P))$ has to be one, since $m_P = (x - x(P), y - y(P)) \triangleleft \mathcal{O}_{W,P}$)
But dx a local generator $\Rightarrow \partial f/\partial y\,(P) \neq 0$
(since $v_P(dx) = 0$ & P smooth $\Rightarrow$ cannot have both $\partial f/\partial y\,(P) = 0 = \partial f/\partial x\,(P)$, which would occur if $\partial f/\partial y\,(P) = 0$ using (*))
Similarly if dy a local generator, then $\partial f/\partial x\,(P) \neq 0$

Hence ω everywhere a local generator on W_0.
i.e. $(\omega)|_{W_0} = 0$.

But $$\frac{dx}{\partial f/\partial y} = \frac{-u^{-2}\,du}{u^{-(d+1)}\,\partial g/\partial v} = -u^{d-3}\frac{du}{\partial g/\partial v}$$

since $\partial f/\partial y = u^{-d}\, \partial g/\partial y = u^{-d}\, \partial v/\partial y\, \partial g/\partial v$

$$= u^{-(d-1)}\, \partial g/\partial v$$

& $v = uy = y/x \Rightarrow \partial v/\partial y = 1/x = u.$

Note that $\omega' = -\dfrac{du}{\partial g/\partial v}$ is everywhere a local generator on W_1 & $\omega = u^{(d-3)}\omega' = \left(\dfrac{X_0}{X_1}\right)^{d-3}\omega' \Rightarrow$

canonical divisor $K_W = ((d-3)H)$ — H given by $X_0 = 0$

& canonical bundle $K_W = \mathcal{O}_W(d-3)$.

Moreover $\deg K_W = (d-3)\deg H|_W = (d-3)d$

$\Rightarrow 2g-2 = d(d-3) \Rightarrow g(W) = \frac{1}{2}(d-1)(d-2)$.

8) Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$

$$(X_0:X_1)\times(Y_0:Y_1) \mapsto (X_0Y_0 : X_1Y_0 : X_0Y_1 : X_1Y_1)$$

has image $Z_{00}Z_{11} = Z_{01}Z_{10}$. Note that it induces an IM of affine pieces $\mathbb{A}^1_0 \times \mathbb{A}^1_0 \xrightarrow{\sim}$ affine piece given by $Z_{00} \neq 0$

via $(x, y) \mapsto (1 : x : y : xy)$

Set $x = X_1/X_0$, $y = Y_1/Y_0$. So on $\mathbb{A}^2$, curve Y is given by $f(x,y) = 0$ & local generator of Ω^1_Y is $\omega = \dfrac{dx}{\partial f/\partial y} = -\dfrac{dy}{\partial f/\partial x}$ as in Q7.

Wlog assume $(\infty,\infty) \notin Y$ i.e. $(1:0)\times(1:0) \notin Y$.

We may therefore reduce to considering affine pieces of X with coords $u = 1/x,\ y$, respectively $x,\ v = 1/y$.

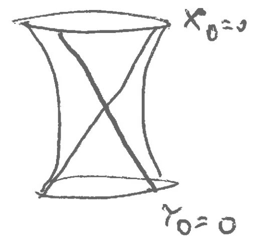
Since $u = 1/x$, $dx = -1/u^2\, du$ &

$$f(x,y) = f(1/u, y) = u^{-a} g(u,y) \text{ say,}$$

where $\partial f/\partial y = u^{-a}\, \partial g/\partial y$.

$$\therefore \quad \omega = -u^{a-2} \frac{du}{\partial g/\partial y} \Rightarrow (\omega) = (a-2) H_1$$

on first affine piece of Y, where H_1 is the line given by $X_0 = 0$ intersected with Y.



Similarly $(\omega) = (b-2) H_2$ on the other affine piece, where H_2 is the line $Y_0 = 0$ intersected with Y

$$\text{Thus } \deg K_Y = (a-2)b + (b-2)a = 2(ab-a-b) = 2g-2.$$

$\Rightarrow g = ab - a - b + 1$ e.g. $a = b = 1 \rightsquigarrow g = 0$

$a = b = 2 \rightsquigarrow g = 1$.

9) The cubic is standard fare and more details may be found in [W], [R], [S] or [H]. The basic idea is that the cubic has genus 1 & that $\exists$

bijection between points of the curve V and $\mathrm{Pic}^0 V = Cl^0(V)$.
Choose a base point P_0 & send $P \in V \xrightarrow{\theta} [P - P_0] \in Cl^0 V$.
Note that if $\deg D = 0$, then $l(D + P_0) = 1$ by R-R
(since $\deg K_V = 0$) $\Rightarrow \exists !\, P \in V$ s.t. $D \sim P - P_0$.
So we have θ a bijection.
Note that $\theta(P) + \theta(Q) + \theta(R) = 0 \iff$
$P + Q + R \sim 3P_0$. If P_0 chosen to be an inflexion
point, then this is $\iff P + Q + R$ is a hyperplane
section (if $H_0 = 3P_0$, then R-R $\Rightarrow l(H_0) = 3$
$\Rightarrow$ divisors linearly equiv to H_0 are precisely the hyperplane
sections) i.e. $\iff P, Q, R$ are collinear
— the standard classical form for group law on cubic

For $g > 1$, choose a base point $P_0 \in V$ as above
& map $\theta : S^g V \longrightarrow Cl^0 V$ by
$\theta(P_1 + \dots + P_g) = [P_1 + \dots + P_g - g P_0] \in Cl^0(V)$.
Given $D \in Cl^0 V$,
$l(D + gP_0) = 1 + l(K_V - D - gP_0) \geq 1$ (R-R)
Hence θ a surjection as claimed.
Recall $l(K_V) = g$. $\therefore$ for general
$D = P_1 + \dots + P_g - g P_0$, $l(K_V - P_1 - \dots - P_g)$

goes down one at each step & so $\ell(K_V - D - gP_0) = 0$, but for specific choices of D (say consisting of $D + gP_0$ being g points from a canonical divisor!) can certainly have $\ell(K_V - D - gP_0) > 0$ & so $\ell(D + gP_0) > 1$ i.e. have θ surjective but only "generically" bijective.

10) We define $\mathrm{Rat}(\mathcal{F})$ to be given by an open (dense) subset $U \subset X$ & a section $s \in \mathcal{F}(U)$, subject to $(U, s) \sim (U', s') \iff \exists$ open (dense) $W \subset U \cap U'$ s.t. $s|_W = s'|_W$. Note that s is determined by $s|_W$, say using the local freeness of $\mathcal{F}$, & that an elt $\sigma \in \mathrm{Rat}(\mathcal{F})$ will therefore determine an elt of $\mathcal{F}_P$ for all but finitely many $P \in V$. (i.e. $\iff \exists$ $W \ni P$ s.t. σ given by (W, s) for some $s \in \mathcal{F}(W)$). Thus we have a morphism of $\mathcal{O}_X$-modules given (for U open $\subset X$) by projection

$$\sigma \in \mathrm{Rat}(\mathcal{F}) \longmapsto \bigoplus_{P \in U} \mathrm{Rat}(\mathcal{F})/\mathcal{F}_P$$

$\therefore$ have s.e.s. of $\mathcal{O}_X$-modules

$$0 \to \mathcal{F} \to \mathrm{Rat}(\mathcal{F}) \to \mathrm{Prin}(\mathcal{F}) \to 0$$

Since $Rat(\mathcal{Y})$ is a constant sheaf, it is flasque & so $H^1(X, Rat(\mathcal{Y})) = 0$. Hence l.e.s. on cohomology is

$$0 \to H^0(\mathcal{Y}) \to Rat(\mathcal{Y}) \to Prin(\mathcal{Y})(X) \to H^1(X, \mathcal{Y})$$

& so $Rat(\mathcal{Y}) \longrightarrow Prin(\mathcal{Y})(X)$ is surjective $\Leftrightarrow H^1(X, \mathcal{Y}) = 0$.

ii) (In fact, follows from more general fact that $\otimes$ commutes with direct limits of modules — Bourbaki, Algebra, Ch II, §6.3).

Let $T(U) = \mathcal{Y}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$ be the presheaf tensor product. For any $P \in U$, the $\mathcal{O}_{X,P}$-module structure on $\mathcal{Y}_P \otimes_{\mathcal{O}_{X,P}} \mathcal{G}_P$ induces an $\mathcal{O}_X(U)$-module structure on $\mathcal{Y}_P \otimes_{\mathcal{O}_{X,P}} \mathcal{G}_P$ in the obvious way. For $P \in U$ open, define

$$\alpha_U : \mathcal{Y}(U) \times \mathcal{G}(U) \longrightarrow \mathcal{Y}_P \otimes_{\mathcal{O}_{X,P}} \mathcal{G}_P$$
$$(s, t) \longmapsto s_P \otimes t_P$$

This induces an $\mathcal{O}_X(U)$-module morphism

$$\mathcal{Y}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U) \longrightarrow \mathcal{Y}_P \otimes_{\mathcal{O}_{X,P}} \mathcal{G}_P$$

& hence a morphism $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_P = T_P \xrightarrow{\theta} \mathcal{F}_P \otimes_{\mathcal{O}_{X,P}} \mathcal{G}_P$

Define $\psi : \mathcal{F}_P \times \mathcal{G}_P \longrightarrow T_P$

$(s_P, t_P) \longmapsto (s|_{U \cap V} \otimes_{\mathcal{O}_X(U \cap V)} t|_{U \cap V})_P$

where $s_P = [(U, s)]$, $t_P = [(V, t)]$. This is bilinear over $\mathcal{O}_{X,P}$ & hence defines a morphism

$\mathcal{F}_P \otimes_{\mathcal{O}_{X,P}} \mathcal{G}_P \longrightarrow T_P$ which is inverse to θ.

12) Given an f-morphism $\phi : \mathcal{G} \to \mathcal{F}$ & $f(P) \in Y$, a system of nbhds W_i of $f(P)$ pulls back to nbhds $f^{-1}W_i$ of P, and hence the morphisms

$\phi(W_i) : \mathcal{G}(W_i) \longrightarrow \mathcal{F}(f^{-1}W_i)$ induce a HM

$\phi_{f(P)} : \mathcal{G}_{f(P)} \longrightarrow \mathcal{F}_P$ (in fact, a composite

$\mathcal{G}_{f(P)} \longrightarrow (f_* \mathcal{F})_{f(P)} \longrightarrow \mathcal{F}_P$).

Given a f-morphism $\phi : \mathcal{G} \to \mathcal{F}$, we can define $\psi(V) : (f^{-1}\mathcal{G})(V) \longrightarrow \mathcal{F}(V)$ for V open in X as follows: Given $s \in LHS$, $\exists$ cover $V = \bigcup V_i$ & W_i open in Y with $V_i \subset f^{-1}W_i$, $t_i \in \mathcal{G}(W_i)$ s.t. $s(P) = (t_i)_{f(P)}$ $\forall P \in V_i$

Now $\phi(W_i)(t_i) \in \mathcal{F}(f^{-1}W_i)$ & so define

$r_i = \phi(W_i)(t_i)|_{V_i} \in \mathcal{F}(V_i)$.

Note that $(t_i)_{f(P)} = s(P) = (t_j)_{f(P)} \quad \forall P \in V_i \cap V_j$

But $(\tau_i)_P = \phi_{f(P)}(t_i)_{f(P)}$ with $\phi_{f(P)}$ defined as above

$\Rightarrow (\tau_i)_P = (\tau_j)_P \quad \forall P \in V_i \cap V_j . \Rightarrow$

$\tau_i|_{V_i \cap V_j} = \tau_j|_{V_i \cap V_j} \quad \forall i, j$. So the τ_i glue together to give a section $\tau \in \mathcal{F}(V)$

i.e. τ is the unique section of $\mathcal{F}(V)$ for which $\tau_P = \phi_{f(P)}(s(P)) \quad \forall P \in V$. Define $\psi(s) = \tau$; clearly compatible with restrictions & so have morphism

$f^{-1}\mathcal{G} \xrightarrow{\psi} \mathcal{F}$.

If now $\sigma \in \mathcal{G}(U)$, $s = \theta(\sigma)$ will be given by $s(P) = \sigma_{f(P)} \quad \forall P \in f^{-1}U$ & so $(\psi \circ \theta)(\sigma)$ is the unique section of $\mathcal{F}(f^{-1}U)$ whose germ at each $P \in f^{-1}U$ is $\phi_{f(P)}(s(P)) = \phi_{f(P)}(\sigma_{f(P)}) = \phi(\sigma)_P$

i.e. $(\psi \circ \theta)(\sigma) = \phi(\sigma) \quad \forall \sigma \Rightarrow \psi \circ \theta = \phi$.

Now clear that the f-morphisms $\mathrm{Hom}_f(\mathcal{G}, \mathcal{F})$ biject with $\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F})$ under this construction, with inverse $\psi \longmapsto \psi \circ \theta$. By definition, clear also that $\mathrm{Hom}_f(\mathcal{G}, \mathcal{F})$ bijects with $\mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$ & so result follows.

13) Given $W^r \hookrightarrow V^n$, have corresponding point

$\mathbb{P}(\Lambda^r W) \hookrightarrow \mathbb{P}(\Lambda^r V)$ given by inclusion of

1-dim^l subspace $\Lambda^r W \hookrightarrow \Lambda^r V$.

If W has basis $\sum_{j=1}^{n} x_{ij} e_j = \underset{\sim}{x}_i \qquad i = 1, \ldots, r$

then have matrix of coefficients

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{r1} & \cdots & x_{rn} \end{pmatrix}$$

Changing the basis of W corresponds to premult^n by a non-sing $r \times r$ matrix Λ

The corresponding point $\underline{x}_1 \wedge \cdots \wedge \underline{x}_r$ of $\mathbb{P}^{\binom{n}{r}-1}$ is

given by all the $r \times r$ subdeterminants of X, say

$X_{i_1 \ldots i_r}$ in obvious notation, $i_1 < i_2 < \cdots < i_r$.

Premultiplying X by Λ multiplies all these by $\det \Lambda$.

Now given such an X, wlog $X_{12\ldots r} \neq 0$ & we

can premultiply by $\begin{pmatrix} x_{11} & \cdots & x_{1r} \\ \vdots & & \vdots \\ x_{r1} & & x_{rr} \end{pmatrix}^{-1}$, & hence wlog

may assume
$$X = \left(\begin{array}{ccc|ccc} 1 & & 0 & x_{1\,r+1} & \cdots & x_{1n} \\ & \ddots & & \vdots & & \\ 0 & & 1 & x_{r\,r+1} & \cdots & x_{rn} \end{array} \right)$$

So the elts with $X_{1\ldots r} \neq 0$ naturally form an

<u>affine</u> space $\mathbb{A}^{r(n-r)}$ with coordinates

$x_{ij} = \pm X_{1\ldots\hat{i}\ldots rj} / X_{12\ldots r}$ for $j > r$, $i \leq r$,

& then any coordinate $X_{i_1\ldots i_r}$ of point in $\mathbb{P}^{\binom{n}{r}-1}$ corresponding to X is a polynomial in the x_{ij} of degree $\leq r$. The corresponding homogeneous polys of degree r then define the image of $G(r,n) \hookrightarrow \mathbb{P}^{\binom{n}{r}-1}$ (wlog we may assume $X_{12\ldots r} = 1$ and then reduce to the affine case, with corresponding matrix of form $\left(\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix}\middle| \begin{matrix} x_{1\,r+1} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{r\,r+1} & \cdots & x_{rn} \end{matrix}\right)$

with the x_{ij} determined as above.)

e.g. $r=2$, $n=4$ $\qquad X = \begin{pmatrix} 1 & 0 & x_1 & x_2 \\ 0 & 1 & y_1 & y_2 \end{pmatrix}$

$\rightsquigarrow$ point $(1 : x_1 : y_1 : x_2 : y_2 : x_1 y_2 - x_2 y_1)$

(modulo signs)

i.e. image is the quadric hypersurface

$$Z_5 Z_0 - Z_1 Z_4 + Z_2 Z_3 = 0 \text{ in } \mathbb{P}^5.$$

The two skew lines l_1, l_2 correspond to 2-dim^l subspaces $U_i < V$ with $U_1 \cap U_2 = \{0\}$, so can choose basis $e_1, \ldots, e_4$ for V s.t. $U_1 = \langle e_1, e_2 \rangle$, $U_2 = \langle e_3, e_4 \rangle$.

So a line $\mathbb{P}(\underline{x}_1 \wedge \underline{x}_2)$ meets $\mathbb{P}\langle e_1 \wedge e_2 \rangle \iff$

$x_1 \wedge x_2 \wedge e_1 \wedge e_2 = 0$ in $\Lambda^4 V$ i.e. a linear equation in the coordinates of $x_1 \wedge x_2$. Similarly for $\mathbb{P}(e_3 \wedge e_4)$ & hence we have the intersection of G with a $\mathbb{P}^3$ (explicitly in the above coords, it will be given by $Z_5 = Z_0 = 0$). & so we have a quadric surface in $\mathbb{P}^3$.

Finally we define a sub-$\mathcal{O}_G$-module $\mathcal{F} \subset \mathcal{O}_G^n$ by $\mathcal{F}(U) = \{ (f_1, \ldots, f_n) \in \mathcal{O}_G(U)^n :$ for all P in U with corresponding r-dimensional subspace $W_P \subset k^n$ we have $(f_1(P), \ldots, f_n(P)) \in W_P \}$

Over an affine piece $\mathbb{A}^{r(n-r)}$, say $U_{1\ldots r}$ given by $X_{1\ldots r} \neq 0$, have for any given point $P = (x_{ij} : i \leq r, j > r) \in \mathbb{A}^{r(n-r)}$, W_P has basis

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \hline x_{1\,r+1} \\ \vdots \\ x_{1n} \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \hline x_{r\,r+1} \\ \vdots \\ x_{rn} \end{pmatrix}$$

& then clearly $\mathcal{F}|_{U_{1\ldots r}} \cong \mathcal{O}_G^r$.

This is sometimes called the <u>tautological bundle</u> on G.

14) If $A = k[X]$, recall that $k[X \times X] \cong A \otimes_k A$ (Ex Sht 2, Q9) where $\sum f_i \otimes g_i$ corresponds to the function $\sum f_i(x)\, g_i(y)$ on $X \times X$. Thus if $I = I_\Delta$, have $\sum f_i \otimes g_i \in I \iff$
$\sum f_i(x) g_i(x)$ vanishes on $\Delta \subset X \times X \iff \sum f_i g_i = 0$ in A

Now define $D : A \to I$ by $Da = 1 \otimes a - a \otimes 1$
We consider $A \otimes_k A$ as an A-module by mult$^=$ on 1st. factor. Note that $D(ab) = 1 \otimes ab - (ab) \otimes 1$

& $aD(b) + bD(a) = a \otimes b - (ab) \otimes 1 + b \otimes a - ab \otimes 1$

So $D(ab) - aD(b) - bD(a)$

$= 1 \otimes (ab) - a \otimes b - b \otimes a + (ab) \otimes 1$

$= (1 \otimes a - a \otimes 1)(1 \otimes b - b \otimes 1) \in I^2$

So induced map to I/I^2 is a derivation & so have induced morphism of A-modules

$\Omega^1_{A/k} \longrightarrow I/I^2$ where $da \longmapsto 1 \otimes a - a \otimes 1$.

Now define $\tilde{\phi} : A \otimes_k A \longrightarrow \Omega^1_{A/k}$ by
$\tilde{\phi}(x \otimes y) = x\, dy$ & extend to map on A-modules (module structure via 1st factor) & let $\phi : I \to \Omega^1_{A/k}$ be restr$^=$.
Note that $\phi(\underbrace{1 \otimes y - y \otimes 1}_{\in I}) = dy$

Also given $\sum x_i \otimes y_i$ & $\sum w_j \otimes z_j \in I$

ie. $\sum x_i y_i = 0$ & $\sum w_j z_j = 0$ in A,

$$\phi\left(\left(\sum x_i \otimes y_i\right)\left(\sum w_j \otimes z_j\right)\right) = \phi\left(\sum_{i,j} (x_i w_j) \otimes (y_i z_j)\right)$$

$$= \sum_{i,j} x_i w_j \, d(y_i z_j)$$

$$= \sum_{i,j} x_i w_j \,(y_i dz_j + z_j dy_i) = 0$$

$\therefore$ ϕ induces $\bar{\phi} : I/I^2 \longrightarrow \Omega^1_{A/k}$ which is inverse to previous morphism $\Omega^1_{A/k} \longrightarrow I/I^2$, with $\bar{\phi} \circ D = d$ ie. we have realized $d : A \to \Omega^1_{A/k}$ as $D : A \longrightarrow I/I^2$ defined by $Da = 1 \otimes a - a \otimes 1$.

The last part is now just the globalization of the affine case. It is clear that $\mathcal{J}/\mathcal{J}^2$ may be regarded as an $\mathcal{O}_\Delta$-module, $\mathcal{O}_\Delta = \mathcal{O}_{X\times X}/\mathcal{J}$. If $\pi_1 : X \times X \longrightarrow X$ is projection on first factor, then $\pi_1|_\Delta : \Delta \longrightarrow X$ is an IM, inverse to the obvious IM $X \longrightarrow \Delta \subset X \times X$ — on affine pieces this corresponds to taking the A-module structure via the first factor of $A \otimes_k A$. Thus regarded as an $\mathcal{O}_X$-module via this IM, $\mathcal{J}/\mathcal{J}^2$ is the global sheaf Ω^1_X defined in lectures.