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Example Sheet 2

(1) Let $(X, \mathcal{O}_X)$ be an abstract variety and A denote the k-algebra $\mathcal{O}_X(X)$. For $f \in A$, we set $X_f = \{P \in X \,:\, f(P) \neq 0\}$; prove that $\mathcal{O}_X(X_f) = A_f$.

(2) In §2, we defined C^∞ and complex manifolds sheaf-theoretically; show these definitions are equivalent to the classical ones, expressed in terms of charts, atlases and transition functions.

(3) If X and Y are irreducible varieties, show that the product $X \times Y$, with the Zariski topology, is an irreducible space (and hence, once one has defined the structure sheaf, is an irreducible variety).

(4) Show that a smooth manifold M is compact if and only if, for any smooth manifold N, the projection map $\pi : M \times N \to N$ is closed.

(5) Show that the forms of degree d in $k[X_0, \ldots, X_N]$, modulo multiplication by non-zero constants, are parametrized by a projective space of dimension $\binom{d+N}{N} - 1$. Show that the reducible forms correspond to a closed subvariety.

(6) Let X be an irreducible variety defined over $\mathbf{C}$. We define the classical topology on X by defining it on the affine pieces U of X in the obvious way : we can embed $U \hookrightarrow \mathbf{C}^n$ and then take the induced topology from the classical topology on $\mathbf{C}^n$. If X is complete, why does Chow's Theorem imply that X is compact in the classical topology?

(7) If X is a variety over k with decomposition $X = X_1 \cup \ldots \cup X_N$ into irreducible components, show that

$$\mathrm{Rat}(X) \cong k(X_1) \times \ldots \times k(X_N)$$

as a product of k-algebras. Prove that two irreducible varieties are birationally equivalent if and only if their function fields are isomorphic over k. Deduce that two general varieties X and Y are birationally equivalent if and only if Rat(X) $\cong$ Rat(Y) as k-algebras.

†(8) If X is an affine variety with coordinate ring A and irreducible components $X_1, \ldots X_N$, and Z is a closed subset of X not containing any component X_i, prove that there exists $f \in I(Z) \subset A$ which does not vanish identically on any X_i. Prove that a basic open set $D(g)$, where $g \in A$, is dense if and only if g is not a zero-divisor. Deduce that $\mathrm{Rat}(X) \cong \mathrm{tot}(A)$ as k-algebras.

(9) If V and W are affine varieties, we consider the k-algebra $A = k[V] \otimes_k k[W]$; show that A is a finitely generated algebra. Let P now denote the k-algebra of all functions $V \times W \to k$, with pointwise addition and multiplication. We can define a morphism of k-algebras $\phi : A \to P$ by $f \otimes g \mapsto \theta$, where $\theta(x, y) = f(x)g(y)$, extended linearly. Show that ϕ is injective and deduce that A is a reduced k-algebra.

Deduce now that $k[V \times W] \cong A$. What universal property is satisfied by the product variety $V \times W$?

(10) If a locally free $\mathcal{O}_X$-module $\mathcal{M}$ of rank r is trivialized with respect to an open cover $\{U_i\}$ by transition functions $\psi_{ji} \in \Gamma(U_{ij}, GL(r, \mathcal{O}_X)$, show that its dual $\mathcal{M}^\vee$ is trivialized with respect to $\{U_i\}$ by transition functions given by the transpose of $\psi_{ji}^{-1} = \psi_{ij}$.

(11) Given a finite open cover $\{U_i\}$ of a ringed space $(X, \mathcal{O}_X)$, and transition functions $\psi_{ji} \in \Gamma(U_{ij}, GL(r, \mathcal{O}_X)$ satisfying the usual compatibility conditions, prove that there exists a locally free $\mathcal{O}_X$-module of rank r with these transition functions.

(12) Let V denote an affine variety with $k[V] = A$, and M an A-module. Let $\mathcal{B}$ denote the basis of open sets of the form $D(f)$ for $f \in A$ and $\tilde{M}$ denote the $\mathcal{B}$-presheaf of $\mathcal{O}_X$-modules defined by $\tilde{M}(D(f)) = M \otimes_A \mathcal{O}_X(D(f))$. Show that $\tilde{M}$ is in fact a $\mathcal{B}$-sheaf (cf (1.6)).

(13) Let $\phi = (f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of varieties. Is the morphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ of sheaves of rings on X in general either surjective or injective?

(14) Let $f : X \to Y$ be a continuous map between topological spaces, and let $\mathcal{F}$ be a sheaf of abelian groups on Y. If $f^{-1}\mathcal{F}$ is the inverse image sheaf as defined in lectures, and $P \in X$, show that $(f^{-1}\mathcal{F})_P \cong \mathcal{F}_{f(P)}$.

For U open in X, we define

$$\mathcal{G}(U) = \lim_{V \supseteq f(U)} \mathcal{F}(V),$$

where the (direct) limit is taken over all open subsets V of Y containing $f(U)$; by defining appropriate restriction maps, show that $\mathcal{G}$ may be made into a presheaf.

Show that there is a natural morphism of presheaves $\mathcal{G} \to f^{-1}\mathcal{F}$. *Show that this morphism defines isomorphisms an stalks.* Deduce that the sheafification of $\mathcal{G}$ is isomorphic to $f^{-1}\mathcal{F}$.

(15) For M an A-module, prove that $M = 0$ if and only if $M_{\mathbf{m}} = 0$ for all maximal ideals $\mathbf{m}$ in A. Deduce that a sequence of A-modules

$$0 \to M \to N \to P \to 0$$

is exact if and only if the sequences of $A_{\mathbf{m}}$-modules

$$0 \to M_{\mathbf{m}} \to N_{\mathbf{m}} \to P_{\mathbf{m}} \to 0$$

are exact for all maximal ideals m.

†(16) Suppose that $\phi : Y \to X$ is a morphism of affine varieties, and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module on X. Show that $\phi_*\phi^*\mathcal{F} \cong \mathcal{F} \otimes_{\mathcal{O}_X} \phi_*\mathcal{O}_Y$. Deduce the same result holds when ϕ is a morphism of abstract varieties with the property that for any affine piece U of X, the inverse image $\phi^{-1}U$ is an affine piece of Y.