Part III Algebraic geometry
Lecturer: Prof. P.M.H. Wilson
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[1]Transcribed by S. Fordham.

Contents

Lecture 1

11th October 12:00

A plan of the course

1. (1) Preliminaries on classical algebraic geometry & commutative algebra [**Rei88**], [**Wil**], [**AM69**]. (4 lectures)
2. (2) Sheaf theory [**Kem93**], [**Har92**]. (4 lectures)
3. (3) Construction and properties of abstract varieties [**Kem93**], [**Har92**]. (4 lectures)
4. (4) Locally free and coherent modules [**Kem93**], [**Har92**]. (4 lectures)
5. (5) Sheaf cohomology [**Kem93**], [**Har92**]. (5 lectures)
6. (6) Differentials and Riemann-Roch for curves [**Kem93**], [**Har92**]. (2 lectures)

The main reference for the course is the book of Kempf [**Kem93**] which essentially covers the whole course. Some introductory reading is the book of Reid [**Rei88**]. You might find it useful to consult my notes on the (old) Part II Algebraic Curves course [**Wil**]. Some useful reading for parts of the course is the book of Shafarevich [**Sha74**]. For some background reading and lots of examples, see the book of Harris [**Har92**]. A more advanced text is the book of Hartshorne [**Har77**] which is a standard reference for the subject. A standard reference for commutative algebra is the book of Atiyah & Macdonald [**AM69**]. And for a historical reference for the material in the course, there is the beautifully written paper of Serre [**Ser55**].

1.1. Some classical algebraic geometry. Throughout the following we take an algebraically closed field $k = \overline{k}$.

Definition 1.1. An affine variety $V \subseteq \mathbb{A}^n(k)$ $(=k^n)$ (same as k^n, except we've just forgotten the coordinates), given by the vanishing of polynomials

$$f_1, \dots, f_n \in k[X_1, \dots, X_n]$$

Equivalently, if $I = \langle f_1, \dots, f_r \rangle \triangleleft k[X]$ then

$$V = V(I) = \{z \in \mathbb{A}^n : f(z) = 0 \text{ for all } f \in I\}.$$

For projective varieties, we have projective space

$$\mathbb{P}^n(k) = (k^{n+1} \backslash \{0\})/k^*$$

where $v \sim \lambda v$ for $\lambda \neq 0$ with homogeneous coordinates $(x_0 : x_1 : \cdots : x_n)$.

Definition 1.2. A projective variety $V \subseteq \mathbb{P}^n$ is given by the vanishing of homogeneous polynomials

$$f_1, \dots, f_r \in k[X_0, \dots, X_n]$$

A homogeneous ideal is an ideal that satisfies: if $f \in J$, then so are its homogeneous parts of each degree. For any homogeneous $I \subset k[X_0, \dots, X_n]$, we define

$$V = V^h(I) = \{z \in \mathbb{P}^n : F(z) = 0, \text{ for all homogeneous } F \in I\}$$

1.2. Coordinate ring of an affine variety. If $V = V(I) \subset \mathbb{A}^n$, set $I(V) = \{f \in k[X] : f(x) = 0, \text{ for all } x \in V\}$. Observe that tautologically $V = V(I(V))$. However, one obviously has $\sqrt{I} \subset I(V(I))$ ($f \in \sqrt{I}$ iff there exists an integer $r > 0$ such that $f^r \in I$) and Hilbert's nullstellensatz says that for algebraically closed fields, $I(V(I)) = \sqrt{I}$ (see [**Rei88**, § III] or [**AM69**, pp. 82-83]).

Definition 1.3. The coordinate ring of a variety V is $k[V] = k[X]/I(V)$, the ring of polynomial functions on V. dfn:coordinate-ring

Remark. $k[V]$ is a finitely generated, reduced[1] k-algebra.

Given a sub-variety $W \subset V$, we have $I(V) \subset I(W)$ defines an ideal of $k[V]$ also denoted $I(W) \lhd k[V]$.

Corollary 1.1 (corollary to nullstellensatz)**.** *If $\mathfrak{m}$ is a maximal ideal of $k[V]$, then one has*

$$\mathfrak{m} = \mathfrak{m}_p = \{f \in k[V]: f(p) = 0\}$$

for some $p \in V$.

Proof. The nullstellensatz implies that $I(V(\mathfrak{m})) = \sqrt{\mathfrak{m}} = \mathfrak{m} \neq k[V]$. So $V(\mathfrak{m}) \neq \varnothing$. Choose $p \in V(\mathfrak{m})$. Then $\mathfrak{m} \subset \mathfrak{m}_p$. But $\mathfrak{m}$ is maximal, so $\mathfrak{m}_p = \mathfrak{m}$. ☐

Remark. Observe that $\{p\} = V(\mathfrak{m}_p) = V(m)$ and so there is a bijection

$$\{\text{points of } V\} \longleftrightarrow \{\text{maximal ideals of } k[V]\}$$

Definition 1.4. A variety W is irreducible if there do not exist proper subvarieties W_1, W_2 of W with $W = W_1 \cup W_2$.

Lemma 1.2. *A sub-variety W of an affine variety V is irreducible iff $\mathfrak{p} = I(W)$ is prime, i.e. $k[W]$ is an integral domain.*

Proof. If $I(W)$ is not prime then there exists $f, g \notin I(W)$ such that $fg \in I(W)$. Set $W_1 = V(f) \cap W$ and $W_2 = V(g) \cap W$ - then W_1, W_2 are proper sub-varieties with $W = W_1 \cup W_2$ i.e. W is not irreducible.

If W_1, W_2 are proper sub-varieties and $W = W_1 \cup W_2$, choose $f \in I(W) \backslash I(W_1)$ and $g \in I(W) \backslash I(W_2)$. Then $fg \in I(W)$, so $I(W)$ is not prime. ☐

For a projective variety $V \subset \mathbb{P}^n$, we have homogeneous ideals

$$I^h(V) \subset k[X_0, \dots, X_n]$$

generated by having polynomials vanishing on V. *Exercise:* show that V is irreducible iff $I^h(V)$ is prime.

Generalising the earlier argument, we have a bijection

$$\left\{\begin{matrix}\text{irreducible sub-varieties } W \\ \text{of an affine variety } V\end{matrix}\right\} \longleftrightarrow \{\text{prime ideals } \mathfrak{p} \subset k[V]\}$$

$$W \longmapsto I(W)$$

Proof. Given a prime ideal $\mathfrak{p} \subset k[V]$, the nullstellensatz implies that $I(V(\mathfrak{p})) = \sqrt{\mathfrak{p}} = \mathfrak{p}$ in $k[V]$ and so we have the inverse map. ☐

Theorem 1.3 (Projective nullstellensatz)**.** *Suppose that I is a homogeneous ideal in $k[X_0, \dots, X_n]$ and $V = V^h(I) \in \mathbb{P}^n$. Then if $\sqrt{I} \neq \langle X_0, \dots, X_n \rangle$ (called the irrelevant ideal), then $I^h(V) = \sqrt{I}$.*

Proof. See Reid's book [**Rei88**, pp. 82] - easy deduction from the affine nullstellensatz. ☐

Theorem 1.4. *Suppose that V is a variety then we can write $V = V_1 \cup \dots \cup V_n$ with V_i irreducible sub-varieties and this decomposition is essentially unique (up-to reordering).*

[1] Reduced means that if $f^n = 0$ then $f = 0$.

Proof. Suppose that V is affine (similar proof works for a projective variety). If such a decomposition does not exist, then there is strictly decreasing sequence of sub-varieties

$$\cdots \subsetneq V_2 \subsetneq V_1 \subsetneq V_0 = V$$

(because if $V = W \cup W'$, then at least one of the W, W' has no such decomposition either, and let this be V_1, then continue by induction using the ascending chain condition (†)). Hence in $k[V]$ we have

$$0 = I(V_0) \subset I(V_1) \subset I(V_2) \subset \dots$$

and Hilbert's basis theorem implies that $k[V]$ is Noetherian, so there is N such that $I(V_{N+r}) = I(V_N)$ for all $r \geq 0$. So $V_{N+r} = V(I(V_{N+r})) = V(I(V_N)) = V_N$ for all $r \geq 0$. So the process described (†) must terminate, and we end up with a decomposition into irreducibles. See Reid's book [**Rei88**, ex. 2.8] for an easy 'topological' argument showing that the decomposition is essentially unique. □

Lecture 2

14th October 12:00

We'll carry on with the preliminary stuff.

2.1. Zariski topology. Let V be a variety (affine or projective). The Zariski topology is the topology on V where the closed sets are the subvarieties. We should check that it is a topology. Without loss of generality, let V be affine. Clearly V and $\varnothing$ are closed. Observe that for ideals $(I_\alpha)_{\alpha\in A}$ on $k[V]$, we have

$$V(\sum_\alpha I_\alpha) = \bigcap_\alpha V(I_\alpha)$$

is closed. Similarly, $V(IJ) = V(I) \cup V(J) = V(I \cap J)$ is closed as well (clearly $V(IJ) \supseteq V(I \cap J) \supseteq V(I) \cup V(J)$. Suppose that $P \in V(IJ) \backslash (V(I) \cup V(J))$ then we can choose $f \in I$ such that $f(P) \neq 0$ and $g \in J$ such that $g(P) \neq 0$. Then $fg \in IJ$ with value non-zero at P, a contradiction.) When V is affine, then we have a basis of open sets $D(f)$ for $f \in k[V]$ where

$$D(f) := \{x \in V : f(x) \neq 0\}$$

and any open set is of the form

$$V \backslash V(f_1, \dots, f_r) = \cup_i D(f_i)$$

If $V = \mathbb{A}^1$, we get the cofinite topology, so the Zariski topology is not Hausdorff. *Exercise:* the Zariski topology is compact, i.e. any open cover of V admits a finite subcover. A little warning here: in some French texts, compactness means compact and Hausdorff, and they say precompact for what we mean by compact.

Example. Let us determine all the closed subsets $X \subset \mathbb{A}^1$. Such a set is given by a system of equations $F_1(T) = \cdots = F_m(T) = 0$ in one variable T. If all the F_i are identically 0 then $X = \mathbb{A}^1$. If the F_i don't have any common factor, then they don't have any common roots, and X does not contain any points. If the highest common factors of all the F_i is $P(T)$ then $P(T) = (T - \alpha_1)\dots(T - \alpha_n)$ and X consists of the finitely many points $T = \alpha_1, \dots, T = \alpha_n$.

Example. Let us determine all the closed subsets $X \subset \mathbb{A}^2$. A closed subset is given by a system of equations

$$F_1(T) = \cdots = F_m(T) = 0, \tag{2.1}$$

where now $T = (T_1, T_2)$. If all the F_i are identically 0 then $X = \mathbb{A}^2$. Suppose this is not the case. If the polynomials $F_1, \dots, F_m$ do not have a common factor then the system of equations (2.1) has only a finite set of solutions (possibly empty).

Finally, suppose that the highest common factor of all the $F_i(T)$ is $P(T)$. Then $F_i(T) = P(T)G_i(T)$, where now the polynomials $G_i(T)$ do not have a common factor. Obviously, then $X = X_1 \cup X_2$ where X_1 is given by $G_1(T) = \dots G_m(T) = 0$ and X_2 is given by the single equations $P(T) = 0$. As we have seen, X_1 is a finite set. The closed sets defined in $\mathbb{A}^2$ by one equation are the algebraic plane curves. Thus a closed set $X \subset \mathbb{A}^2$ either consists of a finite set of points (possibly empty), or the union of an algebraic plane curve and a finite set of points, or the whole of $\mathbb{A}^2$.

Let us recall some stuff from part II.

2.2. Function field of irreducible varieties. If V is an irreducible affine variety, the field of rational functions or the function field is `dfn:rational-function`

$$k(V) := \operatorname{fof} k[V]$$

`dfn:dimension` where fof means field of fractions. In fact, define the dimension of V by $\dim V := \operatorname{tr\,deg}_k k(V)$. For $V \subseteq \mathbb{P}^n$ an irreducible projective variety, define

$$k(V) = \{F/G/: F, G \text{ homogeneous polynomials of same degree, } G \notin I^h(V)\}/\sim$$

where $F_1/G_1 \sim F_2/G_2$ iff $F_1G_2 - F_2G_1 \in I^h(V)$ (need V irreducible, i.e. $I^h(V)$ prime, for transitivity to hold).

Function fields are very crude invariants. One example of this is: if $V \subseteq \mathbb{P}^n$ is irreducible and U is an affine piece of V (say $U = V \cap \{X_0 \neq 0\}$) then U is an affine variety, $U \subset \mathbb{A}^n$ and with the usual affine coordinates, the functions in $X_1/X_0, \dots X_n/X_0$, where the equations for U come from those for V by "putting $X_0 = 1$". There is an easy check now that U is also irreducible and $k(V) \cong k(U)$ (this isomorphism is given by "putting $X_0 = 1$").

`dfn:regular` We say that $h \in k(V)$ is regular at $P \in V$ if it can be written as

- f/g with $f, g \in k[V]$, $g(P) \neq 0$ (affine case).
- F/G with F/G homogeneous of same degree, $G(P) \neq 0$ (projective case)

Now we define

$$\mathcal{O}_{V,P} := \{h \in k(V) \text{ such that } h \text{ is regular at } P\}$$

`dfn:local-ring` the local ring of V at P with unique maximal ideal

$$\mathfrak{m}_{V,P} = \{h \in \mathcal{O}_{V,P}: h(P) = 0\} = \ker(\mathcal{O}_{V,P} \overset{\text{eval}_P}{\to} k)$$

This is the unique maximal ideal since $\mathcal{O}_{V,P} \backslash \mathfrak{m}_{V,P}$ consists of units (i.e. invertible elements) and any proper ideal consists of non-units and so is contained in $\mathfrak{m}_{V,P}$.

2.3. Morphism of affine varieties. I'll do the affine case, you can work out the projective case yourself. When we do abstract varieties then we will do the projective case as well. `dfn:morphism` For $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$, a morphism $\phi: V \to W$ is given by elements $\phi_1, \dots, \phi_m \in k[V]$. This yields a k-algebra (recall this is a ring in which k is embedded) homomorphism $\phi^*: k[W] \to k[V]$ (where $\phi^*(f) = f \circ \phi$; so if y_i is a coordinate function on W, then $\phi^*(y_i) = \phi_i$). Conversely, given a k-algebra homomorphism $\alpha: k[W] \to k[V]$, we define a morphism

$$\alpha^* = \psi: V \to W$$

given by elements $\alpha(y_1), \dots, \alpha(y_m)$ (where $y_i \in k[W]$ is the coordinate function corresponding to polynomial Y_i).

Now there are a couple of remarks to be made here. Observe that for $\phi: V \to W$ then $\phi^{**} = \phi$ and for $\alpha: k[W] \to k[V]$ then $\alpha^{**} = \alpha$. For $\psi: U \to V$, then $\phi\psi: U \to W$ a morphism with $(\phi\psi)^* = \psi^*\phi^*$. For $\beta: k[V] \to k[U]$, then $(\beta\alpha)^* = \alpha^*\beta^*$. So all this formalism allows us to deduce that affine varieties V, W are isomorphic $V \cong W$ iff $k[W] \cong k[V]$ as k-algebras. So formally, there exists an equivalence of categories

$$\left\{\begin{matrix}\text{affine varieties over } k \\ \text{and morphisms}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{finitely generated reduced } k\text{-algebras} \\ \text{and their homomorphisms}\end{matrix}\right\}^{\text{op}}$$

So affine algebraic geometry is a branch of commutative algebra and consequently is not very interesting. Then we could look at arbitrary rings, then this is the theory of affine schemes - this course is really a course in affine schemes.

Lemma 2.5. *For V an irreducible affine variety over k, then*

$$\{f \in k(V) \colon f \text{ regular everywhere}\} = k[V]$$

Proof. Exercise on example sheet 1. □

This is not true for the projective case because the only ones regular everywhere there, are the constant ones - think about holomorphic functions on a compact Riemann surface.

Remark. A projective variety $V \subseteq \mathbb{P}^n$ is covered by finitely many open set which are affine varieties, e.g. the open sets $U_i \cap \{X_i \neq 0\} \subset \mathbb{A}^n$ (affine coordinates $\frac{X_0}{X_i}, \dots, \widehat{\frac{X_i}{X_i}}, \dots, \frac{X_n}{X_i}$)

Lecture 3

16th October 12:00

3.1. A little commutative algebra. A will always denote a commutative ring (with a 1). Result you learnt today in commutative algebra: Nakayama's lemma - if you aren't taking that course then a reference is [**AM69**, pp. 21].

Lemma 3.6 (Nakayama). *If M is a finitely generated A-module over a local ring A with maximal ideal $\mathfrak{m}$ such that $M = \mathfrak{m}M$, then $M = 0$.*

Remark. Different from version in commutative algebra class because that one is more amenable to rephrasing for the non-commutative case.

3.2. Rings & modules of fractions. Let A be a ring, $S \subseteq A$ a multiplicative subset (i.e. $1 \in S$ and if $s, t \in S$ then $st \in S$), we can define an equivalence relation on $A \times S$ by

$$(a, s) \sim (a', s') \Leftrightarrow t(as' - a's) = 0 \text{ for some } t \in S$$

(easy to check that '$\sim$' is an equivalence relation). Let a/s denote the equivalence class of (a, s) and let $S^{-1}A$ denote the set of such elements a/s. Define addition and multiplication on $S^{-1}A$ in the obvious way. Then $S^{-1}A$ is a ring and there exists a homomorphism $\phi \colon A \to S^{-1}A$ given by $a \mapsto a/1$. $S^{-1}A$ is called the ring of fractions of A with respect to S.

There is a universal property: if $g \colon A \to B$ is a homomorphism with $g(S) \subseteq U(B)$ (where $U(B)$ is the set of units of B) then there exists a unique $g' \colon S^{-1}A \to B$ with $g'\phi = g$ (namely $g'(a/s) = g(a)g(s)^{-1} \in B$).

$S^{-1}A$ has a unit ($= 1/1$) and a zero ($= 0/1$). Also we have

$$a/s = 0 \Leftrightarrow ta = 0 \text{ for some } t \in S$$

and $S^{-1}A = 0$ iff $1/1 = 0$ iff $0 \in S$. The map $A \to S^{-1}A$ is an isomorphism iff $S \subseteq U(A)$ (for the '$\Leftarrow$' direction, let $B = A$ and use the universal property).

Let T = the set of non zero-divisors - this is a multiplicative set. Then $A \to S^{-1}A$ is an injection iff $S \subseteq T$. Call $T^{-1}A = \operatorname{tot}(A)$, the total ring of fractions, so we dfn:tot
have an injection $A \hookrightarrow T^{-1}A$. If A is an integral domain, then $\operatorname{tot}(A) = \operatorname{fof}(A)$

($T = A\backslash\{0\}$) (for a reducible affine variety V, we should replace the fraction field $k(V)$ by the ring $\mathrm{Rat}\,(V) \coloneqq \mathrm{tot}\,(k[V])$ of rational functions.)

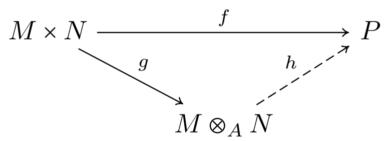
Some relevant examples

(1) If $f \in A$, let $f^{\mathbb{N}} = \{1, f, f^2, \dots\} = S$. Write A_f for $S^{-1}A$ in this case.
(2) If $\mathfrak{p}$ is a prime ideal of A, then take $S = A\backslash\mathfrak{p}$. Then we write $A_{\mathfrak{p}}$ for $S^{-1}A$ and this is called the localisation of A at $\mathfrak{p}$. This is a local ring with unique maximal ideal denoted $\mathfrak{p}A_{\mathfrak{p}}$ consisting of elements a/s with $a \in \mathfrak{p}, s \notin \mathfrak{p}$ (all the other elements are units).

If now M is an A-module and $S \subseteq A$ is a multiplicative subset, then the module of fractions is defined in the obvious way, $S^{-1}M$ (an A and $S^{-1}A$-module).

3.3. Tensor products.

Definition 3.5. Given A-modules M, N, define $M \otimes_A N$ to be an A-module equipped with an A-bilinear map from $M \times N \to M \otimes N$ with the universal property: given any A-bilinear map $f: M \times N \to P$, then there exists a unique morphism of A-modules h making the following diagram commute



and $M \otimes_A N$ is defined up to isomorphism by this property (the existence of such a module is straightforward to prove but boring (see [**AM69**, pp. 24])). We denote $x \otimes y$ to be the image of (x, y) in $M \otimes_A N$.

Some elementary properties of this (see [**AM69**, pp. 26]: for M, N, P A-modules, we have

$$M \otimes N \cong N \otimes M$$
$$(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$$
$$(M \oplus N) \otimes P \cong (M \otimes P) \oplus (N \otimes P)$$
$$A \otimes M \xrightarrow{\sim} M$$

and these are all proved just using the universal property.

3.4. Change of ring. Given a homomorphism $f: A \to B$ (N.B. $f(1) = 1$) of rings we call B an A-algebra. Given an A-algebra structure on B, $f: A \to B$ and an A-module M, set $M_B = B \otimes_A M$ - also a B-module in the obvious way, B acting on the first factor.

Proposition 3.7 (0.4)**.**
(1) *If $\mathfrak{a}$ is an ideal of A then $A/\mathfrak{a} \otimes_A M \cong M/\mathfrak{a}M$.*
(2) *Let $S \subseteq A$ be a multiplicative subset then $S^{-1}A \otimes_A M \cong S^{-1}M$.*

PROOF.
(1) Use the universal property of the tensor product and define the map.
(2) Use the universal property of both S^{-1} and the tensor product - see [**AM69**, pp. 40].

□

Proposition 3.8 (0.5)**.** *If M, N are A-modules, $\mathfrak{a} \lhd A$, $S \subset A$ a multiplicative subset, then*
(1) $A/\mathfrak{a} \otimes (M \otimes_A N) \cong M/\mathfrak{a}M \otimes_{A/\mathfrak{a}} N/\mathfrak{a}N$.
(2) $S^{-1}(M \otimes_A N) \cong S^{-1}M \otimes_{S^{-1}A} S^{-1}N$

PROOF. Exercise on example sheet 1. □

Example. If $\mathfrak{p}$ is a prime ideal of A, then

$$(M \otimes_A N)_{\mathfrak{p}} \cong M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}}$$

(where $M_{\mathfrak{p}} = (A \backslash \mathfrak{p})^{-1} M$).

3.5. R-algebras. Given a ring R and R-algebras A, B, a morphism of R-algebras is given by a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\theta_1} & A \\ & \searrow^{\theta_2} & \downarrow \\ & & B \end{array}$$

Given R-algebras A, B, $A \otimes_R B$ has the structure of an R-algebra, multiplication is given by

$$(a \otimes b)(a' \otimes b') = (aa') \otimes (bb')$$

and a ring homomorphism

$$\begin{aligned} R &\to A \otimes_R B \\ r &\mapsto \theta_1(r) \otimes 1 = 1 \otimes \theta_2(r) \end{aligned}$$

and we have equality here because recall A, B are R-algebras, and we have (for $r \in R$) $r \cdot a := \theta_1(r) a$ and $r \cdot b := \theta_2(r) b$ for $a \in A, b \in B$ so $r(a \otimes b) = \theta_1(r) a \otimes b = a \otimes \theta_2(r) b$.

We also have R-algebra homomorphisms

$$\begin{array}{ccc} A & & B \\ & \searrow^{\alpha} \quad {}^{\beta}\swarrow & \\ & A \otimes_R B & \end{array}$$

given by $a \mapsto a \otimes 1$ and $b \mapsto 1 \otimes b$. So there is a universal property: given R-algebra homomorphisms

$$\begin{array}{ccc} A & & B \\ & \searrow^{\alpha'} \quad {}^{\beta'}\swarrow & \\ & C & \end{array}$$

then there exists a unique R-algebra homomorphism such that $\alpha' = \phi\alpha$ and $\beta' = \phi\beta$ and $A \otimes_R B$ is determined (up to isomorphism) by this universal property.

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & A \otimes_R B & \xleftarrow{\beta} & B \\ & {}_{\alpha'}\searrow & \downarrow{\scriptstyle \phi} & \swarrow_{\beta'} & \\ & & C & & \end{array}$$

Using this, we deduce for instance that for R-algebras A, B, C, then one has

$$A \otimes_R (B \otimes_R C) \cong (A \otimes_R B) \otimes_R C$$

are naturally isomorphisms of R-algebras (rather than just R-modules).

Lecture 4

18th October 12:00

4.1. §1: Sheaf theory. Let X be a topological space.

Definition 4.6. A presheaf $\mathscr{F}$ of algebras (resp. rings) on X consists of data

(1) for every open $U \subseteq X$, an algebra (resp. ring) $\mathscr{F}(U)$,
(2) for inclusion of open sets $V \subseteq U$, a homomorphism (called restriction) $\rho_V^U : \mathscr{F}(U) \to \mathscr{F}(V)$ such that
  (a) $\mathscr{F}(\varnothing) = \{0\}$,
  (b) $\rho_U^U : \mathscr{F}(U) \to \mathscr{F}(U)$ is the identity,
  (c) if $W \subseteq V \subseteq U$ are open, then $\rho_W^V \circ \rho_V^U = \rho_W^U$.

Remark. If U denotes the category of open sets in X (where the morphisms are inclusion) then a presheaf of Abelian groups on X is just a contravariant functor $F : \mathsf{U} \to \mathsf{Ab}$ i.e. an element of the category, $\mathsf{Ab}^{\mathsf{U}^{op}}$ (where Ab is the category with objects Abelian groups and morphisms group homomorphisms).

An element $s \in \mathscr{F}(U)$ is called a section of $\mathscr{F}$ over U. For $s \in \mathscr{F}(U)$ we denote $\rho_V^U(s)$ by $s|_V$ for open $V \subset U$.

Definition 4.7. A presheaf $\mathscr{F}$ on X is a sheaf if it satisfies two further conditions

(A) If U is open and $U = \bigcup_i V_i$ is an open cover, and if $s \in \mathscr{F}(U)$ is such that $s|_{V_i} = 0$ for all i, then $s = 0$.
(B) If $U = \bigcup_i V_i$ as above and we have elements $s_i \in \mathscr{F}(V_i)$ such that for all i, j we have
$$s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$$
then there exists $s \in \mathscr{F}(U)$ such that $s|_{V_i} = s_i$ for all i.

Example.

(1) Let X be a topological space, and A any algebra (resp. a ring). The constant sheaf $\mathscr{A}$ determined by A is defined as follows
  - $\mathscr{A}(\varnothing) = \{0\}$.
  - For $U \neq \varnothing$ open in X then
  $$\mathscr{A}(U) = \{\text{locally constant maps } U \to A\}$$
  an Abelian group (resp. ring) under the the obvious pointwise operations. With the obvious restriction maps, $\mathscr{A}$ is a sheaf (a map is locally constant if every point in the domain has a neighbourhood such that the map is constant when restricted to this neighbourhood).

  If $U \neq \varnothing$ is a connected open set, then $\mathscr{A}(U) = A$, but more generally if $U \neq \varnothing$ is open and its connected components are also open (e.g. true in a locally connected topological space), then $\mathscr{A}(U)$ is a direct sum of copies of A.
(2) If X is a differentiable (say C^∞) manifold, then we can define the sheaf of C^∞-functions (real or complex valued, it doesn't matter) on X, a sheaf of rings. Similarly if X is a complex manifold, we can define the sheaf of holomorphic functions on X. In both cases the sheaf is called the structure sheaf $\mathscr{O}_X$ - checking conditions 1,2 is fine because this sheaf is defined in terms of functions.
(3) The great generalisation of Serre: for V an (irreducible) variety (affine, projective, or quasi-projective[2]), we try to get a structure sheaf: we consider V as a topological space with the Zariski topology and then corresponding to the structure sheaf previously, there is an obvious definition: for U open in V, set $\mathscr{O}_V(U)$ to be the regular functions on U which is
$$\{f \in k(V) : f \text{ is regular on } U\}$$

[2]Quasi-projective: an open subset of a projective variety.

then $\mathscr{O}_V$ is a sheaf of rings with respect to the Zariski topology, the structure sheaf of the variety V. A reminder: exercise 0.2 on the first example sheet implies that $\mathscr{O}_V(V) = k[V]$.

Definition 4.8. If $\mathscr{F}$ is a presheaf on X and $P \in X$, we define the stalk $\mathscr{F}_P$ of $\mathscr{F}$ at P to be

$$\mathscr{F}_P = \varinjlim_{U \ni P} \mathscr{F}(U)$$

i.e. an element of $\mathscr{F}_P$ is represented by a pair (U, s) where $U \ni P$ is an open neighbourhood and $s \in \mathscr{F}(U)$ where (U, s) and (V, t) define the same element of $\mathscr{F}_P$ if there exists an open neighbourhood $W \ni P$ with $W \subseteq U \cap V$ such that $s|_W = t|_W$. The elements of $\mathscr{F}_P$ are called germs. If $\mathscr{F}$ is a sheaf of algebras, rings, ... then $\mathscr{F}_p$ is an algebra, ring, ... in an obvious way.

Example.

(1) For the constant sheaf $\mathscr{A}$, assigned to an algebra, ring, ..., A, then it is clear that $\mathscr{A}_P = A$.
(2) For X a C^∞ (resp. complex) manifold with structure sheaf $\mathscr{O}_X$, the stalk $\mathscr{O}_{X,P}$ at P consists of germs of C^∞ (resp. holomorphic) functions at P.
(3) For V an (irreducible) variety with structure sheaf $\mathscr{O}_V$, the stalk at $P \in V$ is

$$\mathscr{O}_{V,P} = \text{local ring at } P$$

as detailed before.

Definition 4.9. If $\mathscr{F}, \mathscr{G}$ are (pre-)sheaves on X, a morphism $\phi: \mathscr{F} \to \mathscr{G}$ consists of homomorphisms $\mathscr{F}(U) \to \mathscr{G}(U)$ for each open $U \subseteq X$ such that for $V \subseteq U$ the following diagram commutes dfn:morphism-of-sheaves

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\phi(U)} & \mathscr{G}(U) \\ \downarrow{\rho^U_V} & & \downarrow{\rho^U_V} \\ \mathscr{F}(V) & \xrightarrow{\phi(V)} & \mathscr{G}(V) \end{array}$$

i.e. $\rho^U_V \circ \phi(U) = \phi(V) \circ \rho^U_V$ or $\phi(U)(s)|_V = \phi(V)(s|_V)$ for all $s \in \mathscr{F}(U)$. A morphism $\phi: \mathscr{F} \to \mathscr{G}$ induces a homomorphism $\phi_P: \mathscr{F}_P \to \mathscr{G}_P$ for each $P \in X$, i.e.

$$\phi_P[(U, s)] = (U, \phi(U)(s))$$

is well-defined.

Definition 4.10. A morphism $\phi: \mathscr{F} \to \mathscr{G}$ of (pre-)sheaves is injective if $\mathscr{F}(U) \to \mathscr{G}(U)$ is injective for all U, e.g. sheaves of subgroups (resp. subrings) where $\mathscr{F}(U) \subseteq \mathscr{G}(U)$ for all U then if this is the case then $\mathscr{F}$ is called a subsheaf of $\mathscr{G}$. A morphism $\phi: \mathscr{F} \to \mathscr{G}$ is called an isomorphism if there exists an inverse morphism $\psi: \mathscr{G} \to \mathscr{F}$. This is equivalent to the statement that $\phi(U): \mathscr{F}(U) \to \mathscr{G}(U)$ is bijective for all U, similarly we can define $\psi(U) = \phi(U)^{-1}$ gives inverse.

Lemma 4.9 (1.1)**.** *If $\phi: \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then*

(1) *ϕ is injective iff ϕ_P is injective for all $P \in X$.*
(2) *ϕ is an isomorphism iff ϕ_P is an isomorphism for all $P \in X$.*

PROOF. Next time. □

Lecture 5

21st October 12:00

5.1. Stalks & morphisms of sheafs.

Lemma 5.10 (1.1). *If $\phi: \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then*

(1) *ϕ is injective iff ϕ_P is injective for all $P \in X$.*
(2) *ϕ is an isomorphism iff ϕ_P is an isomorphism for all $P \in X$.*

PROOF.

(1) '→' Suppose that there exists a germ $s_P \in \mathscr{F}$ such that $\phi_P(s_P) = 0$ in $\mathscr{G}_P$, i.e. there exists an open neighbourhood $W \subset U$ with $P \in W$ such that $\phi(U)(s)|_W = 0$. Therefore by commutativity of the maps of a morphism, $\phi(W)(s|_W) = 0$. But ϕ is injective therefore $s|_W = 0$.
'←' Let ϕ_P be injective for all P, and let U be open: then it remains to prove that $\phi(U): \mathscr{F}(U) \to \mathscr{G}(U)$ is injective. Suppose that there exists $0 \neq s \in \mathscr{F}(U)$ such that $\phi(U)(s) = 0$ in $\mathscr{G}(U)$. Let s_P denote the germ of s at $P \in U$: $0 = \phi(U)(s)_P = \phi_P(s_P)$ for all $P \in U$. This implies that $s_P = 0$ in $\mathscr{F}_P$ for all $P \in U$ and this implies that for all $P \in U$, there exists an open neighbourhood $W \ni P$ with $W \subseteq U$ such that $s|_W = 0$, and therefore U is covered by open sets U_α such that $s|_{U_\alpha} = 0$ for all α and this implies that $s = 0$ by the identity condition (sheaf condition (A)).
(2) '→' Clear.
'←' $\phi(U): \mathscr{F}(U) \to \mathscr{G}(U)$ is an injection for all open U by part (1). It remains to prove that $\phi(U)$ is also surjective. Suppose then that $t \in \mathscr{G}(U)$ and let $t_P \in \mathscr{G}_P$ be its germ at $P \in U$. Since ϕ_P is surjective, there exists $s_P \in \mathscr{F}_P$ such that $\phi_P(s_P) = t_P$. Suppose that s_P is represented by a pair (V, s) with $P \in V \subseteq U$ and $s \in \mathscr{F}(U)$. Then t_P is represented by $\phi(V)(s)$ i.e. $(U, t) \underset{P}{\sim} (V, \phi(V)(s))$. By shrinking V, we may assume that we have an open neighbourhood $U \supseteq V_P \ni P$ such that
$$\phi(V)(s)|_{V_P} = t|_{V_P}$$
therefore denoting $\sigma = s|_{V_P} \in \mathscr{F}(V_P)$, we have $\phi(V_P)(\sigma) = t|_{V_P}$. In this way, we can cover U by open sets $U = \bigcup_\alpha U_\alpha$ with sections $s_\alpha \in \mathscr{F}(U_\alpha)$ such that $\phi(U_\alpha)(s_\alpha) = t|_{U_\alpha}$. On overlaps $U_{\alpha\beta} = U_\alpha \cap U_\beta$, we have
$$\phi(U_{\alpha\beta})(s_\alpha|_{U_{\alpha\beta}}) = t|_{U_{\alpha\beta}} = \phi(U_{\alpha\beta})(s_\beta|_{U_{\alpha\beta}})$$
Since $\mathscr{F}$ is a sheaf, the s_α patch together to give a section $s \in \mathscr{F}(U)$ such that $s|_{U_\alpha} = s_\alpha$ (condition (B) for $\mathscr{F}$). But then $\phi(U)(s)|_{U_\alpha} = \phi(U_\alpha)(s_\alpha) = t|_{U_\alpha}$ for all α. Then sheaf condition (A) for $\mathscr{G}$ implies that $\phi(U)(s) = t$.

□

Definition 5.11. A morphism of sheaves $\phi: \mathscr{F} \to \mathscr{G}$ is called surjective in the category of sheaves[3]if $\phi_P: \mathscr{F}_P \to \mathscr{G}_P$ is surjective for all P.

Definition 5.12. Given a presheaf/sheaf $\mathscr{F}$ on a topological space X and a continuous map $f: X \to Y$ we have an induced presheaf/sheaf $f_*\mathscr{F}$ on Y defined by
$$(f_*\mathscr{F})(U) = \mathscr{F}(f^{-1}U)$$
for U open in Y with obvious restriction maps
$$\begin{array}{ccc} f_*\mathscr{F}(U) & \longrightarrow & f_*\mathscr{F}(V) \\ \| & & \| \\ \mathscr{F}(f^{-1}U) & \longrightarrow & \mathscr{F}(f^{-1}V) \end{array}$$

[3]One must make a distinction between surjectivity in the category of sheaves and surjectivity in the category of presheaves.

for $V \subseteq U$.

Definition 5.13. A ringed space is a pair $(X, \mathscr{O}_X)$, X a topological space, $\mathscr{O}_X$ a sheaf of rings. Given ringed spaces $(X, \mathscr{O}_X), (Y, \mathscr{O}_Y)$, a morphism of ringed spaces $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a pair $(f, f^\sharp)$ where $f: X \to Y$ is a continuous map and $f^\sharp: \mathscr{O}_Y \to f_* \mathscr{O}_X$ is a morphism of sheaves of rings on Y. So $f^\sharp$ defines homomorphisms $\mathscr{O}_Y(U) \to \mathscr{O}_X(f^{-1}U)$ for all U open in Y compatible with restrictions. dfn:locally-ringed-space

Definition 5.14. If R is a commutative ring (e.g. a field), a ringed space over R is a ringed space $(X, \mathscr{O}_X)$ with $\mathscr{O}_X$ a sheaf of R-algebras (restriction maps are homomorphisms of R-algebras). A morphism of ringed spaces over R is defined in the obvious way. dfn:ringed-space

Definition 5.15. A ringed space $(X, \mathscr{O}_X)$ is a locally-ringed space (another name: geometric space) if the stalks $\mathscr{O}_{X,P}$ are local rings. A morphism of locally-ringed spaces is given by a pair $(f, f^\sharp)$ as above, with the induced maps dfn:locally-ringed-space

$$f^\sharp_P: \mathscr{O}_{Y,f(P)} \to \mathscr{O}_{X,P}$$

being local homomorphisms of local rings.

Remark. Regarding the induced maps above: setting $\phi = f^\sharp$, the homomorphisms $\mathscr{O}_Y(U) \overset{\phi_U}{\to} \mathscr{O}_X(f^{-1}U)$ for $U \ni f(P)$ induces a homomorphism $\mathscr{O}_{Y,f(P)} \overset{\phi_P}{\to} \mathscr{O}_{X,P}$, namely a germ $[(V, s)]$ with $f(P) \in V \subseteq U$ goes to a germ defined by $[(f^{-1}V, \phi_V(s))]$, and this is a well-defined homomorphism.

Definition 5.16. A homomorphism $\phi: (A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ of local rings is called local if $\phi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$, or equivalently (exercise) $\phi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$.

Example.

(1) $(X, \mathbb{Z})$ with the constant sheaf, $\mathbb{Z}$ is a ringed space, but not locally-ringed.

(2) If X is a C^∞-manifold (resp. complex manifold) with structure sheaf $\mathscr{O}_X$, then the pair $(X, \mathscr{O}_X)$ is a locally ringed space over $\mathbb{R}$ (resp. $\mathbb{C}$). A smooth (resp. holomorphic) map $f: X \to Y$ yields a sheaf morphism of $\mathbb{R}$-algebras (resp. $\mathbb{C}$-algebras), namely

$$f: \mathscr{O}_Y \to f_* \mathscr{O}_X$$
$$g \mapsto g \circ f$$

(since a smooth (resp. holomorphic) function on Y pulls back to one on X). Clearly $g(f(P)) = 0$ iff $f^\sharp(g)(P) = 0$ and so $f^\sharp_P(\mathfrak{m}_{Y,f(P)}) \subset \mathfrak{m}_{X,P}$. So $(f, f^\sharp)$ is a morphism of locally ringed spaces on $\mathbb{R}$ (resp. $\mathbb{C}$).

Lecture 6

23rd October 12:00

Continuing with the examples

(3) Let $(V, \mathscr{O}_V)$ be the ringed space given by an (irreducible) affine variety and its structure sheaf. This is a locally ringed space over the base field k. If $\phi: V \to W$ is a morphism of affine varieties in the sense that we have come across classically, then there exists a morphism of locally ringed spaces

$$(\phi, \phi^\sharp): (V, \mathscr{O}_V) \to (W, \mathscr{O}_W)$$

i.e. for $g \in \mathscr{O}_W(U)$, $\phi^\sharp(g) = g \circ \phi \in \mathscr{O}_V(\phi^{-1}U)$ (before we denoted $\phi^\sharp$ as ϕ^*).

Lemma 6.11 (1.2). *If V, W are irreducible affine varieties and*

$$(f, f^\sharp): (V, \mathscr{O}_V) \to (W, \mathscr{O}_W)$$

is a morphism of locally-ringed spaces over k, then f is induced from a morphism of varieties $\phi: V \to W$ with $f^\sharp = \phi^\sharp$ defined as in (3) *above.*

Remark. Once we have defined $\mathscr{O}_V$ on a reducible affine variety (with $\mathscr{O}_V = k[V]$) the same result can be proved for the general case with essentially same proof.

PROOF. Suppose that $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$. Define $g_j = f^\sharp(y_j) \in \mathscr{O}_V(V) \underset{(0.2)}{=} k[V]$[4] where y_j is the j^{th} coordinate function on W. Define $\phi = (g_1, \dots, g_m)$, a morphism $V \to \mathbb{A}^m$. Suppose now that $f(P) = (b_1, \dots, b_m) \in W$ for $P \in V$. We have $y_j - b_j \in \mathfrak{m}_{W, f(P)}$ for all j thus $f^\sharp(y_j - b_j) = y_j \circ f - b_j \in \mathfrak{m}_{V,P}$ because $f^\sharp$ is local which implies that $g_j(P) - b_j = 0$ for all j which implies that $\phi(P) = f(P)$, therefore $\phi: V \to W$ is the same map as f on the topological spaces. Since $f^\sharp(y_j) = g_j = y_j \circ \phi = \phi^\sharp(y_j)$, it follows that $f^\sharp = \phi^\sharp$ on both $k[W]$ and $k(W)$ and hence on any $\mathscr{O}_W(U)$ with $k[W] \subset \mathscr{O}_W(U) \subset k(W)$ for U open in W. □

Because of the result just proven, we make the following definition.

Definition 6.17. So for V, W irreducible quasi-projective varieties, we define a morphism of varieties $V \to W$ to be a morphism of the corresponding locally-ringed spaces over k, $(V, \mathscr{O}_V) \to (W, \mathscr{O}_W)$.

6.1. $\mathscr{O}_X$-modules.

Definition 6.18. Let $\mathscr{M}$ be a sheaf of Abelian groups on a ringed space $(X, \mathscr{O}_X)$, dfn:OX-module then $\mathscr{M}$ is said to be an $\mathscr{O}_X$-module if for every open set $U \subseteq X$, $\mathscr{M}(U)$ is an $\mathscr{O}_X(U)$-module and for any $W \subseteq U$ open, $\alpha \in \mathscr{O}_X(U)$, $m \in \mathscr{M}(U)$, we have

$$(\alpha m)|_W = (\alpha|_W)(m|_W)$$

Similarly we have the obvious definition for a morphism of $\mathscr{O}_X$-modules $\phi: \mathscr{M} \to \mathscr{N}$.

Example. For V an (irreducible) quasi-projective variety with structure sheaf $\mathscr{O}_V$, dfn:ideal-sheaf and $W \subset V$ a closed subvariety, we have a sheaf of ideals $\mathscr{I}_W \subset \mathscr{O}_V$, a subsheaf of $\mathscr{O}_V$ given by

$$\mathscr{I}_W(U) := \{f \in \mathscr{O}_V(U) : f|_{W \cap U} \equiv 0\}$$

This is clearly an $\mathscr{O}_V$-module.

Everything we've done so far goes through unchanged for $\mathscr{O}_X$-modules (apart from one technicality) e.g. if $\mathscr{M}$ is an $\mathscr{O}_X$-module, then any stalk $\mathscr{M}_P$ is an $\mathscr{O}_{X,P}$-module, etc. So what is the technicality: the small change involves the push-forward of an $\mathscr{O}_X$-module $\mathscr{M}$ under a morphism of ringed spaces

$$(X, \mathscr{O}_X) \xrightarrow{\phi = (f, f^\sharp)} (Y, \mathscr{O}_Y)$$

The sheaf $f_*\mathscr{M}$ is then an $f_*\mathscr{O}_X$-module via the morphism $f^\sharp: \mathscr{O}_Y \to f_*\mathscr{O}_X$ - we can consider $f_*\mathscr{M}$ also as an $\mathscr{O}_Y$-module, which we then denote $\phi_*\mathscr{M}$. Explicitly, for U open in Y,

$$f_*\mathscr{M}(U) = \mathscr{M}(f^{-1}U)$$

is a module over $f_*\mathscr{O}_X(U) = \mathscr{O}_X(f^{-1}U)$. But

$$f^\sharp: \mathscr{O}_Y(U) \to f_*\mathscr{O}_X(U) = \mathscr{O}_X(f^{-1}U)$$

and so $\mathscr{M}(f^{-1}U)$ is also a module over $\mathscr{O}_Y(U)$.

6.2. Sheafification. Given a presheaf $\mathscr{F}$ on X, there exists an associated sheaf $\mathscr{F}^+$[5] and a morphism $\theta: \mathscr{F} \to \mathscr{F}^+$ and this morphism has a universal property:

[4]From examples sheet 2.

[5]Some books denote it with two pluses, the one plus case is a "mono-presheaf".

for any sheaf $\mathscr{G}$ and morphism $\phi: \mathscr{F} \to \mathscr{G}$, there exists a unique morphism of sheaves $\psi: \mathscr{F}^+ \to \mathscr{G}$ such that $\phi = \psi \circ \theta$.

We construct $\mathscr{F}^+$ as follows: for U open in X, set

$$\mathscr{F}^+(U) = \left\{ \begin{array}{l} \text{Functions } s: U \to \coprod_{p \in U} \mathscr{F}_P \text{ such that:} \\ \quad (1).\ \text{for each } P \in U,\ s(P) \in \mathscr{F}_P, \text{ and} \\ \quad (2).\ \text{for each } P \in U, \text{ there exists an open neighbourhood} \\ \quad\quad W \text{ of } P \text{ in } U,\ W \subseteq U \text{ and an element } t \in \mathscr{F}(W) \text{ such} \\ \quad\quad \text{that } s(Q) = t_Q \text{ for all } Q \in W. \end{array} \right\}$$

It is clear that $\mathscr{F}^+$ is a sheaf (since the sections are given in terms of functions) and there exists a morphism $\mathscr{F} \to \mathscr{F}^+$ where $\theta(U): \mathscr{F}(U) \to \mathscr{F}^+(U)$ by $\sigma \mapsto s$ where $s(P) = \sigma_P$ for all $P \in U$. Moreover, if we start from a sheaf $\mathscr{F}$, the sheaf conditions imply that a section of $\mathscr{F}^+(U)$ patches together to give a section of $\mathscr{F}(U)$ (noting that if we have $t, t' \in \mathscr{F}(W)$ such that $t_Q = t'_Q$ for all $Q \in W$, then $t = t'$ by sheaf condition (A)), so there exists an inverse morphism and so θ is an isomorphism. In general, it's clear that $\mathscr{F}^+$ has the same stalks as $\mathscr{F}$.

Lecture 7

25th October 12:00

There is an example sheet on the web-page, and there will be an example class at the beginning of the week after next.

7.1. Universal property of sheafification. Let $\theta: \mathscr{F} \to \mathscr{F}^+$ be as last time. Given $\phi: \mathscr{F} \to \mathscr{G}$ with $\mathscr{G}$ a sheaf, we have morphisms of stalks $\phi_P: \mathscr{F}_P \to \mathscr{G}_P$ for all $P \in X$ and hence a unique $\psi: \mathscr{F}^+ \to \mathscr{G}^+ \cong \mathscr{G}$ such that $\phi = \psi \circ \theta$ (on stalks, it's clear that $\phi_P = (\psi \circ \theta)_P = \psi_P \circ \theta_P$ and then use sheaf condition (A) on $\mathscr{G}$ - see example sheet 1, question 7).

The standard argument with universal properties shows that the pair $(\mathscr{F}^+, \theta)$ is unique up to isomorphism.

Corollary 7.12 (1.3)**.** *Suppose that $\mathcal{B}$ is a base of open sets for X which is closed under finite intersections*[6]*, and $\mathscr{F}$ is a $\mathcal{B}$-sheaf (i.e. data specified only with respect to open sets in $\mathcal{B}$ and satisfying sheaf conditions (A) and (B) with respect to open sets in $\mathcal{B}$). Then there exists a sheaf $\mathscr{F}'$ on X and isomorphisms $\mathscr{F}(U) \xrightarrow{\sim} \mathscr{F}'(U)$ for $U \in \mathcal{B}$ compatible with restriction (i.e. $\mathscr{F} \cong \mathscr{F}'|_{\mathcal{B}}$ as $\mathcal{B}$-sheaves and $\mathscr{F}'$ is unique up to isomorphism).*

Proof. The '+' construction used above may be extended to the case where $\mathscr{F}$ is only a $\mathcal{B}$-presheaf yielding a sheaf on X, by only taking open sets $W \in \mathcal{B}$ in condition (2) on sections. If $\mathscr{G}$ is a presheaf on X, it is clear that the sections of $\mathscr{G}^+$ correspond to the sections of $\mathscr{G}_0^+$ (the $\mathcal{B}$-presheaf you get by only looking at the elements of the basis $\mathcal{B}$ i.e. $\mathscr{G}_0$ is the $\mathcal{B}$-presheaf $\mathscr{G}|_{\mathcal{B}}$ i.e. $\mathscr{G}_0^+ = \mathscr{G}^+$). The $\mathscr{F}^+$ described will just be $\mathscr{F}^+$. There exists an obvious morphism

$$\theta(U): \mathscr{F}(U) \to \mathscr{F}^+(U)$$

for $U \in \mathcal{B}$ compatible with restriction. Furthermore, the $\mathcal{B}$-sheaf conditions on $\mathscr{F}$ imply that a section of $\mathscr{F}^+(U)$ for $U \in \mathcal{B}$, patches together to give a section of $\mathscr{F}(U)$. We have that $\theta(U)$ is an isomorphism for all $U \in \mathcal{B}$ i.e. $\mathscr{F} \cong \mathscr{F}'|_{\mathcal{B}}$. If now $\mathscr{F}''$ is a sheaf on X such that $\mathscr{F} \cong \mathscr{F}''|_{\mathcal{B}}$, then

$$\mathscr{F}'' \cong (\mathscr{F}'')^+ = (\mathscr{F}''|_{\mathcal{B}})^+ \cong \mathscr{F}^+ = \mathscr{F}'$$

[6] The condition that it is closed under finite intersections is necessary to make sense as a sheaf - they need to agree when you restrict to intersections, so the intersection had better also be in $\mathcal{B}$ as well.

as sheaves on X. $\square$

7.2. Kernels of sheaf morphisms. If $\phi: \mathscr{F} \to \mathscr{G}$ is a morphism of (pre)-sheaves of abelian groups, define a presheaf $\ker(\phi)$ by

$$(\ker \phi)(U) = \{s \in \mathscr{F}(U) : \phi(U)(s) = 0\}$$

a sub-sheaf of $\mathscr{F}$. If $\mathscr{F}$ is a sheaf, then clearly $\ker \phi$ satisfies sheaf condition (A). If however $U = \bigcup U_\alpha$ and $s_\alpha \in (\ker \phi)(U_\alpha)$ satisfying the compatibility condition on overlaps, then they patch together to give an element $s \in \mathscr{F}(U)$ such that $s_\alpha = s|_{U_\alpha}$. But $\phi(U)(s)|_{U_\alpha} = \phi(U_\alpha)(s_\alpha) = 0$ for all α which implies that $\phi(U)(s) = 0$ from condition (A). If ϕ is a morphism of $\mathscr{O}_X$-modules, then $\ker \phi$ is an $\mathscr{O}_X$-module.

Lemma 7.13 (1.4)**.** *If $\phi: \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves & $\ker \phi$ is defined as above then $(\ker \phi)_P = \ker \phi_P$ for all P.*

PROOF. One has $s_P = [(U,s)] \in \ker(\phi_P: \mathscr{F}_P \to \mathscr{G}_P)$ iff $(\phi(U)(s))_P = 0 \in \mathscr{G}_P$ iff there exists an open neighbourhood $W \subseteq U$ of P such that $\phi(W)(s|_W) = 0$ iff $s|_W \in (\ker \phi)(W)$ for some open neighbourhood $W \subseteq U$ of P, which is true iff $s_P \in (\ker \phi)_P$. $\square$

7.3. Cokernels of sheaf morphisms. If $\phi: \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves of Abelian groups, then we can define a presheaf $\mathscr{C}$ in an obvious way: $\mathscr{C}(U) = \mathscr{G}(U)/\phi(U)\mathscr{F}(U)$ for all U. This is not in general a sheaf.

Definition 7.19. We define the sheaf cokernel of ϕ, $\operatorname{coker} \phi = \mathscr{G}/\phi\mathscr{F} := \mathscr{C}^+$. Consider the morphism of presheaves $\mathscr{G} \to \mathscr{C}$, then for $P \in X$ we have an induced homomorphism on stalks $\mathscr{G}_P \to \mathscr{C}_P$ (it is a trivial check that the kernel we have in this case is just $\phi_P \mathscr{F}_P$). There exists isomorphisms

$$\mathscr{G}_P/\phi_P\mathscr{F}_P \cong \mathscr{C}_P \xrightarrow{\sim} \mathscr{C}_P^+$$

So the sheaf morphism ϕ is surjective (i.e. ϕ_P is surjective for all P) iff $\operatorname{coker} \phi = 0$. In general, a sheaf morphism $\phi: \mathscr{F} \to \mathscr{G}$ induces a sheaf morphism $\mathscr{G} \underset{\psi}{\to} \mathscr{C}^+ = \operatorname{coker} \phi$, a surjection (since it's surjective on stalks).

Definition 7.20. If $\phi: \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then the image sheaf $\operatorname{Im} \phi = \phi(\mathscr{F})$ is the subsheaf of $\mathscr{G}$ given by $\ker(\mathscr{G} \overset{\psi}{\to} \operatorname{coker} \phi)$.

Remark. We did not make the obvious definition, we could just have taken the image presheaf of ϕ i.e. $\mathscr{A}(U) = \phi(U)\mathscr{F}(U)$, but this will not be a sheaf in general, but of course we then could go and sheafify it and then in fact, one has $\mathscr{A}^+ \cong \operatorname{Im} \phi$ (see below). But defining the image sheaf the way that we did gives the new sheaf naturally as a subsheaf of the original sheaf, so it is preferable for this reason.

Claim. $\mathscr{A}^+ \cong \operatorname{Im} \phi$.

PROOF. There exists a morphism of presheaves

$$\mathscr{A} \hookrightarrow \underbrace{\mathscr{G} \longrightarrow \mathscr{C} \longrightarrow \mathscr{C}^+}_{\psi}$$

where the composite is zero. Hence $\mathscr{A} \overset{\theta}{\hookrightarrow} \ker \psi$ induces a morphism of sheaves

$$\mathscr{A}^+ \longrightarrow (\ker \psi)^+ \xrightarrow{\sim} \ker \psi$$

Since $\mathscr{A}(U) = \ker(\mathscr{G}(U) \to \mathscr{C}(U))$ and lemma 7.13 implies that

$$\begin{array}{ccc} \mathscr{A}_P = \ker\big(\mathscr{G}_P & \longrightarrow & \mathscr{C}_P\big) \\ & \searrow^{\psi_P} & \big\downarrow \cong \\ & & \mathscr{C}_P^+ \end{array}$$

for all P so the induced map on stalks

$$\mathscr{A}_P^+ \cong \mathscr{A}_P \xrightarrow{\theta} (\ker\psi)_P \xrightarrow{=} \ker\psi_P$$

an isomorphism on stalks for all P, the morphism $\mathscr{A}^+ \to \ker\psi$ is an isomorphism of sheaves (lemma 5.10 (part 2)). ☐

Lecture 8

28th October 12:00

Examples class: 4pm, Wednesday 6th November, MR5.

Recall that given a morphism $\phi\colon \mathscr{F} \to \mathscr{G}$ of sheaves, we have a morphism $\psi\colon \mathscr{G} \to \operatorname{coker}\phi \coloneqq \mathscr{C}^+$ where $\mathscr{C}$ is the presheaf cokernel. We had

$$\mathscr{G}_P/\phi_P\mathscr{F}_P \cong \mathscr{C}_P \xrightarrow{\sim} \mathscr{C}_P^+ \tag{8.2}$$

and we defined $\operatorname{Im}\phi \coloneqq \ker\psi$.

Remark. If we define $\mathscr{A}(U) = \phi(U)\mathscr{F}(U)$ for all U, then $\operatorname{Im}\phi \cong \mathscr{A}^+$.

Remark. Given $\phi\colon \mathscr{F} \to \mathscr{G}$, a morphism of sheaves, we have a morphism of presheaves $\phi\colon \mathscr{F} \to \mathscr{A}$ and hence a morphism of sheaves $\mathscr{F} \to \mathscr{A}^+ \cong \operatorname{Im}\phi \subset \mathscr{G}$. Now, one has

$$(\operatorname{Im}\phi)_P \coloneqq (\ker\psi)_P \underset{(\text{see } 7.13)}{=} \ker\psi_P = \operatorname{Im}\phi_P$$

for all P since

$$\ker\psi_P = \ker\big(\ \mathscr{G}_P \longrightarrow \mathscr{C}_P^+ \xrightarrow[8.2]{\sim} \mathscr{G}_P/\phi_P\mathscr{F}_P\ \big) = \phi_P\mathscr{F}_P$$

which implies that $\mathscr{F} \to \operatorname{Im}\phi$ is a surjective morphism of sheaves. For all $s \in \mathscr{F}(U)$, one has

$$\phi(U)(s) \in \ker(\mathscr{G} \to \mathscr{C})(U) \subset \ker(\mathscr{G} \to \mathscr{C}^+)(U) = (\operatorname{Im}\phi)(U)$$

and so we have a factorisation $\mathscr{F} \to \operatorname{Im}\phi \hookrightarrow \mathscr{G}$ for ϕ with $\ker(\mathscr{F} \to \operatorname{Im}\phi) = \ker\phi$.

Remark. For sheaves of $\mathscr{O}_X$-modules and their morphisms, we can construct the cokernel/kernel/image as sheaves of Abelian groups and observe that these have a natural structure as $\mathscr{O}_X$-modules. Similarly, if $\mathscr{O}$ is a sheaf of rings, $\mathscr{I}$ a sheaf of ideals in $\mathscr{O}$, then we can define the sheaf $\mathscr{O}/\mathscr{I}$, a sheaf of rings.

Definition 8.21. A sequence of sheaf morphisms

$$\mathscr{F} \xrightarrow{\theta} \mathscr{G} \xrightarrow{\phi} \mathscr{H}$$

is exact (at $\mathscr{G}$) if the corresponding sequence of maps on stalks

$$\mathscr{F}_P \xrightarrow{\theta_P} \mathscr{G}_P \xrightarrow{\phi_P} \mathscr{H}_P$$

is exact at $\mathscr{G}_P$ for all P (i.e. $\operatorname{Im}\theta_P = \ker\phi_P$).

Proposition 8.14 (1.5)**.**

(1) *The sequence* $\mathscr{F} \xrightarrow{\theta} \mathscr{G} \xrightarrow{\phi} \mathscr{H}$ *is exact at* $\mathscr{G}$ *iff* $\operatorname{Im}\theta = \ker\phi$ *as subsheaves of* $\mathscr{G}$.

(2) *If* $0 \longrightarrow \mathscr{F} \xrightarrow{\theta} \mathscr{G} \xrightarrow{\phi} \mathscr{H}$ *is exact, then*

$$0 \longrightarrow \mathscr{F}(U) \xrightarrow{\theta(U)} \mathscr{G}(U) \xrightarrow{\phi(U)} \mathscr{H}(U)$$

is exact for all U open in X.

Proof. (1) '←': if $\operatorname{Im}\theta = \ker\phi$ then $(\operatorname{Im}\theta)_P = (\ker\phi)_P$ for all $P \in X$. Therefore $\operatorname{Im}\theta_P = \ker\phi_P$ for all $p \in X$ (see remark 2, (lemma 7.13)). Therefore

$$\mathscr{F}_P \longrightarrow \mathscr{G}_P \longrightarrow \mathscr{H}_P$$

is exact at $\mathscr{G}_P$ for all P.

(1) '→': conversely, suppose that $\mathscr{F}_P \longrightarrow \mathscr{G}_P \longrightarrow \mathscr{H}_P$ is exact at $\mathscr{G}_P$ for all P, then in particular $(\phi\theta)_P = \phi_P\theta_P = 0$ for all P therefore $\phi\theta = 0$, thus with image presheaf as above, the morphism $\mathscr{A} \to \mathscr{H}$ is zero and hence the morphism of sheaves

$$\mathscr{A}^+ \xrightarrow{\sim} \operatorname{Im}\theta \hookrightarrow \mathscr{G} \longrightarrow \mathscr{H}$$

is zero i.e. $\operatorname{Im}\theta \subset \ker\phi$. Since we have equality on stalks, this implies that $\operatorname{Im}\theta = \ker\phi$ by lemma 5.10.

(2) Let $\mathscr{K} = \ker\phi = \operatorname{Im}\theta \subset \mathscr{G}$ by (1), therefore θ induces a (surjective) morphism of sheaves $\overline{\theta}: \mathscr{F} \to \mathscr{K}$. But $\mathscr{K}_P = \operatorname{Im}\theta_P = \ker\phi_P$, and so the corresponding maps on stalks $\mathscr{F}_P \to \mathscr{K}_P$ are isomorphisms. Lemma 5.10 then implies that $\overline{\theta}: \mathscr{F} \xrightarrow{\sim} \mathscr{K}$ is an isomorphism, hence $\theta(U): \mathscr{F}(U) \xrightarrow{\sim} \mathscr{K}(U)$ for all U where $\mathscr{K}(U) = (\ker\phi)(U)$. Therefore

$$0 \longrightarrow \mathscr{F}(U) \xrightarrow{\theta(U)} \mathscr{G}(U) \xrightarrow{\phi(U)} \mathscr{H}(U)$$

is exact for all U. □

Remark. Even if one starts with a short exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0$$

then the induced maps on sections $\mathscr{G}(U) \to \mathscr{H}(U)$ will not in general be surjective. This is one of the reasons that we introduce sheaf cohomology to try to mend this lack of surjectivity.

Example. To see an example where the surjectivity fails, let $X = \mathbb{P}^1$ and $\mathscr{I}_P$ = ideal sheaf of $P \in X$, k the field of definition, $\mathscr{I}_{P+Q}$ = ideal sheaf of $\{P, Q\}$, say
dfn:skyscraper-sheaf $P \neq Q$. Define the skyscraper sheaf $\mathscr{K}_Q$ at Q by

$$\mathscr{K}_Q(U) = \begin{cases} k & \text{if } Q \in U \\ 0 & \text{if } Q \notin U \end{cases}$$

so $\mathscr{K}_Q$ has stalk k at Q and zero elsewhere. There exists a short exact sequence of $\mathscr{O}_X$-modules (cf. example sheet 1, question 11)

$$0 \longrightarrow \mathscr{I}_{P+Q} \hookrightarrow \mathscr{I}_P \xrightarrow{f \overset{\mathrm{eval}_Q}{\longmapsto} f(Q)} \mathscr{K}_Q \longrightarrow 0$$

Now I want to go back to something that I sort of left hanging.

8.1. Case of reducible varieties. For $V \subseteq \mathbb{A}^n$ any affine variety, we define the regular functions on V to be the polynomial functions, i.e. $\mathscr{O}_V(V) := k[V]$ the coordinate ring. An open set has the form $U = V \backslash V(f_1, \dots, f_N) = \bigcup D(f_i)$ for $f_i \in k[V]$, where dfn:regular-function

$$D(f) = \{P \in V : f(P) \neq 0\}$$

Then $\{D(f) : f \in k[V]\}$ forms a basis of open sets for V, closed under finite intersections. Now $D(f)$ may be given the structure of an affine variety $W \subseteq V \times \mathbb{A}^1 \subset \mathbb{A}^{n+1}$ defined by the equations for V and the extra equation $X_{n+1} f(X_1, \dots, X_n) = 1$. If $\pi : W \to D(f)$ is given by

$$(x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_n)$$

then π is a homeomorphism (with the Zariski topology) with inverse homeomorphism $\alpha : D(f) \to W$ given by $\underline{x} \mapsto (x_1, \dots, x_n, 1/f(\underline{x}))$. Define

$$\mathscr{O}_V(D(f)) = \alpha^* k[W] = \left\{ \begin{array}{c} \text{functions on } D(f) \text{ which are pullbacks} \\ \text{of regular functions on } W \end{array} \right\}$$

(one has that this is $\cong k[V]_f$, noting that $k[W] \cong k[V][X]/(Xf - 1) \cong k[V]_f$).

Remark. In the case when V is irreducible, this corresponds to the previous definition, since an everywhere regular function on $D(f)$ pulls back from one on W, i.e. an element of $k[W]$ by lemma 2.5.

An obvious question now is: is it independent of the choice of f? We will deal with this next time.

Lecture 9

30th October 12:00

Let $V \subset \mathbb{A}^n$ be an affine variety, and let $f \in k[V]$. Let $W \subset V \times \mathbb{A}^1$ be given by the extra equation

$$X_{n+1} f(X_1, \dots, X_n) = 1$$

and take

$$\begin{aligned} D(f) &\to W \\ \underline{x} &\mapsto (x_1, \dots, x_n, 1/f(\underline{x})) \end{aligned}$$

a homeomorphism. Define the following

$$\begin{aligned} \mathscr{O}_V(D(f)) &= \{\text{functions on } D(f) \text{ which are pullbacks of regular functions on } W\} \\ &= \alpha^* k[W] \cong k[V]_f \end{aligned}$$

This does not depend on the choice of $f \in k[V]$: if $D(f) = D(g)$ then $D(f) = D(fg) = D(g)$. Since $g = g(X_1, \dots, X_n)$ is nowhere vanishing on W, then the nullstellensatz implies that g is a unit in $k[W]$. If now we define $W' \subset V \times \mathbb{A}^1$ defined by the extra equation $X_{n+1} fg = 1$ then there is an obvious isomorphism

$$\begin{aligned} \phi : W &\to W' \\ (x_1, \dots, x_{n+1}) &\mapsto (x_1, \dots, x_n, x_{n+1}/g) \end{aligned}$$

with an obvious inverse (which we denote by ψ). The diagram

$$\begin{array}{ccc} D(f) & \xrightarrow{\alpha} & W \\ \| & & \downarrow{\scriptstyle \phi}\, \wr \\ D(fg) & \xrightarrow{\alpha'} & W' \end{array}$$

commutes. Therefore $\mathcal{O}_V(D(f))$ depends only on the open set $D(f)$ and not on the choice of $f \in k[V]$. The basic open sets $D(f)$ are regarded as affine pieces.

Remark. More generally, for V projective (or quasi-projective) we can cover V by a finite number of affine pieces and so we obtain a basis $\mathcal{B}$ of affine open sets, closed under finite intersections. This is true for abstract varieties (see §2).

For arbitrary open $U \subset V$, we can define

$$(9.3) \qquad \mathcal{O}_V(U) := \left\{ \begin{array}{c} \text{continuous functions } f\colon U \to k \text{ such that there exists} \\ \text{a finite open affine cover } U = \bigcup U_i \text{ with } f|_{U_i} \text{ regular for all } i \end{array} \right\}$$

Remark. We have taken the easy way out here (Kempf does this in his book as well) because we use functions to define the structure sheaf here but I will say something about the other way to do this as well (§9.1).

What we need to check for this definition: if U is an affine variety covered by affine pieces $\{U_i\}$ and $f|_{U_i} \in k[U_i]$ for all i, then $f \in k[U]$ (‡)(i.e. definition 9.3 is well-defined). We have seen that $\mathcal{O}_V(D(f))$ is naturally isomorphic to A_f ($A = k[V]$, V an affine variety). Also if $D(f) = D(g)$, then there exists natural isomorphisms

$$A_f \xrightarrow{\sim} A_{fg} \xleftarrow{\sim} A_g$$

Moreover the restriction maps $\mathcal{O}_V(V) \to \mathcal{O}_V(D(f))$ are identified with natural ring homomorphisms

$$A \to A_f$$
$$a \mapsto a/1$$

N.B. restriction maps are not in general injective. The statement (‡) follows from the next thing

Lemma 9.15 (1.6)**.** *Suppose $U = \bigcup_i D(f_i)$ is a (finite) cover of an affine variety U and $g_i \in k[U]_{f_i}$ such that for each i, j, the images $g_i/1$ and $g_j/1$ in $k[U]_{f_i f_j}$ are equal, then there exists $g \in k[U]$ such that $g/1 = g_j$ in $k[U]_{f_j}$ for all j.*

PROOF. Set $A = k[U]$ (since U is compact, we can, without loss of generality, take $U = \bigcup_{i=1}^N D(f_i)$ a finite cover). Moreover, choose r sufficiently large such that $g_i = a_i/f_i^r$ in A_{f_i} with of course $a_i \in A$. Now since my assumption was that $g_i/1 = g_j/1$ in $A_{f_i f_j}$, there exists $n \geq 0$ such that $(f_i f_j)^n (f_j^r a_i - f_i^r a_j) = 0$ in A. As A is reduced, we may take $n = 1$ in this case so

$$(9.4) \qquad f_j^{r+1} f_i a_i - f_i^{r+1} f_j a_j = 0$$

in A for all i, j. So far all we have done is thrown around the definition. So now: since $V(f_i^{r+1}, \dots f_N^{r+1}) = V(f_1, \dots, f_N) = \varnothing$, and so the nullstellensatz implies that this is the whole ring so therefore there exists $e_i \in A$ such that $1 = \sum e_i f_i^{r+1}$. Set $g = \sum e_i f_i a_i \in A$. For each j, then if we look at $f_j^{r+1} g$ we find

$$f_j^{r+1} g = \sum_i e_i f_i f_j^{r+1} a_i = \sum_i e_i f_i^{r+1} f_j a_j = f_j a_j$$

(using 9.4) i.e. $f_j(f_j^r g - a_j) = 0$ which implies that $g/1 = a_j/f_j^r = g_j$ in A_{f_j} for all j. □

9.1. Alternative approach to defining structure sheaf. We have a basis $\mathcal{B}$ of affine open sets closed under finite intersection. We can define a $\mathcal{B}$-presheaf by $\mathcal{O}_V(U) = k[U]$ for all $U \in \mathcal{B}$. It is an easy check (examples sheet 2, question 1) that sheaf condition (A) is satisfied over $\mathcal{B}$. Lemma 9.15 is saying that sheaf condition (B) is satisfied - in fact we can avoid using reducedness in the proof of lemma 9.15. We have a $\mathcal{B}$-sheaf here, thus corollary 7.12 (1.3) implies that there exists a unique extension (up to isomorphism) to a sheaf $\mathcal{O}_V$ on V. This approach is needed if

(1) we work with schemes rather than varieties, or
(2) we're interested in $\mathcal{O}_X$-modules on a variety/scheme (for more details see §3).

9.2. §2: Construction of abstract varieties. If $\mathscr{F}$ is a sheaf on X and U is open in X, we have a sheaf $\mathscr{F}|_U$ on U (restriction of $\mathscr{F}$ to U) defined by $(\mathscr{F}|_U)(W) = \mathscr{F}(W)$ for $W \subset U$ open. If $(X, \mathcal{O}_X)$ is a locally ringed space over R and U is open in X, let $\mathcal{O}_U$ denote $\mathcal{O}_X|_U$, then $(U, \mathcal{O}_U)$ is a locally ringed space over R.

Definition 9.22. If $\phi = (f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of locally ringed spaces over R, then we say that ϕ is an isomorphism if f is a homeomorphism and $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism of sheaves of R-algebras. In this case, there exists an inverse morphism $\phi^{-1} = (g, g^\sharp)$ where $g = f^{-1}$ and $g^\sharp\colon \mathcal{O}_X \to g_*\mathcal{O}_Y$ is defined by setting the required homomorphisms (for W open in X) $\mathcal{O}_X(W) \to \mathcal{O}_Y(g^{-1}W)$ to be the inverse of the isomorphism $\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}U)$ where $U = g^{-1}W$.

Lecture 10

1st November 12:00

Recall that the following are locally-ringed spaces over k:

(1) take $k = \mathbb{R}$, $Y \subset \mathbb{R}^n$ open, then we have $\mathcal{O}_Y$ the sheaf of C^∞ functions;
(2) take $k = \mathbb{C}$, $Y \subset \mathbb{C}^n$ open, then we have $\mathcal{O}_Y$ the sheaf of holomorphic functions on Y;
(3) take $k = \bar{k}$, Y an affine variety over k with the Zariski topology, then we have $\mathcal{O}_Y$ the sheaf of regular functions on Y.

Definition 10.23. A C^∞-manifold is a locally-ringed space $(X, \mathcal{O}_X)$ over $\mathbb{R}$ such that dfn:manifold

(1) X is paracompact[7] and Hausdorff,
(2) for every $P \in X$, there exists an open neighbourhood $P \in U \subset X$ such that $(U, \mathcal{O}_U)$ is isomorphic as a locally-ringed space over $\mathbb{R}$ to $(Y, \mathcal{O}_Y)$ as in (1) above.

For complex manifolds, we substitute locally-ringed over $\mathbb{C}$ and locally isomorphic to $(Y, \mathcal{O}_Y)$ as in (2) above. A C^∞-map (resp. holomorphic) map between C^∞ (resp. complex) manifolds is defined to be a morphism of them as locally-ringed spaces over $\mathbb{R}$ (resp. $\mathbb{C}$).

[7]Let $\{W_\alpha\}_{\alpha \in I}$ be a cover of a topological space X (we do not assume that it is an open cover).

Definition 10.24. A cover $\{T_\beta\}_{\beta \in J}$ is called a refinement of $\{W_\alpha\}_{\alpha \in I}$ if for all $\beta \in J$, there exists $\alpha \in I$ such that $T_\beta \subset W_\alpha$.

Definition 10.25. A collection $\{W_\alpha\}_{\alpha \in I}$ of subsets of X is called locally finite if each $x \in X$ has an open neighbourhood whose intersection with W_α is non-empty for only finitely many α.

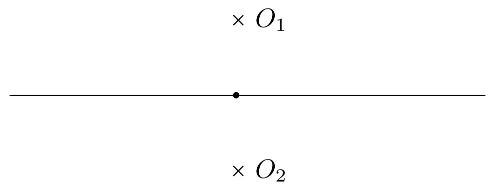
Definition 10.26. A topological space X is called paracompact if every open cover of X has a locally finite refinement.

Exercise: the above definitions correspond to the usual ones in terms of charts and atlases (see examples sheet 1, question 14).

dfn:prevariety **Definition 10.27.** A prevariety over $k = \bar{k}$ is a locally ringed space $(X, \mathscr{O}_X)$ over k such that

(1) $X = \bigcup U_i$ for a finite collection of open sets U_i,
(2) each $(U_i, \mathscr{O}_{U_i})$ is isomorphic, as a locally ringed space over k, to some affine variety $(Y, \mathscr{O}_Y)$ over k. The U_i are called affine pieces.

Example. Consider X to be



the line with two origins i.e. take two copies of $\mathbb{A}^1$ and identify all corresponding points except the origins. The topology is the quotient topology from the Zariski topology on $\mathbb{A}^1$. Let $X = U_1 \cup U_2$ where $U_1 = X \backslash \{O_2\}$ and $U_2 = X \backslash \{O_1\}$ and each U_i is identified with $\mathbb{A}^1_k$. These identifications determine a locally-ringed space structure over k, $(X, \mathscr{O}_X)$. Note that if $U \ni \{O_1, O_2\}$ and $f \in \mathscr{O}_X(U)$, then $f(O_1) = f(O_2)$ i.e. we cannot separate O_1 and O_2 by regular functions. We wish to rule out this kind of prevariety.

Remark. Observe that a topological space is Hausdorff iff the diagonal $\Delta \subset X \times X$ is closed in the product topology, i.e. if you give me a point $(P, Q) \notin \Delta$ then there exists a basic open set $U \times V \ni (P, Q)$ with $(U \times V) \cap \Delta = \varnothing$ i.e. $U \ni P$ and $V \ni Q$ and $U \cap V = \varnothing$.

10.1. Products of pre-varieties. If $V \subset \mathbb{A}^n$, and $W \subset \mathbb{A}^m$ are affine varieties, then the product $V \times W \subset \mathbb{A}^{n+m}$ has the natural structure of an affine variety. On this we have the Zariski topology. Given two pre-varieties $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ with $X = \bigcup U_i$ and $Y = \bigcup V_j$ finite unions with $(U_i, \mathscr{O}_{U_i})$ and $(V_j, \mathscr{O}_{V_j})$ affine for all i, j, we can stipulate a topology on $X \times Y$ by $G \subset X \times Y$ is open iff $G \cap (U_i \times V_j)$ is Zariski open in $U_i \times V_j$ for all i, j. An easy topological check: this doesn't depend on the choice of decomposition of X, Y into open sets (given U'_k, V'_l, we have

$$(U_i \times V_j) \cap (U'_k \times V'_l) = (U_i \cap U'_k) \times (V_j \cap V'_l)$$

open in $U_i \times V_j$). Moreover, there exists a basis of open sets in this topology

$$\mathcal{B} = \{G \subset U_i \times V_j \text{ Zariski open for some } i, j\}$$

and $\mathcal{B}$ is closed under finite intersections. We call the above topology the Zariski topology on $X \times Y$.

dfn:separated **Definition 10.28.** A pre-variety $(X, \mathscr{O}_X)$ is called separated if the diagonal $\Delta_X \subset X \times X$ is closed in the Zariski topology. We then call $(X, \mathscr{O}_X)$ a variety over k.

Remark. An exercise is to show that the line with two origins defined above is not a variety.

The idea now is to show you that the old things we called varieties are also varieties under this new definition.

Example. Given projective varieties $V \subseteq \mathbb{P}^n$ and $W \subseteq \mathbb{P}^m$, we can embed $V \times W \overset{\phi}{\hookrightarrow} \mathbb{P}^{nm+n+m}$ by

$$(x_0 : \cdots : x_n) \times (y_0 : \cdots : y_m) \mapsto (z_{00} : z_{01} : \cdots : z_{nm})$$

where $z_{ij} = x_i y_j$, called the Segre embedding. If V is defined by homogeneous polynomials $\{F_\alpha(x_0 : \cdots : x_n)\}_{\alpha\in A}$, and W is defined by homogeneous polynomials $\{G_\beta(y_0 : \cdots : y_m)\}_{\beta\in B}$, then $\phi(V\times W)$ defined by equations $z_{ij}z_{i'j'} = z_{ij'}z_{i'j}$ and

$$\{F_\alpha(z_{0j} : z_{1j} : \cdots : z_{nj}) \colon \alpha \in A, 0 \le j \le m\}$$
$$\{G_\beta(z_{i0} : z_{i1} : \cdots : z_{im}) \colon \beta \in B, 0 \le i \le n\}$$

and $\phi(\underline{x}\times\underline{y}) = (z_{ij})$ where $\underline{x}\in V$ and $\underline{y}\in W$ and $z_{ij} = x_i y_j$ satisfies these equations, since if $y_j \ne 0$, then

$$(z_{0j} : \cdots : z_{nj}) = (x_0 : x_1 : \cdots : x_n)$$

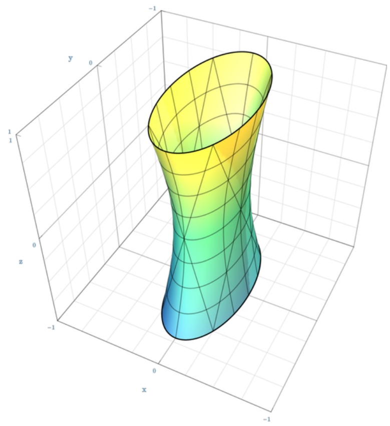
and if $x_i \ne 0$ then

$$(z_{i0} : \cdots : z_{im}) = (y_0 : y_1 : \cdots : y_m)$$

Conversely, if (z_{ij}) satisfies these equations and without loss of generality let $z_{pq} \ne 0$, then

$$\underbrace{(z_{0q} : \cdots : z_{nq})}_{\in V} \times \underbrace{(z_{p0} : \cdots : z_{pm})}_{\in W} \mapsto z_{ij}$$

Finally observe that ϕ is injective, since points $\underline{x}\in V$ and $\underline{y}\in W$ are recovered uniquely by this recipe. So the image $\phi(V\times W)$ has the natural structure of a projective variety e.g. for $V = \mathbb{P}^1 = W$, then $\mathbb{P}^1\times\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$, is the image of the smooth quadric $z_{00}z_{11} = z_{01}z_{10}$.



Lecture 11

4th November 12:00

11.1. More on the Segre embedding; complete varieties. The examples class is on Wednesday at 4pm in MR5.

Let $V\subset\mathbb{P}^n$ and $W\subset\mathbb{P}^m$ be projective varieties, then we talked about the Segre embedding

$$\phi\colon V\times W \hookrightarrow \mathbb{P}^{mn+n+m}$$
$$(\underline{x})\times(\underline{y}) \mapsto (w_{ij})$$

where $w_{ij} = x_i y_j$ with $0\le i\le n$ and $0\le j\le m$. The image $\phi(V\times W)$ is naturally a projective variety. If now $V_0\subset V$, $W_0\subset W$ are the affine pieces of V, W given by $X_0\ne 0$ (resp. $Y_0\ne 0$), the image of $V_0\times W_0$ under Segre in the affine piece of $V\times W$ is given by $Z_{00}\ne 0$. Moreover, there exists an isomorphism of this affine piece with $V_0\times W_0\subset\mathbb{A}^n\times\mathbb{A}^m = \mathbb{A}^{n+m}$

$$\left(\frac{z_{ij}}{z_{00}}\right)\mapsto\left(\frac{z_{10}}{z_{00}},\dots,\frac{z_{n0}}{z_{00}}\right)\times\left(\frac{z_{01}}{z_{00}},\dots,\frac{x_{0m}}{z_{00}}\right)$$

with obvious inverse - this is reassuring. Identifying $V \times W$ with its image under Segre (i.e. $V \times W$ a projective variety) we see that $G \subset V \times W$ is open in the usual projective Zariski topology iff $G \cap (V_i \times W_j)$ is open in the Zariski topology on $V_i \times W_j \subset \mathbb{A}^{n+m}$ for all i, j which is true iff $G \subset V \times W$ is open in the Zariski topology as defined above. This is also reassuring. In particular, it's clear that the diagonal $\Delta_V \subset V \times V$ is closed in this case, since it's just given by the extra equations $z_{ij} = z_{ji}$ $(i, j = 0, \dots, n)$. This is comforting.

Corollary 11.16 (2.1)**.** *A quasi-projective variety $(U, \mathscr{O}_U)$ is a variety.*

Proof. Observe first that U is covered by a finite number of affine pieces. It is sufficient to prove this when U is open in an affine variety $V \subset \mathbb{A}^n$. If $U = V \backslash V(f_1, \dots f_N)$, then $U = \bigcup_{i=1}^N D(f_i)$ for basic open sets $D(f_i)$, where by the definition of $\mathscr{O}_V$, the locally-ringed space $(D(f_i), \mathscr{O}_{D(f_i)})$ over k is isomorphic to an affine variety $W \subset \mathbb{A}^{n+1}$, and where $\mathscr{O}_V(D(f_i)) \cong k[V]_{f_i}$. Hence $(U, \mathscr{O}_U)$ is a pre-variety. If now U is open in a projective variety $\overline{U}$, the topology on $U \times U$ is just the subspace topology $\overline{U} \times \overline{U}$, and so $\Delta_{\overline{U}} \subset \overline{U} \times \overline{U}$ is closed, which implies that $\Delta_U \subset U \times U$ is closed, therefore $(U, \mathscr{O}_U)$ is a variety. ☐

Remark. Given any two varieties $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$, the product space (with the Zariski topology) can be given the structure of a variety. So how do we do this? For each basic open set $G \subset U_i \times V_j$, we can define $\mathscr{O}_{X \times Y}(G) = \mathscr{O}_{U_i \times V_j}(G)$, the regular functions on G. Thus we can define a continuous function on any open subset of $X \times Y$ to be regular if its restriction to these basic affine pieces are regular (alternatively use corollary 7.12 to define $\mathscr{O}_{X \times Y}$ from the $\mathcal{B}$-sheaf on a base $\mathcal{B}$ defined before). For fact that $X \times Y$ is separated, observe

$$\Delta_{X \times Y} = \pi_{13}^{-1}(\Delta_X) \cap \pi_{24}^{-1}(\Delta_Y) \subset X \times Y \times X \times Y$$

where the maps π_{13} and π_{24} are continuous and hence $\Delta_{X \times Y}$ closed.

The Zariski topology is hardly ever Hausdorff so it's a rubbish property, but we have reinterpreted it in a way that is useful to us here. Compactness is also a rubbish property, but we have interpreted that also in a way that has made it useful to us.

Definition 11.29. Given varieties $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$, we then consider the continuous map of topological spaces $\pi\colon X \times Y \to Y$ (in fact a morphism of varieties). The variety $(X, \mathscr{O}_X)$ is called complete if π is a closed map, for any variety Y.

Remark. I said for any Y here, but I can take sort of special Y here: if $Y = \bigcup U_i$, a finite decomposition of Y into open affine pieces, and $Z \subset Y$, then $Y \backslash Z = \bigcup_i (U_i \backslash U_i \cap Z)$, therefore Z is closed iff $Z \cap U_i$ is closed in U_i for all i, therefore without loss of generality we may take Y to be affine, $Y \subset \mathbb{A}^n$. Moreover, since Z is closed in Y, this implies that Z is closed in $\mathbb{A}^m$, we may even take $Y = \mathbb{A}^n$. E.g. $\mathbb{A}^1$ is not complete, since the projection $\mathbb{A}^1 \times \mathbb{A}^1 \xrightarrow{\pi} \mathbb{A}^1$ is not closed (image of $xy = 1$ is $\mathbb{A}^1 \backslash \{0\}$ under the map $(x, y) \overset{\pi}{\mapsto} y$).

Proposition 11.17 (2.2)**.** *Any projective variety $V \subset \mathbb{P}^n$ is complete.*

Proof. It is sufficient to prove that $\mathbb{P}^n$ is complete i.e. if $Z \subset \mathbb{P}^n \times \mathbb{A}^m$ is closed, then the image $\pi(Z)$ is closed in $\mathbb{A}^m$.

Suppose that $\mathbb{P}^n$ has homogeneous coordinates $X_0, \dots, X_n$ and $\mathbb{A}^m$ has affine coordinates $Y_1, \dots, Y_m$ (I won't be careful here in making a distinction between coordinate functions and coordinates). Then $Z \subset \mathbb{P}^n \times \mathbb{A}^m$ is given by the vanishing of some polynomials

$$F_\alpha(X_0, \dots X_n; Y_1, \dots, Y_m)$$

homogeneous in the X_i but not necessarily in the Y_j (follows easily from the above decomposition of $\mathbb{P}^n \times \mathbb{P}^m$ that the closed sets are given by polynomials in the Z_{ij}, and hence bi-homogeneous polynomials in each homogeneous set of variables) i.e. $\underline{x} \times \underline{y} \in Z \subset \mathbb{P}^n \times \mathbb{A}^m$ iff $F_\alpha(\underline{x}, \underline{y}) = 0$ for all α, therefore for $\underline{y} \in \mathbb{A}^m$, the set $Z \cap \pi^{-1}(\underline{y})$ consists of non-zero solutions of $\{F_\alpha(\underline{X}; \underline{Y})\}_{\alpha \in A}$ (for A a finite set), i.e. $\underline{y} \in \pi(Z)$ iff the equations $F_\alpha(\underline{X}; \underline{y}) = 0$ have a non-trivial solution. Let

$$\mathbb{A}^m \backslash \pi(Z) = U = \{\underline{y} \in \mathbb{A}^m : F_\alpha(\underline{X}; \underline{y}) = 0 \text{ have no non-trivial solution}\}$$

It remains to prove that $\mathbb{A}^m \backslash U$ is closed, i.e. U is open. Denote by J_N the space of homogeneous polynomials of degree N in $X_0, \dots, X_n$. The projective nullstellensatz says that the above equations have no non-zero solutions iff the ideal $\langle F_\alpha(\underline{X}; \underline{y}) \rangle_{\alpha \in A}$ in $k[X_0, \dots, X_n]$ has radical containing $(X_0, \dots, X_n)$ the "irrelevant ideal". This is true iff the ideal $\langle F_\alpha(\underline{X}; \underline{y}) \rangle$ contains J_N for some N. Let

$$U_N := \{\underline{y} \in \mathbb{A}^m : \langle F_\alpha(\underline{X}; \underline{y}) \rangle \supseteq J_N\}$$

Since $U = \bigcup_N U_N$, it is sufficient to prove that each $\mathbb{A}^m \backslash U_N$ is closed. We'll finish this in the next class. $\square$

Lecture 12

6th November 12:00

We will finish the proof.

PROOF. Recall: $\{F_\alpha(\underline{X}; \underline{Y})\}_{\alpha \in A}$ a finite set of polynomials, homogeneous in $X_0, \dots, X_n$. For $\underline{y} \in \mathbb{A}^m$, the equations $F_\alpha(\underline{X}; \underline{y}) = 0$ have no non-trivial solutions iff the radical of the ideal $\langle F_\alpha(\underline{X} : \underline{y}) \rangle \lhd k[X_0, \dots, X_n]$ contains $(X_0, \dots, X_n)$ which is true iff $\langle F_\alpha(\underline{X}, \underline{y}) \rangle_{\alpha \in A} \supseteq J_N$ for some $N \gg 0$, where J_N = space of homogeneous polynomials of degree N. Set

$$U_N := \{\underline{y} \in \mathbb{A}^m : \langle F_\alpha(\underline{X}; \underline{y}) \rangle \supseteq J_N\}$$

It is sufficient to prove that each $\mathbb{A}^m \backslash U_N$ is closed. Suppose that $F_\alpha(\underline{X}; \underline{y})$ has degree N_α in the X_j's and let $\{M_{\alpha,i} : i = 1, \dots, s_\alpha\}$ denote the set of monomials of degree $N - N_\alpha$ in $X_0, \dots, X_n$. For $\underline{y} \in \mathbb{A}^m$, consider the linear space spanned by

$$\{M_{\alpha,i}(\underline{X}) F_\alpha(\underline{X}, \underline{y}) : 1 \le i \le s_\alpha, \text{for all } \alpha \in A\} \subset J_N$$

Thus $\underline{y} \notin U_N$ iff this space is not equal to J_N, which is true iff

$$\text{rank}\{\text{coefficients of } M_{\alpha,i}(\underline{X}) F_\alpha(\underline{X}; \underline{y})\} < \dim J_N = r \text{ (say)}$$

with respect to some basis of J_N, which is true iff the $r \times r$ minors of the matrix of coefficients all vanish, which is true iff $\underline{y}$ satisfies a certain set of polynomial equations. $\square$

Corollary 12.18 (1)**.** *The image of a projective variety V under a morphism $\phi: V \to \mathbb{P}^m$ is a projective variety.*

PROOF. Consider the graph of the morphism ϕ denoted by $\Gamma_\phi \subset V \times \mathbb{P}^m$, where

$$\Gamma_\phi = \{(P, \phi(P)) : P \in V\}$$

i.e. $\Gamma_\phi = (\phi \times \text{id})^{-1} \Delta_{\mathbb{P}^m}$, a closed subset. Thus $\pi_2(\Gamma_\phi) = \phi(V)$ closed in $\mathbb{P}^m$ by proposition 11.17 and so is the underlying set of a projective variety. $\square$

Corollary 12.19 (2)**.** *The only everywhere regular functions on an irreducible projective variety are constants.*

PROOF. We prove this for any complete variety X. A global regular function $f \in \mathscr{O}_X(X)$ defines a closed graph $\Gamma_f \subset X \times \mathbb{P}^1$ where

$$\Gamma_f = \{(P, f(P)) \in X \times \mathbb{P}^1\}$$

(clear by looking at affine pieces of X). But X is complete which implies that $\pi_2(\Gamma_f) = f(X) \subseteq \mathbb{A}^1$ is closed in $\mathbb{P}^1$. So either $f(X)$ is $\mathbb{A}^1$ or a finite set. But $\mathbb{A}^1$ is not closed in $\mathbb{P}^1$, so it is a finite set S. We could then write $X = \bigcup_{s \in S} f^{-1}(s)$ which would contradict that X is irreducible. This implies that $f(X)$ is a point which implies that f is constant. □

Corollary 12.20 (3)**.** *A quasi-projective variety U is complete iff it is projective.*

PROOF. Let us suppose that $U \subsetneq V \subseteq \mathbb{P}^n$, with $V = \overline{U}$ projective with

$$P = (1 : 0 : \cdots : 0) \in V \backslash W$$

by choosing our coordinates appropriately. Consider $Z \subset U \times \mathbb{A}^n$ defined by equations $X_0 \cdot y_i = X_i$ $(i = 1, \ldots, n)$. It is clear that $\underline{0} \in \overline{\pi_2(Z)}$ (since π_2 is continuous and $\underline{0} = \pi_2(P) \in \pi_2(\overline{Z})$ where $\overline{Z} \subset V \times \mathbb{A}^n$) but $\underline{0} \notin \pi_2(Z)$ since $P \notin U$. □

Remark. There are obvious topological definitions for a variety being irreducible, connected, etc. Clear (from affine case) that any variety is compact with respect to the Zariski topology. Any variety X has a base of affine open sets.

12.1. Rational functions from the structure sheaf. We want to define rational functions via the structure sheaf rather than vice-versa.

Definition 12.30. A rational function on a variety X is an equivalence class of pairs (U, ϕ) where U is an open dense subset and $\phi \in \mathscr{O}_X(U)$ where $(U, \phi) \sim (V, \psi)$ iff there exists an open dense $W \subset U \cap V$ such that $\phi|_W = \psi|_W$. The rational functions form a ring $\operatorname{Rat}(X)$. The domain of definition of a rational function f, is

$$\operatorname{dom}(f) = \{x \in X : x \in U \text{ for some } (U, \phi) \text{ representing } f\}$$

If U is an open dense subset of X (e.g. open, dense, affine), it is clear that

$$\operatorname{Rat}(X) \xrightarrow{\sim} \operatorname{Rat}(U)$$

If X is irreducible, $\operatorname{Rat}(X)$ is a field, called the function field $k(X)$ of X (observe that X is irreducible iff any non-empty open set contained in it is dense). So for X quasi-projective, this coincides with the previous definition. Then for $P \in X$, there exists an injection $\mathscr{O}_{X,p} \hookrightarrow k(X)$ (since an element of $\mathscr{O}_{X,p}$ is also represented by a pair (U, ϕ) with $P \in U$ and $\phi \in \mathscr{O}_X(U)$). Thus for X irreducible, $\mathscr{O}_X(U) = \bigcap_{p \in U} \mathscr{O}_{X,p} \subset k(X)$. Note that for $W \subset U$ open and $g, h \in \mathscr{O}_U(W)$ agree on an open (dense) subset, then $g = h$.

dfn:birationally-equivalent **Definition 12.31.** Two varieties X and Y are birationally equivalent if there exists open dense subsets $U \subset X$, $V \subset Y$ and an isomorphism of varieties $(U, \mathscr{O}_U) \xrightarrow{\sim} (V, \mathscr{O}_V)$.

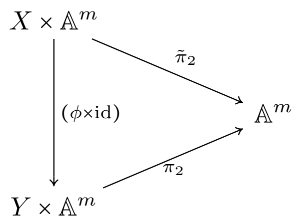
Exercise: (on example sheet 2, question 2) two varieties X and Y are birationally equivalent iff $\operatorname{Rat}(X) \cong \operatorname{Rat}(Y)$ as k-algebras.

I'll tell you a result now that I won't prove, but it gives one example of the importance of this idea.

Theorem 12.21 (Chow)**.** *Let Y be a complete irreducible variety. Then there exists a projective variety X and a birational morphism $\phi: X \to Y$ (i.e. a morphism inducing an isomorphism on suitable open subsets.)*

PROOF. Straightforward - see [**Kem93**, pp. 34]. □

Remark. The converse is clearly true using proposition 11.17 and using the diagram



Lecture 13

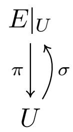
8th November 12:00

13.1. §3: Locally free and coherent $\mathscr{O}_X$-modules. Notation: $(X, \mathscr{O}_X)$ is locally-ringed space over k - in due course we'll specialise to varieties over $k = \overline{k}$.

Given two $\mathscr{O}_X$-modules $\mathscr{M}, \mathscr{N}$ on X, we define $\mathscr{M} \oplus \mathscr{N}$ in the obvious way. To define an arbitrary sum $\bigoplus_{i \in I} \mathscr{M}_i$ of $\mathscr{O}_X$-modules, in general we have to sheafify (because glueing together an infinite set of sections may not be possible in the presheaf sum). If however every open subset of X is compact, then this is not a problem and we can define $\bigoplus_{i \in I} \mathscr{M}_i$ is the simple minded way, since eventually the sheaf coordinates will involve only finitely many non-zero entries). For $U \subset X$ open and $\mathscr{M}$ an $\mathscr{O}_X$-module, we have an $\mathscr{O}_U$-module $\mathscr{M}|_U$.

Definition 13.32. An $\mathscr{O}_X$-module $\mathscr{M}$ is locally free of rank r if for each $P \in X$, there exists an open neighbourhood $U \ni P$ such that $\mathscr{M}|_U \cong \mathscr{O}_U^r$ (i.e. $\mathscr{O}_U^{\oplus r}$).

Example. If $(X, \mathscr{O}_X)$ is a C^∞ (or complex) manifold and $E \to X$ a rank-r C^∞ (resp. holomorphic) vector bundle over X, we can define a locally free $\mathscr{O}_X$-module $\mathscr{E}$ of rank-r, where $\mathscr{E}(U) = C^\infty$ (resp. holomorphic) sections σ of E over U. Since E is locally trivial (i.e. $U \times \mathbb{R}^r$, resp. $U \times \mathbb{C}^r$) it's clear that $\mathscr{E}$ is locally free of rank r.



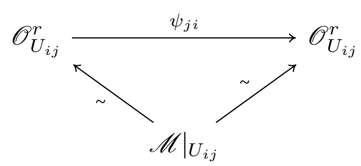
For the case of varieties X, we can define an algebraic vector bundle E over X in the obvious way and obtain a sheaf of regular sections, a locally free $\mathscr{O}_X$-module.

Notation: sections of a presheaf/sheaf $\mathscr{F}$ over an open set U, then we write

$$\mathscr{F}(U) = \Gamma(U, \mathscr{F}) = H^0(U, \mathscr{F})$$

(we will see later that it is the zeroth cohomology group as well, so the latter notation is sensible). So for $E \to X$ a vector bundle, and $\mathscr{E}$ a sheaf of sections, then $\Gamma(U, E) = \Gamma(U, \mathscr{E})$.

For $\mathscr{M}$ a locally free $\mathscr{O}_X$-module of rank r, we have an open cover $\{U_i\}$ of X (when $\mathscr{M}$ is a variety we may take it to be a finite, affine, open cover) and trivialisation $\mathscr{M}|_{U_i} \cong \mathscr{O}_{U_i}^r$. This then gives rise to isomorphisms on the overlaps $U_{ij} \coloneqq U_i \cap U_j$



i.e. elements $\psi_{ji} \in \Gamma(U_{ij}, \mathsf{GL}(r, \mathscr{O}_X))$ satisfying the compatibility conditions

- $\psi_{ii} = \mathrm{id}$ on U_i,
- $\psi_{ij} = \psi_{ji}^{-1}$ on U_{ij},
- $\psi_{kj}\psi_{ji} = \psi_{ki}$ over $U_{ijk} := U_i \cap U_j \cap U_k$.

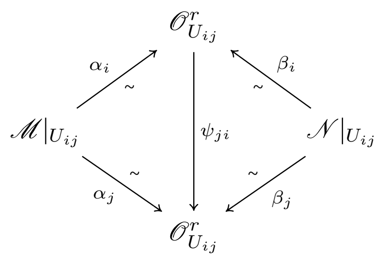
Remark. These maps ψ_{ji} are called transition functions.

Proposition 13.22 (3.1)**.** *Two $\mathscr{O}_X$-modules $\mathscr{M}, \mathscr{N}$ which are locally trivialised by the same data $\{U_i\}, \{\psi_{ji}\}$ are isomorphisms as sheaves of $\mathscr{O}_X$-modules.*

PROOF. Take the given isomorphisms

$$\mathscr{M}|_{U_i} \cong \mathscr{O}_{U_i}^r \cong \mathscr{N}|_{U_i}$$

This induces an isomorphism on stalks $\mathscr{M}_P \to \mathscr{N}_P$ for all $P \in X$ and these isomorphisms are well-defined (independent of U_i) since the transition functions are the same by assumption, i.e. if $\alpha_i: \mathscr{M}|_{U_i} \to \mathscr{O}_{U_i}^r$ and $\beta_i: \mathscr{N}|_{U_i} \to \mathscr{O}_{U_i}^r$, then the following diagram commutes



i.e. $\mathscr{M}|_{U_{ij}} \xrightarrow{\sim} \mathscr{N}|_{U_{ij}}$ is well-defined. Consider now sheaves $\mathscr{M}^+$ and $\mathscr{N}^+$ - $\mathscr{M}^+$ consists of sections $s: U \to \coprod_{P \in U} \mathscr{M}_P$, locally given by a section of $\mathscr{M}$, i.e. $s(Q) = t_Q$ on some neighbourhood. Under the above isomorphisms on stalks, s yields a section $s': U \to \coprod_{p \in U} \mathscr{N}_p$, locally given by a section of $\mathscr{N}$ - i.e. we obtain a morphism of $\mathscr{O}_X$-modules $\mathscr{M}^+ \to \mathscr{N}^+$ inducing isomorphisms on stalks i.e. $\mathscr{M}^+ \cong \mathscr{N}^+$. Since $\mathscr{M} \simeq \mathscr{M}^+$, $\mathscr{N} \simeq \mathscr{N}^+$, we deduce $\mathscr{M} \cong \mathscr{N}$ as $\mathscr{O}_X$-modules. □

Definition 13.33. A locally free $\mathscr{O}_X$-module of rank 1 is called invertible (which then corresponds to a line bundle, and so we often refer to invertible $\mathscr{O}_X$-modules
dfn:line-bundle as line bundles).

Definition 13.34. Given $\mathscr{O}_X$-modules $\mathscr{M}, \mathscr{N}$ on $(X, \mathscr{O}_X)$, define a presheaf $\mathscr{T}$ by $\mathscr{T}(U) = \mathscr{M}(U) \otimes_{\mathscr{O}(U)} \mathscr{N}(U)$ for all open U, and this is clearly a presheaf of $\mathscr{O}_X$-modules. Define $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}$ to be the sheafification $\mathscr{T}^+$, and this is clearly an $\mathscr{O}_X^+$-module and hence an $\mathscr{O}_X$-module.

Remark. We studied this in example sheet 1, question 9 where I told you to write down the universal property satisfied by this.

dfn:hom-sheaf **Definition 13.35.** Define a sheaf $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$[8] by

$$\mathcal{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})(U) = \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{M}|_U, \mathscr{N}|_U)$$

($\nwarrow$ morphisms of $\mathscr{O}_U$-modules)

with obvious restriction maps.

[8]In class, he denotes this by Hom and in Hartshorne it is denoted $\mathscr{H}om$, but $\mathcal{H}om$ looks better I think - this is what Ravi Vakil uses.

Remark. In the above, an element $\phi_U \in \operatorname{Hom}_{\mathscr{O}_U}(\mathscr{M}|_U, \mathscr{N}|_U)$ may be interpreted as an element of $\operatorname{Hom}_{\mathscr{O}_U^+}(\mathscr{M}^+|_U, \mathscr{N}^+|_U)$ i.e.

$$\phi_U = \coprod_{P \in U} \phi_P : \coprod_{P \in U} \mathscr{M}_P \longrightarrow \coprod_{P \in U} \mathscr{N}_P$$

with the property that for any section $s \in \mathscr{M}^+(V)$ for $V \subseteq U$ (i.e. $s: V \to \coprod_{P \in V} \mathscr{M}_P$ with given local coordinates), we have $\phi_U \circ s \in \mathscr{N}^+(V)$. It is clear from this that $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$ is a sheaf. The $\mathscr{O}_X$-module structure is clear: given

$$\phi_U : \mathscr{M}_U \to \mathscr{N}_U$$

and $f \in \mathscr{O}_X(U)$, we have a morphism $f\phi|_U : \mathscr{M}|_U \to \mathscr{N}|_U$.

Definition 13.36. The dual $\mathscr{M}^\vee$ of $\mathscr{M}$, is $\mathscr{M}^\vee := \mathcal{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{O}_X)$.

There exists a morphism of $\mathscr{O}_X$-modules $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{M}^\vee \to \mathscr{O}_X$ defined as follows: (next time!).

Lecture 14

11th November 12:00

Let $\mathscr{M}$ be an $\mathscr{O}_X$-module, then we have the dual $\mathscr{M}^\vee = \mathcal{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{O}_X)$, and then there exists a morphism of $\mathscr{O}_X$-modules $\mathscr{M} \otimes \mathscr{M}^\vee \to \mathscr{O}_X$ defined as follows: let $\mathscr{T}$ be the presheaf of $\mathscr{O}_X$-modules given by $\mathscr{T}(U) = \mathscr{M}(U) \otimes_{\mathscr{O}_X} \mathscr{M}^\vee(U)$ and define a morphism of presheaves $\mathscr{T} \to \mathscr{O}_X$ given by

$$m \otimes \phi \longrightarrow \phi(U)(m)$$

for $U \subset X$ open, and where $m \in \mathscr{M} = \mathscr{M}|_U(U)$ and $\phi \in \mathscr{M}^\vee(U) = \operatorname{Hom}_{\mathscr{O}_U}(\mathscr{M}|_U, \mathscr{O}_U)$, and this respects the $\mathscr{O}_X(U)$-module structure. The universal property of sheafification yields a morphism of sheaves of $\mathscr{O}_X$-modules

$$\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{M}^\vee = \mathscr{T}^+ \longrightarrow \mathscr{O}_X$$

If now $\mathscr{M}$ is locally free of rank 1, then locally we have $\mathscr{M}|_U \cong \mathscr{O}_U$ and

$$\mathscr{M}^\vee|_U \cong \mathcal{H}om_{\mathscr{O}_U}(\mathscr{O}_U, \mathscr{O}_U) \cong \mathscr{O}_U$$

and thus the induced map on stalks $(\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{M}^\vee)_P \to \mathscr{O}_{X,P}$ are all isomorphisms i.e.

$$\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{M}^\vee \xrightarrow{\sim} \mathscr{O}_X$$

by lemma 5.10.

Definition 14.37. The Picard group $\operatorname{Pic}(X)$ of the ringed space $(X, \mathscr{O}_X)$ is the Abelian group whose elements are isomorphism classes of invertible sheaves, composition is given by $\otimes_{\mathscr{O}_X}$, the identity element is $\mathscr{O}_X$ and the inverse of an invertible sheaf $\mathscr{M}$ is its dual $\mathscr{M}^\vee$. `dfn:picard-group`

Remark. By proposition 13.22 we do have a set (!), and the rest is clear.

14.1. Pulbacks of sheaves. Let $\mathscr{F}$ be a sheaf of Abelian groups over a topological space Y and $f: X \to Y$ a continuous map. We define the pullback or inverse image sheaf $f^{-1}\mathscr{F}$ as follows: for $U \subset X$ open, set

$$(f^{-1}\mathscr{F})(U) := \left\{ \begin{array}{c} s: U \to \coprod_{P \in U} \mathscr{F}_{f(P)} \text{ such that } s(P) \in \mathscr{F}_{f(P)}, \text{ for all } P \in U \\ \text{and for any } P \in X, \text{ there exists open neighbourhoods } V, W \\ \text{with } f(P) \in W \text{ and } P \in V \subset f^{-1}W \text{ and a section } t \in \mathscr{F}(W) \\ \text{with } s(Q) = t_{f(Q)} \text{ for all } Q \in V \end{array} \right\}$$

If $\mathscr{F} = \mathscr{O}_Y$, we get a sheaf of rings $f^{-1}\mathscr{O}_Y$ over X. *Claim:* if $(f, f^\sharp): (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of varieties over k (or manifolds), there exists a natural morphism of sheaves of rings on X, $f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$. The obvious question is: is this surjective? The answer is no, as you will see on the second examples sheet.

Proof of claim: given U open in X, define $(f^{-1}\mathscr{O}_Y)(U) \to \mathscr{O}_X(U)$ by

$$\{s: U \to \coprod_{P \in U} \mathscr{O}_{Y,f(P)} \text{ such that} \dots\} \mapsto g$$

where locally $s(Q) = h_{f(Q)}$ for some $h \in \mathscr{O}_Y(W)$ and all $Q \in V \subset f^{-1}W \subset X$ then

$$g(Q) = h(f(Q)) = f^\sharp(h)(Q)$$

for all $Q \in V$ (we have equality by the familiar argument - cf. lemma 6.11 and questions 10, 14 on examples sheet 1 - uses $(f, f^\sharp)$ a morphism of locally-ringed spaces). We have $f^\sharp: \mathscr{O}_Y(W) \to \mathscr{O}_X(f^{-1}W)$ and locally $g = f^\sharp(h)|_V \in \mathscr{O}_V(V)$ as required.

Definition 14.38. Given a morphism of varieties/prevarieties $f: X \to Y$, the inverse image sheaf $f^{-1}\mathscr{M}$ of an $\mathscr{O}_Y$-module $\mathscr{M}$ on Y is naturally an $f^{-1}\mathscr{O}_Y$-module on X. Define the inverse image $\mathscr{O}_X$-module $f^*\mathscr{M}$ by $f^{-1}\mathscr{M} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$.

Remark. We have

$$f^*\mathscr{O}_Y = f^{-1}\mathscr{O}_Y \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X = \mathscr{O}_X$$

Example. If Y is a point, then an $\mathscr{O}_Y$-module is a k-vector space $\mathscr{M}$. Clearly $f^{-1}\mathscr{O}_Y = k$ the constant sheaf (the germs of functions on a point, so it follows), and in this case $f^{-1}\mathscr{O}_Y \hookrightarrow \mathscr{O}_X$ is the obvious inclusion map. $f^{-1}\mathscr{M}$ is the constant sheaf $\mathscr{M}$ on X and $f^*\mathscr{M}$ is the free $\mathscr{O}_X$-module $\mathscr{M} \otimes_k \mathscr{O}_X$.

Definition 14.39. An $\mathscr{O}_X$-module $\mathscr{M}$ on a variety $(X, \mathscr{O}_X)$ over k is said to be

(a) finitely generated[9],
(b) quasi-coherent,
(c) coherent,

if for any $P \in X$, there exists an open neighbourhood $U \ni P$ and an exact sequence of $\mathscr{O}_U$-modules,

(a) $\mathscr{O}_U^p \longrightarrow \mathscr{M}|_U \longrightarrow 0$ where p is finite depending on U,

(b) $\mathscr{O}_U^{\oplus I} \longrightarrow \mathscr{O}_U^{\oplus J} \longrightarrow \mathscr{M}|_U \longrightarrow 0$ where I, J are indexing sets depending on U,

(c) $\mathscr{O}_U^q \longrightarrow \mathscr{O}_U^p \longrightarrow \mathscr{M}|_U \longrightarrow 0$ where p, q are finite, depending on U.

Remark. We'll see later that for varieties one has that (a) and (b) together imply (c) and this is essentially Hilbert's basis theorem.

Definition 14.40. The support of a sheaf $\mathscr{F}$ is defined

$$\operatorname{Supp} \mathscr{F} := \{P \in X: \mathscr{F}_P \neq 0\}$$

Lemma 14.23 (3.3)**.** *If $\mathscr{M}$ is a finitely generated $\mathscr{O}_X$-module on a ringed space $(X, \mathscr{O}_X)$, then* Supp$(\mathscr{M})$ *is closed.*

Proof. It is required to prove that if $\mathscr{M}_P = 0$, then there exists an open neighbourhood on which $\mathscr{M}_Q = 0$ for all $Q \in$ the neighbourhood. There exists an open $U \ni P$ such that $\mathscr{O}_U^q \xrightarrow{\alpha} \mathscr{M}|_U \longrightarrow 0$ is exact i.e. α is surjective on stalks. Therefore there are elements $s_1, \dots, s_q \in \mathscr{M}|_U$ where the germs generates

[9] Sometimes called "of finite type".

$\mathscr{M}_Q$ for all $Q \in U$. Assuming $\mathscr{M}_P = 0$, the germ of each s_i at P is zero implies that there exists a neighbourhood $P \in W \subset U$ such that $s_i|_W = 0$ for all i which implies that $\mathscr{M}_Q = 0$ for all $Q \in W$. □

Lecture 15

13th November 12:00

Examples class 2 will be on Tuesday the 26th November in here (MR4). I will give out the solutions the day before.

15.1. Coherent sheaves on affine varieties. Suppose that $(X, \mathscr{O}_X)$ is an affine variety: from the definition of $\mathscr{O}_X$ we have

(1) $\mathscr{O}_X(X) = k[X]$ (see beginning of §8.1),
(2) for any $f \in k[X]$, $\mathscr{O}_X(D(f)) \cong k[X]_f$ (also see §8.1),
(3) for any $P \in X$ with maximal ideal $\mathfrak{m}_P \lhd k[V]$ then $\mathscr{O}_{X,P} \cong k[X]_{\mathfrak{m}_P}$,

It is an easy check to show that (2) implies (3).

Definition 15.41. If M is a $k[X]$-module and $\mathcal{B}$ is a basis of affine open sets of the form $D(f)$ for $f \in k[X]$, we define a $\mathcal{B}$-sheaf $\tilde{M}$ by

$$\tilde{M}(D(f)) := M \otimes_{k[X]} \mathscr{O}_X(D(f)) \cong M \otimes_{k[X]} k[X]_f \cong M_f$$

Clearly $\tilde{M}$ is a $\mathcal{B}$-presheaf and arguing as in lemma 9.15 (but not reducing to the case of $n = 1$) we deduce that $\tilde{M}$ is a $\mathcal{B}$-sheaf (see examples sheet 2, question 1).

$\tilde{M}$ extends to a sheaf on X by corollary 7.12, and this sheaf also denoted $M \otimes_{k[X]} \mathscr{O}_X$ i.e. the sheafification of the obvious presheaf of $\mathscr{O}_X$-modules. $\tilde{M}$ satisfies

(1) $\tilde{M}(X) = M$,
(2) for any $f \in k[X]$, one has $\tilde{M}(D(f)) \cong M_f$,
(3) for any $P \in X$, one has $\tilde{M}_P \cong M_{\mathfrak{m}_P}$.

If we express M as a cokernel $k[X]^{\oplus I} \longrightarrow k[X]^{\oplus J} \longrightarrow M \longrightarrow 0$ then we have an exact sequence

$$k[X]_f^{\oplus I} \longrightarrow k[X]_f^{\oplus J} \longrightarrow M_f \longrightarrow 0$$

for any $f \in k[X]$ (*exercise:* if $P \to Q \to R$ is an exact sequence of A-modules and S is a multiplicatively closed subset of A, then $S^{-1}P \to S^{-1}Q \to S^{-1}R$ is an exact sequence of $S^{-1}A$-modules - see the commutative algebra class or else [**AM69**]). Therefore we have a corresponding morphism of sheaves $\mathscr{O}_X^{\oplus I} \to \mathscr{O}_X^{\oplus J}$ and $\tilde{M} \cong \operatorname{coker}(\mathscr{O}_X^{\oplus I} \to \mathscr{O}_X^{\oplus J})$ as they are both $\mathcal{B}$-sheaves and hence by corollary 7.12 as sheaves. Therefore

$$\mathscr{O}_X^{\oplus I} \longrightarrow \mathscr{O}_X^{\oplus J} \longrightarrow \tilde{M} \longrightarrow 0$$

is exact and so $\tilde{M}$ is quasi-coherent.

Recall that an A-module is Noetherian if it has the ascending chain condition on submodules. A ring A is Noetherian iff it is Noetherian as an A-module. The standard argument: M is Noetherian iff every submodule is finitely generated. If $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ is a short exact sequence A-modules, then it is easy to see that M_2 is Noetherian iff M_1, M_3 are both Noetherian. In particular, if A is a Noetherian ring, then $A^{\oplus n}$ is a Noetherian module and so too is any finitely generated A-module. For an affine variety X, Hilbert's Basis theorem implies that $k[X]$ is Noetherian.

If M is a finitely generated $k[X]$-module, then there exists a surjection

$$k[X]^p \longrightarrow M \longrightarrow 0$$

with kernel also being finitely generated and thus there exists an exact sequence

$$k[X]^q \longrightarrow k[X]^p \longrightarrow M \longrightarrow 0$$

therefore we see that $\tilde{M}$ is coherent.

Theorem 15.24 (3.4)**.** *Any quasi-coherent $\mathscr{O}_X$-module $\mathscr{M}$ on an affine variety X is of the form $\tilde{M}$ for the $k[X]$-module $M = \mathscr{M}(X)$.*

Remark. For X an affine variety, a sequence of quasi-coherent $\mathscr{O}_X$-modules

$$0 \longrightarrow \mathscr{M}_1 \longrightarrow \mathscr{M}_2 \longrightarrow \mathscr{M}_3 \longrightarrow 0$$

is exact iff the corresponding sequence on global sections

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

is exact. *Proof:* since $\mathscr{M}_i = \tilde{M}_i$ for $i = 1, 2, 3$, the sequence on sheaves is exact iff the sequence on stalks is exact iff the sequence

$$0 \longrightarrow M_{1,\mathfrak{m}} \longrightarrow M_{2,\mathfrak{m}} \longrightarrow M_{3,\mathfrak{m}} \longrightarrow 0$$

is exact for any $\mathfrak{m}$ a maximal ideal of $k[X]$ (by the nullstellensatz, since $\mathfrak{m} = \mathfrak{m}_P$ for some $P \in X$). This is true iff the sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

is exact (standard result from commutative algebra - examples sheet 2, question 11).

PROOF. If $\mathscr{M}$ is quasi-coherent then there exists a finite cover by basic affine open sets $X = \bigcup D(g_i)$ and for each i we have an exact sequence

$$\mathscr{O}_{D(g_i)}^{\oplus I} \xrightarrow{\theta} \mathscr{O}_{D(g_i)}^{\oplus J} \longrightarrow \mathscr{M}|_{D(g_i)} \longrightarrow 0$$

for some I, J (perhaps depending on i). Set $A = k[X]$. For a given i, let $\phi = \theta(D(g_i)) : A_{g_i}^{\oplus I} \longrightarrow A_{g_i}^{\oplus J}$ a homomorphism of A_{g_i}-modules. Therefore

$$A_{g_i}^{\oplus I} \longrightarrow A_{g_i}^{\oplus J} \longrightarrow \operatorname{Coker}\phi \longrightarrow 0$$

and hence we get an exact sequence of sheaves over $D(g_i)$,

$$\mathscr{O}_{D(g_i)}^{\oplus I} \longrightarrow \mathscr{O}_{D(g_i)}^{\oplus J} \longrightarrow (\operatorname{Coker}\phi)^{\sim} \longrightarrow 0$$

and hence an isomorphism $(\operatorname{Coker}\phi)^{\sim} \cong \mathscr{M}_{D(g_i)}$ where

$$\operatorname{Coker}\phi \cong \Gamma(D(g_i), (\operatorname{Coker}\phi)^{\sim}) \cong \Gamma(D(g_i), \mathscr{M}|_{D(g_i)}) = \mathscr{M}(D(g_i))$$

So for all i, we have $\mathscr{M}|_{D(g_i)} \cong (\mathscr{M}(D(g_i)))^{\sim}$. So we have proved the result we want but only for the cover, so we need to 'globalise' the argument.

For any $f \in A$, we have a restriction map $M = \mathscr{M}(X) \to \mathscr{M}(D(f))$. The universal property of modules of fractions implies that we have induced homomorphisms

$M_f \to \mathscr{M}(D(f))$ i.e. homomorphisms $\tilde{M}(D(f)) \xrightarrow{\theta} \mathscr{M}(D(f))$ for all $f \in K[x]$. Our claim is that these are isomorphisms for all f and hence $\mathscr{M} \cong \tilde{M}$ (if we have an isomorphism of $\mathcal{B}$-sheaves, there exists isomorphisms of sheaves via the +-construction corollary 7.12). We'll finish next lecture. □

Lecture 16

15th November 12:00

16.1. More on coherent $\mathscr{O}_X$-modules. The following is quite a crucial result of the course - we continue the proof from last time.

Theorem 16.25 (3.4)**.** *A quasi-coherent $\mathscr{O}_X$-module $\mathscr{M}$ on an affine variety $(X, \mathscr{O}_X)$ is of the form $\mathscr{M} = \tilde{M}$ for $M = \mathscr{M}(X)$.*

Proof. There exists a finite open cover $X = \bigcup D(g_i)$ such that $\mathscr{M}|_{D(g_i)} \cong (\mathscr{M}(D(g_i))^{\sim}$. The natural map

$$M = \mathscr{M}(X) \to \mathscr{M}(D(f))$$

induces a homomorphism

$$M_f \cong \tilde{M}(D(f)) \xrightarrow{\theta} \mathscr{M}(D(f))$$

for all $f \in k[V]$ by the universal property. Our claim is that this is an isomorphism for all $f \in k[V]$.

First we show injectivity: if $m \in \mathscr{M}(X)$ such that $m|_{D(f)} = 0$, then it is required to prove that there exists l such that $f^l m = 0$ in M. Since[10]

$$m|_{D(fg_i)} \in \mathscr{M}(D(fg_i)) = \mathscr{M}|_{D(g_i)}(D(fg_i)) \cong \mathscr{M}(D(g_i))_f,$$

and since $m|_{D(f)} = 0$ and $D(fg_i) = D(f) \cap D(g_i)$ thus $m|_{D(fg_i)} = 0$ inside $\mathscr{M}(D(g_i))_f$ so there exists l such that $f^l m|_{D(g_i)} = 0$. Choosing $l \gg 0$, we can assume $f^l m|_{D(g_i)} = 0$ for all i, then sheaf condition (A) for $\mathscr{M}$ implies that $f^l m = 0$.

Next we show that θ is surjective, which we are going to have to work a little harder for. Given $x \in \mathscr{M}(D(f))$, consider the restriction

$$x|_{D(fg_i)} \in \mathscr{M}(D(g_i))_{f/1}$$

say $x|_{D(fg_i)} = t_i/f^l$ for $t_i \in \mathscr{M}(D(g_i))$ then pick $l \gg 0$ large enough so that it works for all i (just choose an l for each i then take the maximum). Consider $(t_i - t_j)_{D(g_i g_j)} \in \mathscr{M}(D(g_i g_j))$. Since

$$(t_i - t_j)|_{D(fg_ig_j)} = f^l(x|_{D(fg_ig_j)} - x|_{D(fg_ig_j)}) = 0,$$

the injectivity argument previous implies that there exists $m \gg 0$ (independent of i, j) such that $f^m(t_i - t_j) = 0$ in $\mathscr{M}(D(g_ig_j))$. Therefore we have sections $f^m t_i \in \mathscr{M}(D(g_i))$ which agree on overlaps, then this implies (using sheaf condition (B)) that there exists $s \in \mathscr{M}(X) = M$ such that $s|_{D(g_i)} = f^m t_i$. Taking the image σ in $\mathscr{M}(D(f))$ of $s/f^{m+l} \in M_f$, we have

$$\sigma|_{D(fg_i)} = (f^m t_i)|_{D(fg_i)}/f^{m+l} = t_i|_{D(fg_i)}/f^l = x|_{D(fg_i)}$$

for all i. Then using the sheaf condition (A) we have that $\sigma = x$ in $\mathscr{M}(D(f))$. □

Corollary 16.26 (3.5)**.** *The coherent $\mathscr{O}_X$-modules on the affine variety X are of the form $\tilde{M}$ for M a finitely generated $k[X]$-module.*

[10] Also see Hartshorne [**Har77**, II, lem. 5.3].

PROOF. We have done most of this, there is only one extra thing to prove: if $\tilde{M}$ is coherent, then the global sections $M = \mathscr{M}(X)$ is finitely-generated (the converse has already been proved). We know that there exists an open cover $X = \bigcup_{j=1}^{r} D(f_i)$ such that $\tilde{M}|_{D(f_j)}$ is generated by elements say $\alpha_{j1}, \dots, \alpha_{js(j)} \in \tilde{M}(D(f_j)) \cong M_{f_j}$. This tells me that $\alpha_{j1}, \dots, \alpha_{js(j)}$ generate M_{f_j} as an A_{f_j}-module. Write $\alpha_{ji} = a_{ji}/f_j^l$ with $a_{ji} \in M$ for all i, j, and $l \gg 0$ is chosen sufficiently large so that it is independent of i, j. Consider the finitely generated submodule $N = \sum_{i,j} Aa_{ji} \subset M$ (a submodule of M). Now we observe that if we restrict $\tilde{N}|_{D(f_j)} = \tilde{M}|_{D(f_j)}$ for all j. So what does this now say? The induced inclusion of $\mathscr{O}_X$-modules $\tilde{N} \hookrightarrow \tilde{M}$ is in fact an isomorphism. Therefore $N = M$ is finitely generated. □

Just briefly, we will say some things about tensor products

(1) if M is an A-module, then $\otimes_A M$ is right exact on the category of A-modules, i.e. given an exact sequence of A-modules

$$N \longrightarrow P \longrightarrow Q \longrightarrow 0$$

then the sequence

$$N \otimes_A M \longrightarrow P \otimes_A M \longrightarrow Q \otimes_A M \longrightarrow 0$$

is exact (see [**AM69**, prop. 2.18]).

(2) If $(X, \mathscr{O}_X)$ is a ringed space and $\mathscr{F}, \mathscr{G}$ are $\mathscr{O}_X$-modules, then

$$(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_P = \mathscr{F}_P \otimes_{\mathscr{O}_{X,P}} \mathscr{G}_P$$

for all $P \in X$. There is a more general categorical statement here: tensoring commutes with direct limits. Let $\mathscr{T}(U) = \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U)$ define the presheaf tensor product. For any $U \ni P$, the $\mathscr{O}_{X,P}$-product structure on $\mathscr{F}_P \otimes_{\mathscr{O}_{X,P}} \mathscr{G}_P$ induces an $\mathscr{O}_X(U)$-module structure on $\mathscr{F}_P \otimes_{\mathscr{O}_{X,P}} \mathscr{G}_P$. For $U \ni P$ open, define

$$\alpha_U : \mathscr{F}(U) \times \mathscr{G}(U) \to \mathscr{F}_P \otimes_{\mathscr{O}_{X,P}} \mathscr{G}_P$$
$$(s, t) \mapsto s_P \otimes t_P$$

This induces an $\mathscr{O}_X(U)$-module morphism

$$\mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U) \to \mathscr{F}_P \otimes_{\mathscr{O}_{X,P}} \mathscr{G}_P$$

and hence a morphism

$$(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_P = \mathscr{T}_P \xrightarrow{\theta} \mathscr{F}_P \otimes_{\mathscr{O}_{X,P}} \mathscr{G}_P$$

Define $\psi : \mathscr{F}_P \times \mathscr{G}_P \to \mathscr{T}_P$ by

$$(s_P, t_P) \mapsto (s|_{U \cap V} \otimes t|_{U \cap V})_P$$

(where $s_P = (U, s)$ and $t_P = (V, t)$) and this is plainly bilinear over $\mathscr{O}_{X,P}$ and hence defines a morphism

$$\mathscr{F}_P \otimes_{\mathscr{O}_{X,P}} \mathscr{G}_P \to \mathscr{T}_P$$

which is inverse to θ.

(3) For $\mathscr{M}$ an $\mathscr{O}_X$-module, (1) and (2) together imply that $\otimes_{\mathscr{O}_X} \mathscr{M}$ is right exact.

Lecture 17

18th November 12:00

Proposition 17.27 (3.6)**.** *If $\phi: X \to Y$ is a morphism of affine varieties, inducing a homomorphism of k-algebras $A = k[Y] \to B = k[X]$, then for any A-module M, we have $\phi^* \tilde{M} \cong (M \otimes_A B)^\sim$.*

Proof. There exists an exact sequence

$$A^{\oplus I} \longrightarrow A^{\oplus J} \longrightarrow M \longrightarrow 0 \tag{17.5}$$

inducing an exact sequence

$$\mathscr{O}_Y^{\oplus I} \longrightarrow \mathscr{O}_Y^{\oplus J} \longrightarrow \tilde{M} \longrightarrow 0$$

which implies that

$$\phi^{-1}\mathscr{O}_Y^{\oplus I} \longrightarrow \phi^{-1}\mathscr{O}_Y^{\oplus J} \longrightarrow \phi^{-1}\tilde{M} \longrightarrow 0$$

is also exact (because ϕ^{-1} is an exact functor, since the stalk sequence at P of $\phi^{-1}\mathscr{F} \to \phi^{-1}\mathscr{G} \to \phi^{-1}\mathscr{H}$ is by construction just the stalk sequence of $\mathscr{F} \to \mathscr{G} \to H$ at $\phi(P)$). Tensoring with $\otimes_{\phi^{-1}\mathscr{O}_X} \mathscr{O}_Y$, we obtain

$$\mathscr{O}_X^{\oplus I} \longrightarrow \mathscr{O}_X^{\oplus J} \longrightarrow \phi^*\tilde{M} \longrightarrow 0 \tag{17.6}$$

by fact 3. However, this sequence is also obtained by tensoring $\otimes_A B$ the sequence (17.5)

$$B^{\oplus I} \longrightarrow B^{\oplus J} \longrightarrow B \otimes_A M \longrightarrow 0$$

which is exact by fact 1, yields

$$\mathscr{O}_X^{\oplus I} \longrightarrow \mathscr{O}_X^{\oplus J} \longrightarrow (B \otimes_A M)^\sim \longrightarrow 0 \tag{17.7}$$

Since the first morphisms of (17.6) and (17.7) are the same, we deduce that $\phi^*\tilde{M} \cong (B \otimes_A M)^\sim$. □

Corollary 17.28 (3.7)**.** *For $\phi: (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ a morphism of varieties, $\mathscr{M}$ a (quasi-)coherent $\mathscr{O}_Y$-module, then $\phi^*\mathscr{M}$ is a (quasi-)coherent $\mathscr{O}_X$-module.*

Proof. Reduce to the affine case. □

17.1. Closed subvarieties. Let $(X, \mathscr{O}_X)$ be a variety over k and $Y \subset X$ a closed subset. Let $\mathscr{I}$ be the sheaf of ideals given by

$$\mathscr{I}(U) = \{f \in \mathscr{O}_X(U) : f|_{Y \cap U} \equiv 0\}$$

One has $\mathscr{I}_Y \subset \mathscr{O}_X$. On the affine piece $U \subset X$, set $\mathscr{I}(U) = I = I(U \cap Y) \lhd k[U]$ and then $\mathscr{I}|_U = \tilde{I}$ (one would just check this on the basic open sets $D(f)$). Hilbert's basis theorem implies that I is finitely generated so $\mathscr{I}$ is coherent. Moreover there exists a short exact sequence

$$0 \longrightarrow \mathscr{I} \hookrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_X/\mathscr{I} \longrightarrow 0$$

and hence $\mathscr{O}_X/\mathscr{I}$ is also coherent. The stalks of $\mathscr{O}_X/\mathscr{I}$ at $P \notin Y$ are zero. Let $\iota: Y \hookrightarrow X$, then for any open set $U \subset X$, the definition of ι^{-1} yields natural isomorphisms

$$(\mathscr{O}_X/\mathscr{I})(U) \xrightarrow{\sim} (\mathscr{O}_X/\mathscr{I})^+(U) \xrightarrow[\text{restriction}]{\sim} \iota^{-1}(\mathscr{O}_X/\mathscr{I})(U \cap Y)$$

Define $\mathscr{O}_Y = \iota^{-1}(\mathscr{O}_X/\mathscr{I})$, a sheaf of rings on Y. If $U \subseteq X$ is open and affine, then by the remark following theorem 15.24, the sequence

$$0 \longrightarrow I \longrightarrow k[U] \longrightarrow (\mathscr{O}_X/\mathscr{I})(U) \longrightarrow 0$$

is exact and so $\mathscr{O}_Y(Y \cap U) \cong (\mathscr{O}_X/\mathscr{I})(U) \cong k[U]/I = k[Y \cap U]$. So $\mathscr{O}_Y$ restricts to the correct structure on affine pieces and therefore $(Y, \mathscr{O}_Y)$ is a variety (it is clearly separated since Y is closed in X). Such varieties are called closed subvarieties of X.

Remark. Any sheaf of ideals $\mathscr{T} \subset \mathscr{O}_X$ determines a closed subvariety; it determines a closed subset $\operatorname{Supp}(\mathscr{O}_X/\mathscr{T}) = Y$ by lemma 14.23 and hence a closed subvariety.

Remark. If $U \subset X$ is open, then $(U, \mathscr{O}_U)$ is also a variety, and an open subvariety of X.

Definition 17.42. Given $Y \overset{\iota}{\hookrightarrow} X$ closed, $\mathscr{F}$ any sheaf on Y, we call $\iota_*\mathscr{F}$ the extension of $\mathscr{F}$. Clearly

$$(\iota_*\mathscr{F})_P = \begin{cases} \mathscr{F}_P & \text{if } P \in Y \\ 0 & \text{otherwise} \end{cases}$$

therefore $\operatorname{Supp}(\iota_*\mathscr{F}) \subseteq Y$. Often we don't distinguish between a sheaf on Y and its extension by zero on X since

$$(\iota_*\mathscr{F})(U) = \mathscr{F}(U \cap Y)$$

for all U open in X.

Example. If Y is a closed subvariety of X, then $\iota_*\mathscr{O}_Y \cong \mathscr{O}_X/\mathscr{I}$ ($\mathscr{I}$ = sheaf of reduced ideals corresponding to Y) (clearly true on any open affine piece). Hence there exists a short exact suquence of $\mathscr{O}_X$-modules

$$0 \longrightarrow \mathscr{I}_Y \hookrightarrow \mathscr{O}_X \longrightarrow \iota_*\mathscr{O}_Y \longrightarrow 0$$

(usually written as

$$0 \longrightarrow \mathscr{I}_Y \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_Y \longrightarrow 0 \quad)$$

Any $\mathscr{O}_Y$-module $\mathscr{F}$ yields an $\mathscr{O}_X$-module $\iota_*\mathscr{F}$ via $\mathscr{O}_X \to \iota_*\mathscr{O}_Y$.

17.2. The invertible sheaves $\mathscr{O}_{\mathbb{P}^n}(m)$. Suppose X is an irreducible variety;
dfn:locally-principal an (irreducible) subvariety $Y \subset X$ is called locally principal if there exists an open affine cover $X = \bigcup_i U_i$ such that

$$I(Y \cap U_i) = (f_i) \lhd k[U_i]$$

Remark. $\operatorname{codim}(Y) = 1$ by the standard dimension theory - e.g. Krull's principal ideal theorem.

Then theorem 15.24 implies that $\mathscr{I}_Y|_{U_i} = f_i\mathscr{O}_{U_i}$ and so $\mathscr{I}_Y$ is invertible. The transition functions are obtained by $f_ig_i = f_jg_j$ on $U_i \cap U_j$ i.e. $g_j = \psi_{ji}g_i$ with $\psi_{ji} = f_i/f_j$. We also have a sheaf of rational functions with at worst a simple pole along Y (and regular elsewhere) denoted $\mathscr{O}_X(Y)$ where

$$\Gamma(U, \mathscr{O}_X(Y)) = \{h \in k(X) : f_ih \in \mathscr{O}_X(U \cap U_i) \text{ for all } i\}$$

$\mathscr{O}_X \subset k(X)$ the constant sheaf and $\mathscr{O}_X(Y)|_{U_i} = \frac{1}{f_i}\mathscr{O}_{U_i}$. So $\mathscr{O}_X(Y)$ is invertible with transition functions $\psi_{ji} = f_j/f_i$ i.e. $\mathscr{I}_Y$ and $\mathscr{O}_X(Y)$ are dual. Notation: we usually denote $\mathscr{I}_Y$ in this case as $\mathscr{O}_X(-Y)$.

Suppose $X = \mathbb{P}^n$ with the standard open affine cover $\{U_i\}$, U_i given by $X_i \neq 0$ and H a hyperplane given by $L = 0$ for some homogeneous linear form L. Therefore

H is given locally on U_i by $L/X_i \in \mathcal{O}_{\mathbb{P}^n}(U_i)$, thus $\mathcal{O}_{\mathbb{P}^n}(-H)$ defined by transition functions $\psi = x_j/x_i$ and $\mathcal{O}_{\mathbb{P}^n}(H)$ defined by transition functions $\psi_{ji} = x_i/x_j$. So we see that the isomorphism classes of $\mathcal{O}_{\mathbb{P}^n}(H)$ (respectively $\mathcal{O}_{\mathbb{P}^n}(-H)$) doesn't depend on H. Denote these invertible sheaves $\mathcal{O}_{\mathbb{P}^n}(1)$ (respectively $\mathcal{O}_{\mathbb{P}^n}(-1)$). For $m \in \mathbb{Z}$, denote

$$\mathcal{O}_{\mathbb{P}^n}(m) = \mathcal{O}_{\mathbb{P}^n}(mH) \cong \begin{cases} \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes m} & m \geq 0 \\ \mathcal{O}_{\mathbb{P}^n}(-1)^{\otimes |m|} & m < 0 \end{cases}$$

Remark. Suppose we have $Y \subset \mathbb{P}^n$ a hyperplane instead, defined by irreducible homogeneous polynomials $F(X_0, \ldots, X_n)$ of degree m, then $\mathcal{O}_{\mathbb{P}^n}(Y) \cong \mathcal{O}_{\mathbb{P}^n}(m)$ and $\mathcal{O}_{\mathbb{P}^n}(-Y) \cong \mathcal{O}_{\mathbb{P}^n}(-m)$.

Lemma 17.29 (3.7)**.** *$\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = 0$ if $m < 0$ and is isomorphic to the vector space of homogeneous polynomials of degree m if $m \geq 0$.*

PROOF. Exercise on examples sheet 3. □

Remark. The result implies $\mathcal{O}_{\mathbb{P}^n}(r) \cong \mathcal{O}_{\mathbb{P}^n}(s)$ iff $\mathcal{O}_{\mathbb{P}^n}(r-s) \cong \mathcal{O}_{\mathbb{P}^n}$ iff $r = s$.

It is a fact that any invertible sheaf on $\mathbb{P}^n$ is of the form $\mathcal{O}_{\mathbb{P}^n}(m)$ for some $m \in \mathbb{Z}$ i.e. $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}$.

Lecture 18

20th November 12:00

Addendum to last time:

Definition 18.43. Given a projective variety $V \overset{\iota}{\hookrightarrow} \mathbb{P}^n$, we have invertible sheaves $\mathcal{O}_V(m)$ on V defined by $\mathcal{O}_V(m) = \iota^* \mathcal{O}_{\mathbb{P}^n}(m)$, restrictions of $\mathcal{O}_{\mathbb{P}^n}(m)$. Aliter: $\mathcal{O}_V(U) = \mathcal{O}_V(V \cap H)$, etc. - the transition functions are just restrictions of these for $\mathcal{O}_{\mathbb{P}^n}(m)$.

18.1. §4: Sheaf cohomology.

Definition 18.44. A sheaf $\mathcal{F}$ of Abelian groups on a topological space X is called flabby (flasque) if for all U open in X, the retriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective.

An important example: if $\mathcal{F}$ is an arbitrary sheaf/presheaf, we define a sheaf $D(\mathcal{F})$ by

$$D(\mathcal{F})(U) := \{s: U \to \coprod_{P \in U} \mathcal{F}_P \text{ such that } s(P) \in \mathcal{F}_P\}$$

- cf. the definition of $\mathcal{F}^+$. $D(\mathcal{F})$ is clearly a flabby sheaf.

We want to construct resolutions by flasque sheaves - the next result helps us to do this. The idea is that to understand the cohomology of some sheaf, we try to understand the cohomology of the resolution of that sheaf by flasque sheaves.

Lemma 18.30 (4.1)**.** *Suppose that $0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$ is a short exact sequence of sheaves of Abelian groups on X, then*

(1) *if $\mathcal{F}_1$ is flabby, then the sequence*

$$0 \longrightarrow \mathcal{F}_1(U) \longrightarrow \mathcal{F}_2(U) \longrightarrow \mathcal{F}_3(U) \longrightarrow 0,$$

is exact for all $U \subset X$ open,

(2) *if $\mathcal{F}_1, \mathcal{F}_2$ are flabby, then so too is $\mathcal{F}_3$.*

PROOF. (1). It is required to prove that $\sigma \in \mathscr{F}_3(U)$ lifts to a section of $\mathscr{F}_2(U)$ (then the rest follows from proposition 8.14). Let τ_V be a lifting of $\sigma|_V$, where $V \subset U$ is open. If for any proper subset $V \subsetneq U$ and lifting τ_V of $\sigma|_V$, we can always find an extension of τ_V to a strictly larger open subset $V' \subset U$ such that $\tau_{V'}$ is a lifting of $\sigma|_{V'}$, then we keep extending until we get a lifting to all of U (for general case, there is an implicit use of Zorn's lemma here - for X compact, X a variety, then Zorn's lemma is not needed).

So it is required to prove such an extension of τ_V always exists. Given a point in $U\backslash V$, there exists an open neighbourhood W such that $\sigma|_W$ lifts to $\tau_W \in \mathscr{F}_2(W)$ (map on stalks is surjective). If $\tau_W|_{V\cap W} = \tau_V|_{V\cap W}$, we can patch to get a section $\tau_{V\cup W}$ which lifts $\sigma|_{V\cup W}$. If not, we modify our choice of τ_W: let $\rho = \tau_V|_{V\cap W} - \tau_W|_{V\cap W} \in \mathscr{F}_2(V\cap W)$, then the image of ρ in $\mathscr{F}_3(V\cap W)$ is zero and hence ρ comes from a section of $\mathscr{F}_1(V\cap W)$ by proposition 8.14 (b). Since $\mathscr{F}_1$ is flabby, we may extend this to a section of $\mathscr{F}_1(X)$, and hence we have $\rho' \in \mathscr{F}_2(X)$ which extends $\rho \in \mathscr{F}_2(V\cap W)$, with ρ' coming from a section of $\mathscr{F}_1(X)$. Set $\tau'_W = \tau_W + \rho'|_W \in \mathscr{F}_2(W)$; the exactness of

$$0 \longrightarrow \mathscr{F}_1(W) \longrightarrow \mathscr{F}_2(W) \longrightarrow \mathscr{F}_3(W),$$

implies that the image of τ'_W is still $\rho|_W$ and now by construction

$$\tau'_W|_{V\cap W} = \tau_W|_{V\cap W} + \rho'|_{V\cap W} = \tau_W|_{V\cap W} + \rho = \tau_V|_{V\cap W}.$$

Hence we can patch $\tau'|_W$ and τ_V to give a lift of $\sigma|_{V\cup W}$, proving part (1).

(2) It is required to prove that $r: \mathscr{F}_3(X) \to \mathscr{F}_3(U)$ is surjective for all U open in X. We have a commutative diagram

$$\begin{array}{ccc} \mathscr{F}_2(X) & \xrightarrow{\phi(X)} & \mathscr{F}_3(X) \\ \downarrow{\scriptstyle r'} & & \downarrow{\scriptstyle r} \\ \mathscr{F}_2(U) & \xrightarrow{\phi(U)} & \mathscr{F}_3(U) \end{array}$$

where part (1) implies that $\phi(U)$ is surjective since $\mathscr{F}_1$ is flabby. But r' is surjective since $\mathscr{F}_2$ is flabby. Hence r is surjective. □

The construction is as follows. Given a sheaf of Abelian groups X, we construct a short exact sequence of sheaves

$$0 \longrightarrow \mathscr{F} \hookrightarrow D(\mathscr{F}) \longrightarrow D(\mathscr{F})/\mathscr{F} \longrightarrow 0$$

and the idea is to iterate the sequence (the $D(\mathscr{F})/\mathscr{F}$ isn't flabby) - iterate as follows:

$$C^0(\mathscr{F}) = \mathscr{F}$$
$$D^i(\mathscr{F}) = D(C^i\mathscr{F})$$
$$C^{i+1}(\mathscr{F}) = D(C^i\mathscr{F})/C^i\mathscr{F}$$

and then the sequence

$$0 \longrightarrow C^i(\mathscr{F}) \longrightarrow D(C^i\mathscr{F}) \longrightarrow C^{i+1}(\mathscr{F}) \longrightarrow 0$$

is exact for all i. Putting these sequences together, we get an (exact) resolution

$$0 \longrightarrow \mathscr{F} \longrightarrow D^0(\mathscr{F}) \longrightarrow D^1(\mathscr{F}) \longrightarrow \dots$$

of $\mathscr{F}$ by a complex $D^*(\mathscr{F})$ where the sheaves $D^i\mathscr{F}$ are all flabby.

Definition 18.45. The ith cohomology group $H^i(X, \mathscr{F})$ is the ith cohomology of the complex $\Gamma(X, D^*\mathscr{F})$ i.e.

$$H^i(X, \mathscr{F}) = \frac{\ker(\Gamma(D^i\mathscr{F}) \to \Gamma(D^{i+1}\mathscr{F}))}{\operatorname{im}(\Gamma(D^{i-1}\mathscr{F}) \to \Gamma(D^i\mathscr{F}))}$$

Since we have

$$0 \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X, D^0\mathscr{F}) \longrightarrow \Gamma(X, C^1\mathscr{F}) \hookrightarrow \Gamma(X, D^1\mathscr{F})$$

(is exact at $\Gamma(X, D^0\mathscr{F})$ by proposition 8.14 (b)) and we have

$$H^0(X, \mathscr{F}) = \ker(\Gamma(D^0\mathscr{F}) \to \Gamma(D^1\mathscr{F}) \cong \Gamma(X, \mathscr{F})$$

So the global sections of a sheaf $\mathscr{F}$ are denoted $\mathscr{F}(X)$, $\Gamma(X, \mathscr{F})$ and $H^0(X, \mathscr{F})$

Suppose that

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

is a short exact sequence of sheaves. We then have an exact sequence of morphisms of sheaves

$$0 \longrightarrow D^*\mathscr{F}_1 \longrightarrow D^*\mathscr{F}_2 \longrightarrow D^*\mathscr{F}_3 \longrightarrow 0$$

i.e. a diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{F}_1 & \longrightarrow & \mathscr{F}_2 & \longrightarrow & \mathscr{F}_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D^0\mathscr{F}_1 & \longrightarrow & D^0\mathscr{F}_2 & \longrightarrow & D^0\mathscr{F}_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D^1\mathscr{F}_1 & \longrightarrow & D^1\mathscr{F}_2 & \longrightarrow & D^1\mathscr{F}_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

(the construction of the diagram follows easily from the universal property of cokernels: the first step is to observe that

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{F}_1 & \longrightarrow & \mathscr{F}_2 & \longrightarrow & \mathscr{F}_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D^0\mathscr{F}_1 & \longrightarrow & D^0\mathscr{F}_2 & \longrightarrow & D^0\mathscr{F}_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & C^1\mathscr{F}_1 & \longrightarrow & C^1\mathscr{F}_2 & \longrightarrow & C^1\mathscr{F}_3 & &
\end{array}$$

and then deduce that $0 \to C^1\mathscr{F}_1 \to C^1\mathscr{F}_2 \to C^1\mathscr{F}_3 \to 0$ is exact by checking on stalks - i.e. use the corresponding result in the category of modules. Then iterate.)

Taking global sections and using lemma 18.30 (a), one obtains

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(D^0\mathscr{F}_1) & \longrightarrow & \Gamma(D^0\mathscr{F}_2) & \longrightarrow & \Gamma(D^0\mathscr{F}_3) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma(D^1\mathscr{F}_1) & \longrightarrow & \Gamma(D^1\mathscr{F}_2) & \longrightarrow & \Gamma(D^1\mathscr{F}_3) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma(D^2\mathscr{F}_1) & \longrightarrow & \Gamma(D^2\mathscr{F}_2) & \longrightarrow & \Gamma(D^2\mathscr{F}_3) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

which by the standard homological algebra (examples sheet II, question 14) yields a long exact sequence of cohomology groups:

$$\begin{array}{l}
0 \longrightarrow H^0(\mathscr{F}_1) \longrightarrow H^0(\mathscr{F}_2) \longrightarrow H^0(\mathscr{F}_3) \xrightarrow{\delta^1} \\
 H^1(\mathscr{F}_1) \longrightarrow H^1(\mathscr{F}_2) \longrightarrow H^1(\mathscr{F}_3) \xrightarrow{\delta^2} \\
 H^2(\mathscr{F}_1) \longrightarrow \dots
\end{array}$$

where the $\delta = \delta^i : H^{i-1}(X, \mathscr{F}_3) \to H^i(X, \mathscr{F}_1)$ are called the connecting maps. Remark: our construction of cohomology is functorial.

Lecture 19

22nd November 12:00

19.1. Local vanishing principle.

Lemma 19.31 (4.2)**.** *If $\mathscr{F}$ is a flabby sheaf, then $H^i(X, \mathscr{F}) = 0$, for all $i > 0$.*

PROOF. Using lemma 18.30, we have that $D(\mathscr{F})/\mathscr{F}$ is flabby and the exact sequence

$$0 \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X, D(\mathscr{F})) \longrightarrow \Gamma(X, D\mathscr{F}/\mathscr{F}) \longrightarrow 0$$

Repeating this argument shows that the whole complex

$$0 \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X, D^0\mathscr{F}) \longrightarrow \Gamma(X, D^1\mathscr{F}) \longrightarrow \dots$$

is exact and hence $H^i(X, \mathscr{F}) = 0$ for all $i > 0$. □

Lemma 19.32 (4.3, Resolution principle)**.** *Let*

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}^0 \longrightarrow \mathscr{F}^1 \longrightarrow \dots$$

be a resolution of $\mathscr{F}$ by sheaves $\mathscr{F}^i$ such that $H^j(X, \mathscr{F}^i) = 0$ for all i and for all $j > 0$. Then $H^i(X, \mathscr{F})$ is naturally isomorphic to the ith cohomology of the complex

$$\Gamma(X, \mathscr{F}^0) \longrightarrow \Gamma(X, \mathscr{F}^1) \longrightarrow \dots$$

PROOF. We have a short exact sequence $0 \to \mathscr{F} \to \mathscr{F}^0 \to \mathscr{G} \to 0$ and an (exact) resolution of $\mathscr{G}$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{G} & \longrightarrow & \mathscr{F}^1 & \longrightarrow & \mathscr{F}^2 & \longrightarrow & \dots \\
 & & & & \searrow & & \nearrow & & \\
 & & & & & \mathscr{F}^1/\mathscr{G} & & &
\end{array}
$$

and proposition 8.14 (b) proves the claim for $i = 0$ as before. For $i = 1$, since $H^i(\mathscr{F}^0) = 0$ by hypothesis, the above construction implies that we have an exact sequence

$$\Gamma(X, \mathscr{F}^0) \longrightarrow \Gamma(X, \mathscr{G}) \longrightarrow H^1(X, \mathscr{F}) \longrightarrow 0. \tag{19.8}$$

But proposition 8.14 (b) implies that

$$0 \longrightarrow \Gamma(X, \mathscr{G}) \longrightarrow \Gamma(X, \mathscr{F}^1) \longrightarrow \Gamma(X, \mathscr{F}^2), \tag{19.9}$$

is exact and therefore we have induced isomorphisms

$$
\begin{aligned}
H^1(X, \mathscr{F}) &\cong \Gamma(X, \mathscr{G})/\operatorname{Im}\Gamma(X, \mathscr{F}^0) && \text{from (19.8)} \\
&\cong \frac{\ker(\Gamma(X, \mathscr{F}^1) \to \Gamma(X, \mathscr{F}^2))}{\operatorname{Im}(\Gamma(X, \mathscr{F}^0) \to \Gamma(X, \mathscr{F}^1))} && \text{from (19.9)} \\
&= \text{first homology of complex } \Gamma(X, \mathscr{F}^*)
\end{aligned}
$$

If $i > 1$, then $\delta: H^{i-1}(\mathscr{G}) \to H^i(\mathscr{F})$ is an isomorphism by hypothesis on $H^j(X, \mathscr{F}^i)$, and by induction on i, $H^{i-1}(\mathscr{G})$ is the $(i-1)$-homology of the complex

$$\Gamma(X, \mathscr{F}^1) \longrightarrow \Gamma(X, \mathscr{F}^2) \longrightarrow \dots$$

i.e. the ith homology of $\Gamma(X, \mathscr{F}^*)$ □

Now we prove a locally vanishing principle. If U is an open subset of a topological space X and $i: U \hookrightarrow X$, we have a sheaf

$${}_U\mathscr{F} := i_*(\mathscr{F}|_U)$$

i.e. ${}_U\mathscr{F}(V) = \mathscr{F}(U \cap V)$ for V open in X. There exists an obvious morphism of sheaves $\mathscr{F} \to {}_U\mathscr{F}$ given by $\mathscr{F}(V) \to {}_U\mathscr{F}(V) = \mathscr{F}(U \cap V)$ for all open V in X.

Remark. ${}_U(-)$ is not right exact, e.g. take $U = \mathbb{C}\backslash\{0\} \subset \mathbb{C}$ and the exponential short exact sequence.

Proposition 19.33 (4.4)**.** *Let $\mathcal{B}$ be a basis of open sets in X, closed under finite intersections, $\mathscr{F}$ a sheaf of Abelian groups on X, and suppose $H^j(V, \mathscr{F}|_V) = 0$ for $0 < j < i$ and for all $V \in \mathcal{B}$. Then for any $\sigma \in H^i(X, \mathscr{F})$, we can find an open cover $X = \bigcup_\alpha W_\alpha$ with $W_\alpha \in \mathcal{B}$ such that the image of σ in $H^i(X, {}_{W_\alpha}\mathscr{F})$ is zero for each α.*

Remark. The proof in Kempf's book [**Kem93**] is quite hard to understand because he doesn't draw the commutative diagram out (see below).

PROOF. Suppose that $i = 1$ - in which case there are no conditions on $H^j(V, \mathscr{F}|_V)$. The universal property of cokernels implies that for any open W in X, there exists a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{F} & \longrightarrow & D\mathscr{F} & \longrightarrow & D\mathscr{F}/\mathscr{F} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & {}_W\mathscr{F} & \longrightarrow & {}_WD\mathscr{F} & \longrightarrow & ({}_WD\mathscr{F})/({}_W\mathscr{F}) & \longrightarrow & 0 \\
 & & & & & \searrow & \downarrow & & \\
 & & & & & & {}_W(D\mathscr{F}/\mathscr{F}) & &
\end{array}$$

(to see $({}_WD\mathscr{F})/({}_W\mathscr{F}) \hookrightarrow {}_W(D\mathscr{F}/\mathscr{F})$ is injective - take U open in ${}_WD\mathscr{F}$ - if we have something in the image zero, then its preimage via ${}_WD\mathscr{F}$ must locally be zero by our construction here[11]). The cohomology class $\sigma \in H^1(X,\mathscr{F})$ is represented by a section τ of $D(\mathscr{F})/\mathscr{F}$ over X since by lemma 19.31

$$H^0(X, D\mathscr{F}/\mathscr{F}) \twoheadrightarrow H^1(X,\mathscr{F}),$$

because $D\mathscr{F}$ is flabby. Choose a cover $X = \bigcup W_\alpha$ with $W_\alpha \in \mathcal{B}$ such that $\tau|_{W_\alpha}$ lifts to a section of $D(\mathscr{F})(W_\alpha)$ for all α. Set $W = W_\alpha$ in the above diagram and take sections over X. So the image of τ in $\Gamma(X, {}_W(D\mathscr{F}/\mathscr{F})) = \Gamma(W, D\mathscr{F}/\mathscr{F})$ lifts to an element of $\Gamma(X, {}_WD\mathscr{F}) = \Gamma(W, D\mathscr{F})$, and so the image of τ in $\Gamma(X, {}_WD\mathscr{F}/{}_W\mathscr{F})$ comes from this same element (using the fact that $\Gamma(X, {}_WD\mathscr{F}/{}_W\mathscr{F}$ injects into $\Gamma(X, {}_W(D\mathscr{F}/\mathscr{F})))$. Hence by the functoriality of our construction of cohomology, the image of σ in $H^1(X, {}_W\mathscr{F})$ must be zero (since it comes from an element of $H^0(X, {}_WD\mathscr{F})$) for $W = W_\alpha$ for all α.

Suppose now $i > 1$ and $W \in \mathcal{B}$. Our claim is

(a) there exists a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{F} & \longrightarrow & D\mathscr{F} & \longrightarrow & D\mathscr{F}/\mathscr{F} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & {}_W\mathscr{F} & \longrightarrow & {}_WD\mathscr{F} & \longrightarrow & {}_W(D\mathscr{F}/\mathscr{F}) & \longrightarrow & 0
\end{array}$$

(b) the sheaf $D(\mathscr{F})/\mathscr{F}$ satisfies the assumption of the proposition with i replaced by $(i-1)$

Proof: (a) Recalling that $\Gamma(V, {}_W\mathscr{G}) = \Gamma(W \cap V, \mathscr{G})$ for any $\mathscr{G}$, we obtain

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{F} & \longrightarrow & D\mathscr{F} & \longrightarrow & D\mathscr{F}/\mathscr{F} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & {}_W\mathscr{F} & \longrightarrow & {}_WD\mathscr{F} & \longrightarrow & {}_W(D\mathscr{F}/\mathscr{F}) & &
\end{array}$$

It is required to prove that ${}_WD(\mathscr{F}) \to {}_W(D\mathscr{F}/\mathscr{F})$ is a surjection. For any $V \in \mathcal{B}$, we have $W \cap V \in \mathcal{B}$ and so $H^1(W \cap V, \mathscr{F}) = H^1(W \cap V, \mathscr{F}_{W\cap V}) = 0$ by assumption. Therefore

$$0 \longrightarrow \Gamma(W\cap V, \mathscr{F}) \longrightarrow \Gamma(W \cap V, D\mathscr{F}) \longrightarrow \Gamma(W\cap V, D\mathscr{F}/\mathscr{F}) \longrightarrow 0$$

is exact, i.e.

$$0 \longrightarrow \Gamma(V, {}_W\mathscr{F}) \longrightarrow \Gamma(V, {}_WD\mathscr{F}) \longrightarrow \Gamma(V, {}_W(D\mathscr{F}/\mathscr{F})) \longrightarrow 0$$

[11] From examples class 2.

is exact. Since this is true for all $V \in \mathcal{B}$, we deduce the stalk sequences are exact and hence claim (a) follows.

Furthermore, $H^j(V, D\mathscr{F}/\mathscr{F}) \cong H^{j+1}(V, \mathscr{F})$ for $j > 0$ from the cohomology of the flasque sheaf (lemma 19.31) and so (b) follows from our previous assumptions on $\mathscr{F}$.

As $D(\mathscr{F})$ and ${}_W D(\mathscr{F})$ are flabby, claim (a) above implies that there exists isomophisms in the commutative diagrams as shown

$$\begin{array}{ccc} H^{i-1}(X, D\mathscr{F}/\mathscr{F}) & \xrightarrow{\sim} & H^i(X, \mathscr{F}) \\ \downarrow & & \downarrow \\ H^{i-1}(X, {}_W(D\mathscr{F}/\mathscr{F})) & \xrightarrow{\sim} & H^i(X, {}_W\mathscr{F}) \end{array}$$

and so the proposition follows from claim (b) and induction on i. □

Lecture 20

25th November 12:00

Theorem 20.34 (4.5, Serre)**.** *Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module on an affine variety X. Then $H^i(X, \mathscr{F}) = 0$ for all $i > 0$.*

Proof. By induction on i. We consider our standard basis of affine open sets $\mathcal{B}$ consisting of sets $D(f)$ for $f \in k[X]$. $\mathcal{B}$ is certainly closed under finite intersections and we now apply the result we proved last time (proposition 19.33): given $\alpha \in H^i(X, \mathscr{F})$, there exists a finite open cover $U_1, \ldots, U_d$ by elements of $\mathcal{B}$ such that the image in $H^i(X, {}_{U_l}\mathscr{F})$ is zero for each l (using the induction hypothesis). Consider the short exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \textstyle\bigoplus_l {}_{U_l}\mathscr{F} \longrightarrow \mathscr{G} \longrightarrow 0$$

where $\mathscr{F}(U) \hookrightarrow \bigoplus_l \mathscr{F}(U \cap U_l)$ for all U (sheaf condition (A)). For each $D(f) \in \mathcal{B}$, ${}_{D(f)}\mathscr{F}$ is also a quasi-coherent $\mathscr{O}_X$-module (namely, if $\mathscr{F}$ corresponds to a $k[X]$-module M, the ${}_{D(f)}\mathscr{F}$ corresponds to the $k[X]$-module M_f) and so using theorem 15.24 it follows that $\mathscr{F}$ is also quasi-coherent (in fact $\mathscr{G} \cong \operatorname{coker}(M \to \bigoplus_j M_{f_j})^\sim$). So by the cohomology long exact sequence $\alpha = \delta(\beta)$ for some $\beta \in H^{i-1}(X, \mathscr{G})$. If $i > 1$, then induction yeilds that $\beta = 0$ so $\alpha = 0$. If $i = 1$, then $\alpha = 0$ since $\Gamma(X, -)$ is an exact functor, for quasi-coherent sheaves on an affine variety X (see the remark following theorem 15.24) and the fact that the long exact sequence on cohomology is exact. Therefore $\alpha = 0$ in this case too. □

Definition 20.46. A morphism $\phi: X \to Y$ is called affine if for every open affine piece V of Y, $\phi^{-1}(V)$ is an open affine piece of X.

Remark. You may see a different definition in the books, but this is equivalent (see [**Kem93**, §II.4]).

Lemma 20.35 (4.6)**.** *Let $\phi: X \to Y$ be an affine morphism of varieties and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module. Then there exists a natural isomorphisms $H^i(X, \mathscr{F}) \cong H^i(Y, \phi_*\mathscr{F})$.*

Proof. Let $o \to \mathscr{F} \to \mathscr{F}^*$ be a flabby-resolution of $\mathscr{F}$. Then for all affine open sets V in Y, by theorem 20.34 above, we have $H^i(\phi^{-1}V, \mathscr{F}) = 0$ for all $i > 0$. and so

$$0 \longrightarrow \Gamma(\phi^{-1}V, \mathscr{F}) \longrightarrow \Gamma(\phi^{-1}V, \mathscr{F}^0) \longrightarrow \Gamma(\phi^{-1}V, \mathscr{F}^i) \longrightarrow \ldots$$

is exact. Hence

$$0 \longrightarrow \phi_*\mathscr{F} \longrightarrow \phi_*\mathscr{F}^0 \longrightarrow \phi_*\mathscr{F}^1 \longrightarrow \dots$$

induces an exact sequence on stalks, i.e. $0 \to \phi_*\mathscr{F} \to \phi_*\mathscr{F}^*$ a flabby resolution of $\phi_*\mathscr{F}$ (clearly if $\mathscr{F}^1$ is flabby then $\phi_*\mathscr{F}^i$ is flabby). Therefore $H^i(Y, \phi_*\mathscr{F})$ is the ith cohomology of $\Gamma(Y, \phi_*\mathscr{F}^*)$, which is the ith cohomology of $\Gamma(X, \mathscr{F}^*)$ which is $= H^i(X, \mathscr{F})$. □

Remark. For $\phi: X \to Y$ affine, $\mathscr{F}$ quasi-coherent/coherent on X, then theorem 15.24 implies that $\phi_*\mathscr{F}$ is quasi-coherent/coherent on Y (for any affine piece U of Y with $A = k[U]$, then $\phi^{-1}(U)$ has coordinate ring B with $\phi^*: A \to B$ and if $\mathscr{F}$ is given by a B-module M, then $\phi_*\mathscr{F}$ is given by M considered as an A-module).

Example. Let $\iota: X \hookrightarrow \mathbb{P}^n$, X a projective variety, $\mathscr{F}$ a quasi-coherent sheaf on X, then $H^i(X, \mathscr{F}) \cong H^i(\mathbb{P}^n, \iota_*\mathscr{F})$ i.e. we may take the cohomology on X or $\mathbb{P}^n$ and get the same results.

20.1. Cohomology of $\mathbb{A}^n \backslash \{0\}$ and $\mathbb{P}^n$. I am going to give a sketch of this and refer you to Kempf [**Kem93**, §IX.1] for the details. Since $\mathscr{O}_{\mathbb{A}^n}(U)$ is a $k[\mathbb{A}^n] = k[X_1, \dots, X_n]$-module for any open $U \subset \mathbb{A}^n \backslash \{0\}$, the cohomology groups

$$H^i(\mathbb{A}^n \backslash \{0\}, \mathscr{O}_{\mathbb{A}^n})$$

will all be $k[X_1, \dots, X_n]$ modules in a natural way.

Proposition 20.36 (4.7)**.** *One has the following*

1. $H^i(\mathbb{A}^n \backslash \{0\}, \mathscr{O}_{\mathbb{A}^n}) = 0$ *unless* $i = 0$ *or* $i = n-1$*,*
2. *If* $n = 1$*, then* $H^0(\mathbb{A}^1 \backslash \{0\}, \mathscr{O}_{\mathbb{A}^1}) = k[X_1, X_1^{-1}]$ *and* $H^i(\mathbb{A}^1 \backslash \{0\}, \mathscr{O}_{\mathbb{A}^1}) = 0$ *for* $i > 0$*,*
3. *if* $n > 1$*,* $H^0(\mathbb{A}^n \backslash \{0\}, \mathscr{O}_{\mathbb{A}^n}) = k[X_1, \dots, X_n]$ *and*

$$H^{n-1}(\mathbb{A}^n \backslash \{0\}, \mathscr{O}_{\mathbb{A}^n}) = \bigoplus_{p \in \mathbb{Z}, p_i \le -1} k X_1^{p_1} \dots X_n^{p_n}$$

(with the obvious module structure, i.e. X_i^a *kills* $kX_1^{p_1} \dots X_n^{p_n}$ *if* $p_i + a \ge 0$*).*

Proof: (sketch). See Kempf [**Kem93**, §IX.1.1] for the details. We induct on n. The case $n = 1$: $\mathbb{A}^1 \backslash \{0\} = D(X_1)$ an affine variety with $H^0(D(X_1), \mathscr{O}_{\mathbb{A}^1}) = k[X_1, X_1^{-1}]$ and $H^i(D(X_1), \mathscr{O}_{\mathbb{A}^1}) = 0$ for $i > 0$. The case $n > 1$: there exists a short exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{A}^n \backslash \{0\}} \longrightarrow {}_{D(X_n)}\mathscr{O}_{\mathbb{A}^n \backslash \{0\}} \longrightarrow \bigoplus_{p_n \le -1} \mathscr{O}_{\mathbb{A}^n \backslash \{0\}} X_n^{p_n} \longrightarrow 0$$

(over affine pieces $D(X_i)$, $i \ne n$, this corresponds to

$$k[X_1, \dots, X_n, X_i^{-1}] \hookrightarrow k[X_1, \dots, X_n, X_i^{-1}, X_n^{-1}] \to \bigoplus_{p_n \le -1} k[X_1, \dots, X_{n-1}, X_i^{-1}] X_n^p$$

Over $D(X_n)$, we have $(\mathbb{A}^{n-1} \backslash \{0\}) \cap D(X_n) = \varnothing$ and so sections of right-hand sheaf are zero and we just have

$$k[X_1, \dots, X_n, X_n^{-1}] \xrightarrow{\sim} k[X_1, \dots, X_n, X_n^{-1}] \to 0)$$ □

Lecture 21

27th November 12:00

Proposition 21.37 (4.7). *One has the following*

(a) $H^i(\mathbb{A}^n\backslash\{0\}, \mathscr{O}_{\mathbb{A}^n}) = 0$ *unless* $i = 0$ *or* $i = n-1$,
(b) *If* $n = 1$, *then* $H^0(\mathbb{A}^1\backslash\{0\}, \mathscr{O}_{\mathbb{A}^1}) = k[X_1, X_1^{-1}]$ *and* $H^i(\mathbb{A}^1\backslash\{0\}, \mathscr{O}_{\mathbb{A}^1}) = 0$ *for* $i > 0$,
(c) *if* $n > 1$, $H^0(\mathbb{A}^n\backslash\{0\}, \mathscr{O}_{\mathbb{A}^n}) = k[X_1, \dots, X_n]$ *and*

$$H^{n-1}(\mathbb{A}^n\backslash\{0\}, \mathscr{O}_{\mathbb{A}^n}) = \bigoplus_{p\in\mathbb{Z}, p_i\leq -1} kX_1^{p_1}\dots X_n^{p_n}$$

(with the obvious module structure, i.e. X_i^a *kills* $kX_1^{p_1}\dots X_n^{p_n}$ *if* $p_i + a \geq 0$*).*

PROOF: (SKETCH). $n > 1$. We have a sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{A}^n\backslash\{0\}} \longrightarrow {}_{D(X_n)}\mathscr{O}_{\mathbb{A}^n\backslash\{0\}} \longrightarrow \oplus_{p_n\leq -1}\mathscr{O}_{\mathbb{A}^{n-1}\backslash\{0\}}X_n^{p_n} \longrightarrow 0$$

We have that this sequence is exact on each $D(X_i)$ for all i and so we have a short exact sequence of sheaves on $\mathbb{A}^n\backslash\{0\}$. Since $D(X_n)$ is affine and the inclusion $D(X_n) \hookrightarrow \mathbb{A}^{n-1}\backslash\{0\}$ is an affine map, Serre duality 20.34 and lemma 20.35 implies the middle sheaf has no higher H^i for $i > 0$ and so we have

$$H^0(\quad) = k[X_1, \dots, X_n, X_n^{-1}]$$

The long exact sequence on cohomology enables us to prove results by induction on n - for more details see Kempf [**Kem93**, pp. 113-114]. □

Corollary 21.38 (4.8).

(a) $H^i(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)) = 0$ *unless* $i = 0, n$,
(b) $H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)) \cong k[X_0, \dots, X_n]_{\deg d}$ *((corollary 17.28) e.g. = 0 for* $d < 0$ *in particular this is zero if* $d > -(n+1)$*),*
(c) $H^n(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(-n-1))$ *is 1-dimensional and the multiplication*

$$H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)) \times H^n(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(-n-1-d)) \to H^n(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(-n-1)) \cong k$$

is non-degenerate for all $d \geq 0$.

Remark. People who know about duality may recognise statement (c) as a manifestation of Serre duality - the thing on the right $H^n(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(-n-1))$ is the 'canonical' sheaf on projective space.

PROOF:(SKETCH). See Kempf [**Kem93**, IX.1.2] for the 'gory' details. Consider the projection map $\pi\colon \mathbb{A}^{n+1}\backslash\{0\} \to \mathbb{P}^n$. There exists a basis $\mathcal{B}$ of open affine $\underbrace{D^h(F)}_{=\mathbb{P}^n\backslash V^h(F)}$ of $\mathbb{P}^n$ (F a homogeneous polynomial) such that $\pi^{-1}(D^h(F)) = D(F) \subset \mathbb{A}^{n+1}\backslash\{0\}$ is an open affine subset of $\mathbb{A}^{n+1}\backslash\{0\}$ therefore the conclusion and the proof of (4.6) apply. Now show that

$$\pi_*\mathscr{O}_{\mathbb{A}^{n+1}} \cong \bigoplus_{d\in\mathbb{Z}} \mathscr{O}_{\mathbb{P}^n}(d),$$

where $\mathscr{O}_{\mathbb{P}^n}(d) = \mathscr{O}_{\mathbb{P}^n}(dH)$ where without loss of generality H is the hyperplane $X_0 = 0$ (see example sheet III, question 5). Moreover, if we let k^* act on $\mathbb{A}^{n+1}\backslash\{0\}$, and hence on $\mathscr{O}_{\mathbb{A}^{n+1}\backslash\{0\}}$ in the obvious way, we identify $\mathscr{O}_{\mathbb{P}^n}(d)$ as the part of $\pi_*\mathscr{O}_{\mathbb{A}^{n+1}\backslash\{0\}}$ which is homogeneous of degree d i.e. $\lambda \in k^*$ acts by $s \mapsto \lambda^d s$. Therefore lemma 20.35 implies that

$$H^i(\mathbb{A}^{n+1}\backslash\{0\}, \mathscr{O}_{\mathbb{A}^{n+1}}) \cong H^i(\mathbb{P}^n, \pi_*\mathscr{O}_{\mathbb{A}^{n+1}\backslash\{0\}}) = \bigoplus_{d\in\mathbb{Z}} H^i(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d))$$

(for all i) where the degree d part of the left hand side corresponds to $H^i(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d))$. This enables Kempf to deduce corollary 21.38 from proposition 21.37. □

21.1. Čech cohomology. Let X be a topological space, $\mathcal{U} = \{U_0, \dots, U_d\}$ an open cover. For $\mathscr{F}$ a sheaf of Abelian groups, define a complex

$$\check{C}^*(\mathcal{U}, \mathscr{F})\colon \check{C}^0(\mathcal{U}, \mathscr{F}) \xrightarrow{\delta^0} \check{C}^1(\mathcal{U}, \mathscr{F}) \xrightarrow{\delta^1} \check{C}^2(\mathcal{U}, \mathscr{F}) \xrightarrow{\delta^2} \dots$$

where

$$\check{C}^n(\mathcal{U}, \mathscr{F}) = \bigoplus_{i_0 < i_1 < \dots < i_n} \Gamma(\mathcal{U}_{i_0 \dots i_n}, \mathscr{F})$$

where $(\mathcal{U}_{i_0 \dots i_n} = U_{i_0} \cap \dots \cap U_{i_n})$ and differentials δ_n given by $\delta_n(\alpha) = \beta$, where

$$\beta_{i_0, \dots, i_{n+1}} = \sum_{0 \le j \le n} (-1)^j \alpha_{i_0, \dots, \hat{i}_j, \dots, i_n} \Big|_{\mathcal{U}_{i_0, \dots, i_{n+1}}}$$

As usual, $\delta^2 = \delta_{n+1}\delta_n = 0$ and $\check{C}^*(\mathcal{U}, \mathscr{F})$ is a complex. Moreover, we have an injection

$$0 \to \Gamma(X, \mathscr{F}) \xrightarrow{\varepsilon} \check{C}^0(\mathcal{U}, \mathscr{F})$$

where $s \mapsto (s_0, \dots, s_d)$, $s_i = s|_{U_i}$. Moreover, note that

$$\delta(\varepsilon(s))_{ij} = s|_{\mathcal{U}_{ij}} - s|_{\mathcal{U}_{ij}} = 0$$

Definition 21.47. The Čech cohomology $\check{H}^i(\mathcal{U}, \mathscr{F})$ is the ith cohomology of the complex $\check{C}^*(\mathcal{U}, \mathscr{F})$.

Remark. $\check{H}^0(\mathcal{U}, \mathscr{F}) = \ker \delta_0$ consists of giving section $(s_1, \dots, s_d)$ with $s_i \in \mathscr{F}(U_i)$ such that $s_i|_{\mathcal{U}_{ij}} = s_j|_{\mathcal{U}_{ij}}$ for all i, j and so the sheaf conditions imply that ε induces an isomorphism

$$\Gamma(X, \mathscr{F}) \xrightarrow{\sim} \check{H}^0(\mathcal{U}, \mathscr{F})$$

Lemma 21.39 (4.9)**.** *If $U_l = X$ for some $1 \le l \le d$, then the complex $\check{C}^*(\mathcal{U}, \mathscr{F})$ is exact and $\check{H}^i(\mathcal{U}, \mathscr{F}) = 0$ for $i > 0$.*

PROOF. For $n > 0$, we define maps

$$k_n\colon \check{C}^n(\mathcal{U}, \mathscr{F}) \to \check{C}^{n-1}(\mathcal{U}, \mathscr{F})$$

such that

$$\delta_{n-1}k_n + k_{n+1}\delta_n = \mathrm{id}_{\check{C}^n} \tag{21.10}$$

namely $k_n(\alpha) = \beta$ where

$$\beta_{i_0, \dots, i_{n-1}} = \begin{cases} (-1)^k \alpha_{i_0, \dots, l, \dots i_{n-1}} & \text{if } i_{k-1} < l < i_k \\ 0 & \text{otherwise} \end{cases}$$

(aside: this is called a chain homotopy) (n.b. Kempf is missing the $(-1)^k$ factor) (it is an easy check that (21.10) is true (see example sheet III, question 7)). So given $\alpha \in \check{C}^n(\mathcal{U}, \mathscr{F})$ such that $\delta_n \alpha = 0$, we have $\alpha = \delta_{n-1}(k_n \alpha)$ and so the complex is exact. □

Proposition 21.40 (4.10)**.** *If $\mathscr{F}$ is a quasi-coherent sheaf on a variety X which has an open affine cover $\mathcal{U} = \{U_1, \dots, U_d\}$, then $H^i(X, \mathscr{F}) \cong \check{H}^i(\mathcal{U}, \mathscr{F})$ for all i.*

PROOF. For the given $\mathcal{U}$, we define sheaves $\check{C}^i(\mathscr{F})$ for $i = 0, 1, \dots$ as follows for $V \subset X$ open, set $\check{C}^i(\mathscr{F})(V) = \check{C}^i(\mathcal{U}', \mathscr{F})$, where $\mathcal{U}'$ is the open cover of V given by $U_0 \cap V, \dots, U_d \cap V$ and where the restriction maps are obvious. Since $\mathscr{F}$ is a sheaf, so too are the $\check{C}^i(\mathscr{F})$. We then get a complex of sheaves $\check{C}^*(\mathscr{F})$, and a sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \check{C}^0(\mathscr{F}) \longrightarrow \check{C}^1(\mathscr{F}) \longrightarrow \dots$$

We'll finish this next time. □

Lecture 22

29th November 12:00

22.1. Cohomology of coherent sheaves.

Proposition 22.41 (4.10). *If $\mathscr{F}$ is a quasi-coherent sheaf on a variety X with open affine cover $\mathfrak{U} = \{U_1, \dots, U_d\}$, then $H^i(X, \mathscr{F}) \cong \check{H}^i(\mathfrak{U}, \mathscr{F})$.*

Proof. We constructed a complex of sheaves

$$0 \longrightarrow \mathscr{F} \longrightarrow \check{C}^0(\mathscr{F}) \longrightarrow \check{C}^1(\mathscr{F}) \longrightarrow \dots$$

If we can show that $\mathscr{F} \to \check{C}^*(\mathscr{F})$ is a resolution and $H^j(X, \check{C}^i(\mathscr{F})) = 0$ for all i and all $j > 0$, the result follows from lemma 19.32. The first statement is local, so we may assume that $X = U_l$ and the exactness follows lemma 21.39. To show vanishing note that

$$\check{C}^i(\mathscr{F}) \cong \bigoplus_{k_0 < k_1 < \dots < k_i} {}_{U_{k_0} \cap \dots \cap U_{k_i}} \mathscr{F}$$

As X is separated, $U_i \cap U_j \cong \Delta_X \cap (U_i \times U_j)$ is affine and so too is $U_{k_0} \cap \dots \cap U_{k_i}$, and moreover the inclusion $\iota : U_{k_0} \cap \dots \cap U_{k_i} \to X$ is an affine map. For U open affine in X, $U_{k_0} \cap \dots \cap U_{k_0} \cap U$ is affine. We have

$$\begin{aligned} H^j(X, {}_{(U_{k_0} \cap \dots U_{k_i})}\mathscr{F}) &= H^j(X, \iota_*(\mathscr{F}|_{U_{k_0} \cap \dots \cap U_{k_i}})) \\ &\cong H^j(U_{k_0} \cap \dots \cap U_{k_i}, \mathscr{F}_{U_{k_0} \cap \dots \cap U_{k_i}}) \quad \text{(by lemma 20.35)} \\ &= 0 \qquad (\text{for } j > 0) \quad \text{(by theorem 20.34)} \quad \square \end{aligned}$$

Remark. In particular, $H^j(X, \mathscr{F}) = 0$ for $j > d$, e.g. if X is a projective variety, $\mathscr{F}$ a quasi-coherent sheaf on X, then $H^j(X, \mathscr{F}) = 0$ for $j > \dim X$ (since by a basic result in dimension theory, all components of a general hyperplane section of X have codimension 1, and so we can cover X by $\dim X + 1$ affine pieces).

Lemma 22.42 (4.11). *Suppose $V \subset \mathbb{P}^N$ is a projective variety with affine pieces $U_i = V \cap \{X_i \neq 0\}$ and $\mathscr{F}$ a quasi-coherent sheaf on X. Given $\sigma \in \Gamma(U_0, \mathscr{F})$, there exists $m \geq 0$ such that σ extends to a section in $\Gamma(V, \mathscr{F}(m))$, where $\mathscr{F}(m) = \mathscr{F} \otimes_{\mathscr{O}_V} \mathscr{O}_V(m)$.*

Proof. Consider $\mathscr{F}(m)$ as $\mathscr{F}(mH) = \mathscr{F} \otimes_{\mathscr{O}_V} \mathscr{O}_V(mH)$ for H a hyperplane $X_0 = 0$. So $\mathscr{F}|_{U_0} \cong \mathscr{F}(m)|_{U_0}$ for all m. Set $\tau_0 = \sigma \in \Gamma(U_0, \mathscr{F})$. For $j > 0$, let $A = k[U_j]$ and $M = \mathscr{F}(U_j)$ - if $f = X_0/X_j \in A$, $\sigma|_{U_0 \cap U_j}$ corresponds to an element of M_f and so $f^m \sigma|_{U_0 \cap U_j}$ corresponds for some $m > 0$ to an element of M i.e. extends to a section $\tau_j \in \mathscr{F}(U_j)$. Without loss of generality, choose $m \gg 0$ to work for all $j > 0$. Now, we have

$$\tau_j|_{U_{ij} \cap U_0} = \left(\frac{X_i}{X_j}\right)^m \tau_i|_{U_{ij} \cap U_0}$$

for all $i, j > 0$. Choosing U_{ij} as an affine piece with $U_{ij} \cap U_0$ is given by $X_0/X_j \neq 0$, then for some $r \geq 0$ we have

$$\begin{aligned} \left(\frac{X_0}{X_j}\right)^r \tau_j|_{U_{ij}} &= \left(\frac{X_i}{X_j}\right)^m \left(\frac{X_0}{X_j}\right)^r \tau_i|_{U_{ij}} \\ &= \left(\frac{X_i}{X_j}\right)^{m+r} \left(\frac{X_0}{X_i}\right)^r \tau_i|_{U_i \cap U_j} \end{aligned}$$

Without loss of generality, the same r works for all pairs $i,j>0$ and we may replace τ_j by

$$\left(\frac{X_0}{X_j}\right)^r \tau_j = \left(\frac{X_0}{X_j}\right)^{m+r} \sigma|_{U_{ij}}$$

and replace m by $m+r$. So for some $m \gg 0$, we have sections $\tau_j \in \mathscr{F}(U_j)$ such that

$$\tau_j|_{U_{ij}} = \left(\frac{X_i}{X_j}\right)^m \tau_i|_{U_{ij}}$$

for all $i,j>0$. By construction

$$\tau_j|_{U_{0j}} = \left(\frac{X_0}{X_j}\right)^m \tau_0|_{U_{0j}}$$

for all j. This however, is precisely the data needed to define a global section of $\mathscr{F}(m)$, by examples sheet III, question 8. □

Theorem 22.43 (4.12)**.** *Let $\mathscr{F}$ be a coherent sheaf on a projective variety $X \subset \mathbb{P}^N$. Then*

(a) *$H^i(X,\mathscr{F})$ finite dimensional over k for all i,*
(b) *there exists n_0 such that $H^i(X,\mathscr{F}(n)) = 0$ for $i>0$, $n \geq n_0$.*

Remark. This result and the previous are the two most significant ones from FAC [**Ser55**].

Proof. First we do some mumbling, so that we can just take $X = \mathbb{P}^N$. Given any coherent $\mathscr{O}_X$-module $\mathscr{G}$, there exists natural morphisms of $\mathscr{O}_{\mathbb{P}^n}$-modules (examples sheet II, question 11). Consider

$$\iota_*\mathscr{F} \otimes_{\mathscr{O}_{\mathbb{P}^N}} \iota_*\mathscr{G} \longrightarrow \iota_*(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})$$

By considering affine pieces, some commutative algebra gives us that we get an isomorphism on stalks and hence an isomorphism on sheaves. Take $\mathscr{G} = \mathscr{O}_X(m) = \iota^*\mathscr{O}_{\mathbb{P}^N}(n)$. By examples sheet II, question 13, we have

$$\iota_*\mathscr{G} = \iota_*\mathscr{O}_X \otimes_{\mathscr{O}_{\mathbb{P}^N}} \mathscr{O}_{\mathbb{P}^N}(n)$$

Noting that

$$\iota_*\mathscr{F} \otimes_{\mathscr{O}_{\mathbb{P}^N}} \iota_*\mathscr{O}_X \xrightarrow{\sim} \iota_*\mathscr{F}$$

(clear have $B \to A$, therefore $M \otimes_B A \cong M$) we deduce that

$$\iota_*\mathscr{F} \otimes_{\mathscr{O}_{\mathbb{P}^N}} \mathscr{O}_{\mathbb{P}^N}(n) \cong \iota_*(\mathscr{F}(n))$$

Moreover, by corollary 16.26, $\iota_*\mathscr{F}$ is a coherent $\mathscr{O}_{\mathbb{P}^N}$-module (over $\mathbb{A}^N$ it corresponds to a finitely generated $k[X_1,\ldots,X_N]$-module). Therefore without loss of generality we can take $X = \mathbb{P}^N$.

The proof is by downward induction on i. If $i > N$, we know the cohomology is zero. For $i \leq N$: on $\mathbb{A}^N = U_0$, we have a surjection of sheaves

$$\mathscr{O}^p_{\mathbb{A}^n} \xrightarrow{\ \phi\ } \mathscr{F}|_{\mathbb{A}^n} \longrightarrow 0$$

by corollary 16.26 i.e. $\mathscr{F}|_{\mathbb{A}^N}$ is generated by sections $s_1,\ldots,s_p \in \Gamma(\mathbb{A}^n,\mathscr{F})$. By lemma 22.42 above, then extend to give sections $\tilde{s}_1,\ldots,\tilde{s}_p$ of $\mathscr{F}(m)$ for some $m>0$.

We deduce that ϕ extends to a morphism $\mathscr{O}^p_{\mathbb{P}^N} \xrightarrow{\tilde{\phi}} \mathscr{F}(m)$, surjective on stalks over U_0. Taking the sum of such maps for $i = 0,\ldots,N$, we get a surjective morphism

$$\mathscr{O}^q_{\mathbb{P}^N} \xrightarrow{\ \psi\ } \mathscr{F}(m)$$

for some suitable $m \gg 0$. Now consider the short exact sequence

$$0 \longrightarrow \mathscr{G} \longrightarrow \mathscr{O}^q_{\mathbb{P}^N}(-m) \longrightarrow \mathscr{F} \longrightarrow 0 \tag{22.11}$$

where $\mathscr{G}$ is coherent by corollary 16.26.

Now we prove (a). By corollary 21.38 we have $H^i(\mathbb{P}^N, \mathscr{F}) \cong H^{i+1}(\mathbb{P}^N, \mathscr{G})$ for $1 \le i < N-1$. We have

$$H^{N-1}(\mathbb{P}^N, \mathscr{F}) \hookrightarrow H^N(\mathbb{P}^N, \mathscr{G})$$

There exists a surjection

$$\bigoplus_{i=1}^{q} H^N(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}}(-m)) \twoheadrightarrow H^N(\mathbb{P}^N, \mathscr{F})$$

Local map implies that $H^N(\mathbb{P}^N, \mathscr{F})$ is finite dimensional (since the left-hand-side is finite dimensional by corollary 21.38) Therefore (a) is true for $i = N$. For $i < N$, the result follows by downward induction from $H^{i+1}(\mathbb{P}^N, \mathscr{G})$ being finite dimensional.

(b) The result follows from $\mathscr{O}_{\mathbb{P}^N}(n)$ by corollary 21.38 and induction on i, tensoring equation (22.11) by $\mathscr{O}_{\mathbb{P}^N}(n)$ and passing to the long exact sequence of cohomology. □

Definition 22.48. For $\mathscr{F}$ a coherent sheaf on a projective variety X of dimension d, we define the Euler characteristic of $\mathscr{F}$ to be

$$\chi(X, \mathscr{F}) := \sum_{i=0}^{d} (-1)^i h^i(X, \mathscr{F})$$

where $h^i(X, \mathscr{F}) = \dim_k H^i(X, \mathscr{F})$ (which makes sense because of the last result).

Lecture 23

2nd December 12:00

Examples class III is Friday the 17th January at 4pm in MR3 (pavilion E). The solutions to the problems will be in my pigeon-hole from Wednesday the 15th January - help yourself to one copy.

23.1. §5: Differentials & Riemann-Roch for curves. Suppose that V is an irreducible (it is possible to do this also in the general case) variety of dimension $n = \operatorname{tr\,deg}_k k(V)$. The space of rational 1-forms on V, denoted $\Omega^1_{k(V)/k}$ is the $k(V)$-vector space which is universal with respect to k-derivations into $k(V)$ vector spaces M.

$$\begin{array}{ccc} k(V) & \xrightarrow{\ \mathrm{d}\ } & \Omega^1_{k(V)/k} \\ & {\scriptstyle D}\searrow \quad \swarrow{\scriptstyle \exists!} & \\ & M & \end{array}$$

Concretely, it is the $k(V)$-vector space generated by elements $\mathrm{d}g$ for $g \in k(V)$ quotiented by the relations

$$\begin{aligned} \mathrm{d}(f+g) &= \mathrm{d}f + \mathrm{d}g, \\ \mathrm{d}(fg) &= f\,\mathrm{d}g + g\,\mathrm{d}f, \\ \mathrm{d}a &= 0, \end{aligned}$$

for $f, g \in k(V)$ and $a \in k$ (we say that such a d is a k-derivation). The space of related r-forms is

$$\Omega^r_{k(V)/k} := \underbrace{\Omega^1_{k(V)/k} \wedge \ldots \wedge \Omega^1_{k(V)/k}}_{r \text{ times}}$$

Exercise: if $x_1, \dots, x_n$ is a separating transcendence basis for $k(V)/k$ i.e.

$$k(V)/k(x_1, \dots, x_n)$$

is separable, and finite, then $\mathrm{d}x_1, \dots, \mathrm{d}x_n$ is a basis for $\Omega^1_{k(V)/k}$ over $k(V)$. It follows that

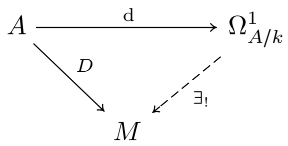
$$\dim_{k(V)} \Omega^r_{k(V)/k} = \binom{n}{r}$$

Definition 23.49. An r-form ω is regular at $P \in V$ if it can be written

$$\omega = \sum_{\underline{i}} f_{\underline{i}} \, \mathrm{d}g_{i_1} \wedge \cdots \wedge \mathrm{d}g_{i_r}$$

with $f_{\underline{i}}$ and g_j in $\mathscr{O}_{V,P}$.

Given U open in V, set $\Omega^r(U)$ to be the regular r-forms on U, a module over $\mathscr{O}_V(U)$. So we get a sheaf of regular r-forms Ω^r_V, an $\mathscr{O}_V$-module. Suppose that $V \subseteq \mathbb{A}^N$ affine and $A = k[V]$. Let $\Omega^1_{A/k}$ be the universal A-module with respect to k-derivations in A-modules M, i.e. given $D\colon A \to M$ a k-derivation, then there exists a factorisation



and we define $\Omega^r_{A/k} \coloneqq \wedge^r \Omega^1_{A/k}$.

If $A = k[X_1, \dots, X_N]/I$, then $\Omega^1_{A/k}$ is the A-module generated by $\mathrm{d}X_1, \dots, \mathrm{d}X_N$ with relations (†) $\mathrm{d}f = 0$ for all $f \in I$.

Claim: $\omega \in \Omega^r_{k(V)/k}$ is regular at $P \in V$ iff $\omega \in \Omega^r_{A/k} \otimes \mathscr{O}_{V,P} \subset \Omega^r_{k(V)/k}$.

PROOF. The direction '←' is clear. For the direction '→', note that for $g = h/u \in \mathscr{O}_{V,P}$ $(h, u \in A, u(P) \neq 0)$, then we have $\mathrm{d}g = \frac{1}{u}\,\mathrm{d}h - \frac{h}{u^2}\,\mathrm{d}u \in \Omega^1_{A/k} \otimes \mathscr{O}_{V,P}$. □

Lemma 23.44 (5.1)**.** *If V is an irreducible affine, $A = k[V]$, then $\Omega^1_{A/k} = \Gamma(V, \Omega^1_V)$ and similarly for r-forms.*

PROOF. We have $\Omega^1_{A/k} \subset \Omega^1_V(V)$ as A-modules. The previous claim implies that localisations at each maximal ideal are equal so $\Omega^1_{A/k} = \Omega^1_V(V)$ (cf. example sheet II, question 12). □

Definition 23.50. The Zariski tangent space of V at P is defined

$$T_{V,P} \coloneqq \mathrm{Hom}_k(\mathfrak{m}_P/\mathfrak{m}_P^2, k)$$

as a $k = \mathscr{O}_{V,P}/\mathfrak{m}_P$ vector space.

Assuming V is affine, $A = k[V]$, $\Omega^1_{V,P} \cong \Omega^1_{A/k} \otimes \mathscr{O}_{V,P}$. The derivation $\mathrm{d}\colon A \to \Omega^1_{A/k}$ induces a derivation

$$\mathrm{d}\colon \mathfrak{m}_P \longrightarrow \Omega^1_{A/k} \otimes \mathscr{O}_{V,P} = \Omega^1_{V,P}$$

and hence a linear map of $k = \mathscr{O}_{V,P}/\mathfrak{m}_P$ vector spaces

$$\mathfrak{m}_P/\mathfrak{m}_P^2 \cong \mathfrak{m}_P \otimes \mathscr{O}_{V,P}/\mathfrak{m}_P \overset{\mathrm{d}_P:\mathrm{d}\otimes 1}{\longrightarrow} \Omega^1_{A/k} \otimes \mathscr{O}_{V,P}/\mathfrak{m}_P \cong \Omega^1_{V,P}/\mathfrak{m}_P\Omega^1_{V,P}$$

(and d_P is zero on $\mathfrak{m}_P^2$ since $\mathrm{d}_P(fg) = f(P)\,\mathrm{d}g + g(P)\,\mathrm{d}f$).

Proposition 23.45 (5.2)**.** *d_P is an isomorphism of k-vector spaces.*

PROOF. Exercise sheet III, question 10. □

Definition 23.51. We say that $P \in V$ is smooth (or non-singular) if $\dim T_{V,P} = n$ ($= \dim V$) and hence iff $\dim \mathfrak{m}_P/\mathfrak{m}_P^2 = \dim \Omega^1_{V,P}/\mathfrak{m}_P\Omega^1_{V,P} = n$.

We are saying that the local ring is regular.

23.2. Case of curves. (For more details see my abbreviated lecture notes from the algebraic curves course [**Wil**]). When $n = 1$, then P is smooth iff $\dim_K \mathfrak{m}_P/\mathfrak{m}_P^2 = 1$ which is true iff (by Nakayama's lemma) $\mathfrak{m}_P = (t)$, where t is any element of $\mathfrak{m}_P \backslash \mathfrak{m}_P^2$. And this is the case iff (by [**AM69**, §9.2]) every element $a \in \mathscr{O}_{V,P}$ is of the form ut^n for a unit u of $\mathscr{O}_{V,P}$ and $n \geq 0$, i.e. $\mathscr{O}_{V,P}$ is a discrete valuation ring with valuation

$$v_P : \mathscr{O}_{V,P}\backslash\{0\} \to \mathbb{N}$$
$$a = ut^n \mapsto n$$

So proposition 23.45 implies that $\mathrm{d}_P t = \overline{\mathrm{d}t}$ generates $\Omega^1_{V,P}/\mathfrak{m}_P\Omega^1_{V,P}$. Thus by Nakayama's lemma, $\mathrm{d}t$ generates $\Omega^1_{V,P}$ as an $\mathscr{O}_{V,P}$-module, whence $\mathrm{d}t$ generates $\Omega^1_{V,Q}$ as an $\mathscr{O}_{V,Q}$-module for all Q in some neighbourhood $U \ni P$ (without loss of generality $V \subset \mathbb{A}^N$, then $\mathrm{d}X_1, \ldots, \mathrm{d}X_N$ do generate and they can be expressed in term of $\mathrm{d}t$ with coefficients in $\mathscr{O}_{V,P}$). Therefore, there exists a surjection of sheaves $\mathscr{O}_U \twoheadrightarrow \Omega^1_U$ inducing isomorphisms on stalks for all $Q \in U$ i.e. $\Omega^1_U \cong \mathscr{O}_U$. Thus for V a smooth curve, Ω^1_V is invertible. For a smooth irreducible variety V of dimension n, a similar proof shows that Ω^n_V is an invertible sheaf, the canonical sheaf K_V of V.

Theorem 23.46 (Serre duality). *If V is a smooth projective variety of dimension n, $\mathscr{F}$ a locally free $\mathscr{O}_V$-module of rank r, then there exists a perfect pairing*

$$H^i(\mathscr{F}) \times H^{n-i}(K_V \otimes \mathscr{F}^r) \longrightarrow H^n(K_V) \cong k$$

for all i (cf. 21.38(d) for invertible sheaves on $\mathbb{P}^n$) (where $K_{\mathbb{P}^n} \cong \mathscr{O}_{\mathbb{P}^n}(-n-1)$).

23.3. Divisors on smooth curves and Riemann-Roch theorem. For V a smooth irreducible curve, $P \in V$, there exists a valuation $v_P : \mathscr{O}_{V,P}\backslash\{0\} \to \mathbb{N}$. A local parameter t at P is an element t with $v_P = 1$ i.e. $\mathfrak{m}_P = (t)$. There exists an extension to a valuation $v_P : k(V)^* \to \mathbb{Z}$ i.e. $v_P(f) \geq 0$ iff $f \in \mathscr{O}_{V,P}$. Given $f \in k(V)^*$, there exists only finitely many points P such that $v_P(f) \neq 0$ (write $f = F/G$ and then $v_P(f) \neq 0$ only when P is a zero of F or G - this is just a finite set of points by example sheet III, question 3).

Lecture 24

4th December 12:00

24.1. Curves and Riemann-Roch. Next term: Prof. Mark Gross: Introduction to Mirror Symmetry, a graduate course.

A divisor D on V is a finite sum $\sum n_i P_i$ with $n_i \in \mathbb{Z}$ and $P_i \in V$ called a Cartier divisor on V. The group of divisors $\mathrm{Div}(V)$ is the free Abelian group on points of V and the degree of a divisor is given by $\deg D = \sum n_i \in \mathbb{Z}$. For $f \in k(V)^*$, the principal divisor associated to f is

$$(f) := \sum_{P \in V} v_P(f) P$$

Two divisors D_1, D_2 are linearly equivalent if there exists $f \in k(V)^*$ such that $D_1 = D_2 + (f)$. If V is projective, then $(f) = 0$ iff $f \in k^*$ (corollary to proposition 11.17).

Definition 24.52. We define the divisor class group $\mathrm{Cl}(V)$ by

$$\mathrm{Cl}(V) = \mathrm{Div}(V)/\sim$$

where the equivalence relation is linear equivalence.

We say that $D = \sum n_i P_i$ is effective (written $D \geq 0$) if $n_i \geq 0$ for all i. Given $D = \sum n_i P_i$, define a subsheaf $\mathscr{O}_V(D) \subset k(V)$ by

$$\Gamma(U, \mathscr{O}_V(D)) = \left\{ \begin{matrix} f \in k(V)^* : v_{P_i}(f) \geq -n_i \text{ for all } P_i \in U \\ \text{and regular elements on } U \end{matrix} \right\} \cup \{0\}$$

If t_i is a local parameter at P_i, then $\mathscr{O}_V(D)$ is invertible and locally generated by $t_i^{-n_i}$ (cf. Cartier divisors). The dual of $\mathscr{O}_V(D)$ is $\mathscr{O}_V(-D)$.

Lemma 24.47 (5.3)**.** *One has that $\mathscr{O}_V(D) \cong \mathscr{O}_V$ iff D is principal (and thus $\mathscr{O}_V(D_1) \cong \mathscr{O}_V(D_2)$ iff $D_1 \sim D_2$). Given an invertible sheaf $\mathscr{L}$, there exists a divisor D on V such that $\mathscr{L} \cong \mathscr{O}_V(D)$ (thus $\mathrm{Cl}(V) \xrightarrow{\sim} \mathrm{Pic}(V)$ given by $D \mapsto \mathscr{O}_V(D)$).*

PROOF. If $D = (f)$, then multiplication by f^{-1} yields an isomorphism $\mathscr{O}_V \xrightarrow{\sim} \mathscr{O}_V(D)$. Conversely, if $\mathscr{O}_V \cong \mathscr{O}_V(D)$, let f^{-1} be a global section of $\mathscr{O}_V(D)$ corresponding to $1 \in \mathscr{O}_V(V)$. Then $D = (f)$ so $D_1 \sim D_2$ iff $D_1 - D_2$ is principal which is true iff $\mathscr{O}_V(D_1 - D_2) \cong \mathscr{O}_V$ which is true iff $\mathscr{O}_V(D_1) \cong \mathscr{O}_V(D_2)$. Moreover addition of divisor classes corresponds to tensor products of corresponding invertible sheaves. Finally, any invertible sheaf $\mathscr{L}$ comes from a divisor class on V - use examples sheet III, question 1. □

Given a non-zero rational 1-form ω on V and $P \in V$, choose a local parameter $t \in \mathfrak{m}_{V,P}$, a generator of the maximal ideal. Since we saw that $\mathrm{d}t$ is a local generator of the 1-forms Ω^1_V and $\ker \Omega^1_V$ is 1-dimensional basis over $k(V)$ (which follows from the result last time about transcendence bases or just from the remarks just made). We deduce that there is $f \in K(V)^*$ such that $\omega = f\,\mathrm{d}t$, and we define $v_P(\omega) := v_P(f)$ (this is independent of the choice of t, but it is not a trivial thing to see this and t is zero except at finitely many points - see lemmas 3.1 & 3.2 from the algebraic curves course notes [**Wil**]).

Definition 24.53. A canonical divisor D_V is of the form $(\omega) = \sum_{P \in V} v_P(\omega) P$. This defines a unique divisor class (because the space of 1-forms is one-dimensional) - if $\omega' = f\omega$, then $(\omega') = (f) + (\omega)$.

Definition 24.54. For V a smooth projective variety, the genus is defined

$$g(V) := \dim_k \Omega^1_V(V)$$

Proposition 24.48 (5.4)**.** *Let $K_V = (\omega)$ then $\Omega^1_V \cong \mathscr{O}_V(K_V)$.*

PROOF. For any $U \subset V$, $\omega' \in \Gamma(U, \Omega^1_V)$ iff $v_P(\omega') \geq 0$ for all $P \in U$ and this is true iff $\omega' = f\omega$ with $(K_V + (f))|_U \geq 0$ which is true iff $f \in \Gamma(U, \mathscr{O}_V(K_V))$. We write $h^i(V, D)$ for $h^i(V, \mathscr{O}_V(D))$ for $i = 0, 1$. Serre duality then implies that $h^1(V, D) = h^0(K_V - D)$. For curves, there are also several classical proofs of Serre duality. □

Theorem 24.49 (5.5, Riemann-Roch for curves)**.** *Let V be a smooth projective curve and let D be a divisor on V, then*

$$h^0(D) - h^0(K_V - D) = 1 - g(V) + \deg D.$$

Putting $D = K_V$, we deduce in particular that $\deg(K_V) = 2g - 2$.

PROOF. We are required to prove that $\chi(V, \mathscr{O}_V(D)) = 1 - g + \deg D$ (*). Write $D = D_1 - D_2$ ($D_1, D_2 \geq 0$ and disjoint). Suppose that $D_2 = \sum n_i P_i$, then there exists a short exact sequence of sheaves

$$0 \longrightarrow \mathscr{O}_V(-D_2) \longrightarrow \mathscr{O}_V \longrightarrow \mathscr{O}_{D_2} \longrightarrow 0$$

where $\mathscr{O}_{D_2}$ denotes the skyscraper sheaf supported on the P_i, with stalks

$$\mathscr{O}_{V,P}/\mathfrak{m}_{P_i}^{n_i} \cong k^{n_i}$$

of P_i (since $\mathscr{O}_{V,P_i}$ is a discrete valuation ring, it is easy to see that $\mathscr{O}_{V,P_i}/\mathfrak{m}_{P_i}^{n_i}$ has dimension n_i over k). Therefore $h^0(\mathscr{O}_{D_2}) = \deg D_2$ and $h^i(\mathscr{O}_{D_2}) = 0$ for $i > 0$ (e.g. $\mathscr{O}_{D_2}$ is flabby). We tensor the previous sequence to get

$$0 \longrightarrow \mathscr{O}_V(D) \longrightarrow \mathscr{O}_V(D_1) \longrightarrow \mathscr{O}_{D_2} \longrightarrow 0$$

The long exact sequence on cohomology implies that $\chi(V, \mathscr{O}_V(D)) = \chi(V, \mathscr{O}_V(D_1)) - \deg D_2$. We now tensor the sequence

$$0 \longrightarrow \mathscr{O}_V(-D_1) \longrightarrow \mathscr{O}_V \longrightarrow \mathscr{O}_{D_1} \longrightarrow 0$$

by $\mathscr{O}_V(D_1)$ to get a short exact sequence

$$0 \longrightarrow \mathscr{O}_V \longrightarrow \mathscr{O}_V(D_1) \longrightarrow \mathscr{O}_{D_1} \longrightarrow 0$$

and hence $\chi(\mathscr{O}_V(D_1)) = \chi(\mathscr{O}_V) + \deg D_1$ (from the long exact sequence on cohomology). Putting these two formulæ together, we obtain (*). Setting $D = K_V$, and using $h^0(\mathscr{O}_V) = 1$ (corollary 12.19 to proposition 11.17), get $g - 1 = 1 - g + \deg K_V$ and thus $\deg K_V = 2g - 2$. $\square$

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