Part III Algebraic Geometry 2017 pmhw@dpmms.cam.ac.uk

Example Sheet 4

(1) If $\phi : X \to Y$ is an affine morphism of varieties, $\mathcal{F}$ a quasi-coherent sheaf on X and $\mathcal{G}$ a quasi-coherent sheaf on Y, show that $\phi_*(\mathcal{F} \otimes_{\mathcal{O}_X} \phi^*\mathcal{G}) \cong \phi_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}$. [This result may be compared with the *projection formula*, in which ϕ is an arbitrary morphism but $\mathcal{G}$ is a locally free sheaf of finite rank.]

(2) Let X be an irreducible variety; an $\mathcal{O}_X$-module $\mathcal{M}$ is called *torsion-free* if for any open set U, and $m \in \mathcal{M}(U)$, $f \in \mathcal{O}_X(U)$ such that $fm = 0$, we have either $m = 0$ or $f = 0$. If X is a smooth curve, show that any coherent torsion-free $\mathcal{O}_X$-module is locally free.

(3) Let A denote the coordinate ring over k for the affine cuspidal cubic $y^2 = x^3$ in $\mathbf{A}^2$; show that the A-module $\Omega^1_{A/k}$ has torsion.

(4) Let P be a point on a variety V, and

$$d_P : m_P/m_P^2 \to \Omega^1_{V,P}/m_P\Omega^1_{V,P}$$

be the linear map of vector spaces over k defined in Chapter 5. Prove that d_P is an isomorphism.

(5) Show that the canonical sheaf of $\mathbf{P}^n$ is isomorphic to $\mathcal{O}_{\mathbf{P}^n}(-n-1)$.

†(6) For integers $d_0, d_1, \ldots, d_n \geq 0$, show that the homogeneous ideal $\langle X_0^{d_0}, X_1^{d_1}, \ldots, X_n^{d_n}\rangle$ of $k[X_0, \ldots, X_n]$ contains all homogeneous polynomials of degree $d = \sum_{i=0}^n d_i$. Using Čech cohomology, deduce that $H^n(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}) = 0$ for all $n > 0$.

Assuming standard general properties of cohomology, by arguing by inductively on $n > 0$, deduce that

$$H^n(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(-r)) = 0$$

for all $r \leq n$. Why is this consistent with the statement of Serre duality on $\mathbf{P}^n$?

(7) Given a smooth projective curve $W \subset \mathbf{P}^2$ of degree $d > 0$ (defined by an irreducible homogeneous polynomial of degree d), prove that $K_W \cong \mathcal{O}_W(d-3)$, and hence that $g(W) = \frac{1}{2}(d-1)(d-2)$.

(8) Prove that a smooth quadric surface $X \subset \mathbf{P}^3$ (i.e. defined by a quadratic form of rank 4) may be regarded with respect to suitable coordinates as the image of $\mathbf{P}^1 \times \mathbf{P}^1$ in $\mathbf{P}^3$ under the Segre embedding. Suppose now Y is a smooth curve on X, defined by an irreducible polynomial $F(X_0, X_1; Y_0, Y_1)$, homogeneous of degree a in the first set of variables, and homogeneous of degree b in the second set. Find a formula for the genus $g(Y)$.

(9) For $V \subset \mathbf{P}^2$ a smooth plane cubic, investigate the group $\mathrm{Pic}(V)$. In particular, if $\mathrm{Pic}^0(V)$ denotes the kernel of the degree map $\mathrm{Pic}(V) \to \mathbf{Z}$, show that there is a natural bijection between $\mathrm{Pic}^0(V)$ and the points of V. For V a smooth projective curve of arbitrary genus $g > 0$, show that there is a natural surjection $S^gV \to \mathrm{Pic}^0(V)$, which however when

$g > 1$ is not a bijection. [We can in fact give both the symmetric product $S^g V$ and $\mathrm{Pic}^0(V)$ the structures of a smooth varieties, in the latter case it is called the *Jacobian variety*, and then the above surjection is a birational morphism that contracts certain subvarieties.]

(10) Let X be an irreducible smooth curve and $\mathcal{F}$ a locally free $\mathcal{O}_X$-module; by analogy with the general definition of rational functions on a variety, we can define the k-algebra of *rational sections* $\mathrm{Rat}(\mathcal{F})$ of $\mathcal{F}$. We define the sheaf of *principle parts* $\mathrm{Prin}(\mathcal{F})$ of $\mathcal{F}$ by $\mathrm{Prin}(\mathcal{F})(U) = \bigoplus_{P\in U} \mathrm{Rat}(\mathcal{F})/\mathcal{F}_P$, for U open in X. The $\mathcal{O}_{X,P}$-module $\mathrm{Rat}(\mathcal{F})/\mathcal{F}_P$ is called the *module of principle parts* at P. We consider the property that for any finite collection of points $P_i \in X$ and any choice of the principal parts at the P_i, there is an element of $\mathrm{Rat}(\mathcal{F})$ inducing the given principal part at each point; show that this property is equivalent to the condition $H^1(X, \mathcal{F}) = 0$. [There are a number of possible proofs of this: one uses a long exact sequence of cohomology; another argues directly from Čech cohomology.]

(11) If $(X, \mathcal{O}_X)$ is a ringed space and $\mathcal{F}$, $\mathcal{G}$ are $\mathcal{O}_X$-modules, show that $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_P \cong \mathcal{F}_P \otimes_{\mathcal{O}_{X,P}} \mathcal{G}_P$ for all $P \in X$.

†(12) Suppose $f : X \to Y$ is a continuous map on topological spaces, that $\mathcal{F}$ is a sheaf of abelian groups on X and $\mathcal{G}$ a sheaf of abelian groups on Y. An *f-morphism* $\phi : \mathcal{G} \to \mathcal{F}$ is defined by giving homomorphisms of abelian groups $\phi(U) : \mathcal{G}(U) \to \mathcal{F}(f^{-1}U)$ for all U open in Y which are compatible with restrictions — thus the f-morphisms correspond to sheaf morphisms in $\mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$. Show that any such f-morphism ϕ induces induces homomorphisms on stalks $\phi_{f(P)} : \mathcal{G}_{f(P)} \to \mathcal{F}_P$ for all $P \in X$.

With notation as above, show that there is a natural f-morphism $\theta : \mathcal{G} \to f^{-1}\mathcal{G}$ with the property that any f-morphism $\phi : \mathcal{G} \to \mathcal{F}$ determines a unique morphism $\psi : f^{-1}\mathcal{G} \to \mathcal{F}$ of sheaves on X with $\phi(U) = \psi(f^{-1}U) \circ \theta(U)$ for all U open in Y. [*Hint: To define the image of a section $s \in (f^{-1}\mathcal{G})(V)$ for V open in X, you may need to glue together certain sections of $\mathcal{F}$ over some open cover of V.*]

Deduce that there is a natural bijection between $\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F})$ (morphisms of sheaves on X) and $\mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{G})$.

(13) Let $G(r, n)$ denote the set of r-dimensional vector subspaces of $V = k^n$, or equivalently $(r-1)$-dimensional linear subspaces of $\mathbf{P}(V) = \mathbf{P}^{n-1}$. Prove that $G(r, n)$ has the natural structure of a projective variety, embedded in $\mathbf{P}(\wedge^r V)$, of dimension $r(n-r)$. Show that $G = G(2, 4)$ is a quadric hypersurface in $\mathbf{P}^5$. If l_1, l_2 are skew lines in $\mathbf{P}^3$, show that the set $Q \subset G$ representing lines meeting both is the intersection of G with a 3-plane $\mathbf{P}^3 \subset \mathbf{P}^5$, and so is a quadric surface. Show in general that there is a naturally defined locally free sheaf of rank r on $G(r, n)$.

(14) Let X be an affine variety and $\Delta \subset X \times X$ the diagonal; which elements of the coordinate ring $k[X \times X] = k[X] \otimes_k k[X]$ vanish on Δ? Let I denote the ideal corresponding to Δ. Considering $k[X] \otimes_k k[X]$ as a $k[X]$-module via action on the first factor, show that there is a natural identification of the $k[X]$-module $\Omega^1_{k[X]/k}$ with I/I^2, under which da corresponds to the class of $1 \otimes a - a \otimes 1 \in I/I^2$.

For an arbitrary variety X we let $\mathcal{I}$ denote the ideal sheaf corresponding to the diagonal Δ (closed in $X \times X$); deduce that the quotient $\mathcal{I}/\mathcal{I}^2$ may be regarded as an $\mathcal{O}_\Delta$-module, which corresponds to the sheaf Ω^1_X under the obvious isomorphism of X with Δ.