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Example Sheet 1

(1) If V is an affine or projective variety, show that the Zariski topology on V is compact.

(2) Suppose that V is an irreducible affine variety. Show that the ring of everywhere regular rational functions on V is just the coordinate ring. Show that the only everywhere regular rational functions on $\mathbf{P}^n$ are the constants.

(3) For $\phi : V \to W$ a morphism of affine varieties, and $\phi^* : k[W] \to k[V]$ the induced map on coordinate rings, show

(a) ϕ is *dominating* (i.e. its image is dense in W) iff ϕ^* is injective.

(b) ϕ is an isomorphism onto a subvariety of W iff ϕ^* is surjective.

(4) Using the universal property of tensor product, show that for A-modules M, N, P, there exists an isomorphism

$$\mathrm{Hom}_A(M \otimes_A N, P) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P)).$$

(5) Suppose M is an A-module, N is an (A, B)-bimodule and P is a B-module. Show that $N \otimes_B P$ is an A-module, and $M \otimes_A N$ is a B-module. Using the universal property of tensor product, prove that

$$(M \otimes_A N) \otimes_B P \cong M \otimes_A (N \otimes_B P).$$

(6) Let M, N be A-modules, I an ideal of A and S a multiplicative subset of A. Using the result from Question 5, or otherwise, prove that

(a) $A/I \otimes_A (M \otimes_A N) \cong M/IM \otimes_{A/I} N/IN$,

(b) $S^{-1}(M \otimes_A N) \cong S^{-1}M \otimes_{S^{-1}A} S^{-1}N$.

(7) Let $\mathcal{F}$ be a presheaf on a topological space X and $\mathcal{G}$ a presheaf on X satisfying sheaf condition (A). If $\phi, \psi : \mathcal{F} \to \mathcal{G}$ are two presheaf morphisms such that $\phi_x = \psi_x : \mathcal{F}_x \to \mathcal{G}_x$ for all $x \in X$, show that $\phi = \psi$.

(8) Show that the sheafification of a presheaf is unique up to isomorphism.

†(9) State, and prove carefully, the universal property satisfied by:

(a) the sum of two sheaves of abelian groups,

(b) the tensor product of two $\mathcal{O}_X$-modules.

Show then that these universal properties determine the sheaves in (a) and (b) up to isomorphism.

†(10) Given a morphism $\phi : \mathcal{F} \to \mathcal{G}$ of sheaves of abelian groups on a topological space, verify the universal properties satisfied by the kernel and cokernel sheaves of ϕ. If now $\phi = \theta\psi$ with $\psi : \mathcal{F} \to \mathcal{H}$ and $\theta : \mathcal{H} \to \mathcal{G}$ morphisms of sheaves, show that there is a unique morphism $\alpha : \operatorname{Coker}\phi \to \operatorname{Coker}\theta$, a surjection making the obvious diagram commute. Deduce that $\operatorname{Im}\phi \subseteq \operatorname{Im}\theta$, and prove that equality holds if and only if α is an isomorphism.

(11) Let U be an open subset of $\mathbf{C}$, and $\mathcal{O}_U$ the sheaf of holomorphic functions, and $\mathcal{O}_U^*$ the multiplicative sheaf of nowhere vanishing holomorphic functions. Define an exact sequence of sheaves $0 \to \mathbf{Z} \to \mathcal{O}_U \to \mathcal{O}_U^* \to 1$, where $\mathbf{Z}$ here denotes the constant sheaf, and where by convention we use a 1 rather than 0 on the right since the operation in $\mathcal{O}_U^*$ is multiplication. [Hint: The exponential function will be needed here.] When is the map of sections $\Gamma(U, \mathcal{O}_U) \to \Gamma(U, \mathcal{O}_U^*)$ surjective?

(12) *Show that the monomials for degree d in $X_0, X_1, \ldots, X_n$ define an embedding $v_d : \mathbf{P}^n \hookrightarrow \mathbf{P}^N$, where $N + 1 = \binom{n+d}{n}$, i.e. v_d defines an isomorphism of $\mathbf{P}^n$ with its image.*

For $V \subset \mathbf{P}^n$ a projective variety, and $F \in k[X_0, \ldots, X_n]$ a homogeneous polynomial of degree $d > 0$, prove that the open subset $V \setminus V^h(F)$ has the natural structure of an affine variety.

(13) We saw in lectures that given a smooth/complex, manifold X, we had a corresponding locally ringed space $(X, \mathcal{O}_X)$ over $\mathbf{R}$, respectively $\mathbf{C}$. Moreover, a smooth/holomorphic map between smooth/complex manifolds gave rise to a morphism between the corresponding locally ringed spaces over $\mathbf{R}$, respectively $\mathbf{C}$; show that all morphisms of the locally ringed spaces over $\mathbf{R}$, respectively $\mathbf{C}$, arise in this way.

(14) (Extended exercise) Let $V \subset \mathbf{P}^n$ and $W \subset \mathbf{P}^m$ be (irreducible) projective varieties over an algebraically closed field k. We can define a *rational map* $\phi : V - \to \mathbf{P}^m$ to be given by an $(m+1)$-tuple $(h_0 : h_1 : \ldots : h_m)$ of elements of $k(V)$, not all zero, subject to two $(m+1)$-tuples giving the same rational map if they are multiples of each other by some non-zero $h \in k(V)$. A rational map ϕ is said to be *regular* at a point $P \in V$ if there exists such a representation with h_j regular at P for all j, with at least one $h_j(P) \neq 0$. A *regular map* $\phi : V \to W$ is then defined to be a rational map to $\mathbf{P}^m$, which is regular at all points $P \in V$ with $\phi(P) \in W$. Show that this concept of regular map is equivalent to that of a morphism of projective varieties as defined in lectures.