Part III Algebraic Geometry 2017 pmhw@dpmms.cam.ac.uk

Example Sheet 3

(1) Given a commutative diagram of abelian groups (with exact rows and with columns being complexes)

$$\begin{array}{ccccccccc}
0 & \to & U_0 & \to & V_0 & \to & W_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & U_1 & \to & V_1 & \to & W_1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & U_2 & \to & V_2 & \to & W_2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

prove that there is a long exact sequence on the cohomology groups of the complexes

$$0 \to H^0(U_*) \to H^0(V_*) \to H^0(W_*) \to H^1(U_*) \to H^1(V_*) \to H^1(W_*) \to H^2(U_*) \to \ldots.$$

(2) Given an exact sequence of sheaves of abelian groups $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ on a topological space X, show that the sequence of sheaves $D\mathcal{F} \to D\mathcal{G} \to D\mathcal{H}$ is also exact. I hope that you will agree that this is rather an easy question; why have I set it?

(3) Let $(X, \mathcal{O}_X)$ be a variety (or indeed a manifold) and let $\mathcal{U}$ be an open cover of X. Show that the subgroup of $\operatorname{Pic} X$ consisting of isomorphism classes of line bundles which are trivialised with respect to the cover $\mathcal{U}$ is isomorphic to the Čech cohomology group $\check{H}^1(\mathcal{U}, \mathcal{O}_X^*)$, where $\mathcal{O}_V^*$ denotes the multiplicative sheaf of units in the structure sheaf.
[In fact, we can define Čech cohomology on X as a direct limit over open covers, which for the spaces being considered will be isomorphic to the flabby cohomology, and it can then be deduced that $\operatorname{Pic} X$ is isomorphic to $H^1(X, \mathcal{O}_X^*)$.]

†(4) Let V be an irreducible variety, and $\mathcal{K}^*$ the (multiplicative) constant sheaf of non-zero rational functions on V. Letting $\mathcal{O}_V^*$ denote the multiplicative sheaf of nowhere vanishing regular functions, what do the global sections of $\mathcal{K}^*/\mathcal{O}_V^*$ look like? Prove that $H^0(\mathcal{K}^*/\mathcal{O}_V^*)/H^0(\mathcal{K}^*) \cong H^1(V, \mathcal{O}_V^*)$. Show that the former group is isomorphic to $\operatorname{Pic} V$.

(5) Let $f : X \to Y$ be a morphism of varieties, and suppose that Y can be covered with open sets U_i such that the induced morphism $f^{-1}(U_i) \to U_i$ is an isomorphism of varieties for each i; show that f is an isomorphism.

Let X be an algebraic variety and Y an affine variety with coordinate ring B; show that the morphisms $f : X \to Y$ correspond to the k-algebra homomorphisms $B \to \mathcal{O}_X(X)$. Given $f : X \to Y$ with Y affine, suppose now that Y has a finite cover by open affine pieces $U_1, \ldots, U_m$ with the property that $f^{-1}U_i$ is an affine variety for all i. Prove that the ring $A = \mathcal{O}_X(X)$ is a finitely generated algebra over k, and deduce that X itself is an affine variety. [Hint: *You will need the result from Question 1 on Example Sheet 2.*]

(6) If W is an irreducible subvariety of an irreducible variety V, prove that $\dim(W) \leq \dim(V)$, with equality if and only if $V = W$. If $V \subset \mathbf{A}^n$ is an irreducible affine variety, prove that $\dim(V) = n - 1$ if and only if V is a hypersurface (i.e. $V = V(f)$ for some polynomial f). Prove the corresponding result also for $V \subset \mathbf{P}^n$ an irreducible projective variety.

(7) Prove that any invertible sheaf on $\mathbf{P}^n$ is of the form $\mathcal{O}_{\mathbf{P}^n}(m)$, for some integer m.

(8) Let X be a variety and $\mathcal{F}_i$ for $i \in I$ be sheaves of abelian groups on X. If $\mathcal{F}$ denotes the direct sum $\mathcal{F} = \bigoplus_{i \in I} \mathcal{F}_i$, prove that for any $r \geq 0$, we have

$$H^r(X, \mathcal{F}) = \bigoplus_{i \in I} H^r(X, \mathcal{F}_i).$$

(9) Consider the projection $\pi : \mathbf{A}^{n+1} \setminus \{0\} \to \mathbf{P}^n$. Prove that π is an affine morphism, and show that

$$\pi_* \mathcal{O}_{\mathbf{A}^{n+1} \setminus \{0\}} \cong \bigoplus_{d \in \mathbf{Z}} \mathcal{O}_{\mathbf{P}^n}(d),$$

where $\mathcal{O}_{\mathbf{P}^n}(d) = \mathcal{O}_{\mathbf{P}^n}(dH)$, with H the hyperplane $X_0 = 0$.

[HINT: Consider the cover of $\mathbf{P}^n$ by basic open sets $D^h(F)$ with F homogeneous, where

$$D^h(F) = \{\mathbf{x} \in \mathbf{P}^n \ : \ F(\mathbf{x}) \neq 0\}$$

— cf. Example Sheet 1, Question 12.]

(10) Let $\mathcal{F}$ be a sheaf on a topological space X and G a group; what does it mean to say that G acts on $\mathcal{F}$? If G acts on $\mathcal{F}$, show that it acts on the cohomology groups. If now $\mathcal{F}$ is an $\mathcal{O}_X$-module on a variety X and k^* acts on $\mathcal{F}$ by $\lambda \mapsto \theta_\lambda$, where for all open sets U and all sections $s \in \mathcal{F}(U)$, we have $\theta_\lambda(s) = \lambda^r s$, show that k^* acts on the sheaf cohomogy in a similar way.

(11) Assuming Proposition 4.7 from lectures, deduce Corollary 4.8.

(12) With the maps k_n and δ_n defined as in the proof of (4.9), prove that $\delta_{n-1} k_n + k_{n+1} \delta_n$ is the identity map on $\check{C}^n(\mathcal{U}, \mathcal{F})$.

†(13) Suppose that $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of varieties, and that $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module. In the case when both X and Y are affine, prove that $f_* \mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module. Suppose now that Y is affine but X is a general variety, and that $X = \bigcup U_i$ is a finite cover of X by open affines. By considering an appropriate morphism

$$\bigoplus_i f_*(\mathcal{F}|_{U_i}) \to \bigoplus_{i,j} f_*(\mathcal{F}|_{U_i \cap U_j}),$$

or otherwise, show that $f_* \mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module. Deduce that this still holds when X and Y are arbitrary varieties.

(14) Let $V \subset \mathbf{P}^N$ be a projective variety, with standard affine pieces $U_0, \ldots, U_N$, and suppose $\mathcal{F}$ is a quasi-coherent sheaf on V. Show that giving a global section s of $\mathcal{F}(m)$ is equivalent to giving local sections $s_i \in \mathcal{F}(U_i)$, with the property that $s_j|_{U_{ij}} = (X_i/X_j)^m s_i|_{U_{ij}}$ for all i, j.