Monochromatic Infinite Sumsets

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Abstract

We show that there is a rational vector space V such that, whenever V is finitely coloured, there is an infinite set X whose sumset $X+X$ is monochromatic. Our example is the rational vector space of dimension $\sup\{\aleph_0, 2^{\aleph_0}, 2^{2^{\aleph_0}}, \dots\}$. This complements a result of Hindman, Leader and Strauss, who showed that the result does not hold for dimension below $\aleph_\omega$. So our result is best possible under GCH.

1 Introduction

It is a well-known consequence of Ramsey's theorem that, whenever the naturals are finitely coloured, there is an infinite set X such that all pairwise sums of distinct elements of X have the same colour. If one asks for a stronger conclusion, that the entire sumset $X + X = \{x + y : x, y \in X\}$ is monochromatic, then the answer is no: this is because such a sumset automatically contains two numbers with one roughly twice the other, and this can easily be ruled out by a suitable 3-colouring (see e.g. [3]).

We mention in passing that it is, surprisingly, unknown as to whether or not this can be achieved with a 2-colouring: this is called Owings' problem [5]. For background on this, and other results mentioned in this introduction, see [4] – although we mention that this paper is self-contained and does not rely on any results from [4].

What happens if one passes to a larger ambient space, for example the rationals? Here again, the answer is no: there is a finite colouring of $\mathbb{Q}$ with no infinite sumset monochromatic (see e.g. [4]). What about for the reals?

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Hindman, Leader and Strauss [4] showed that, for every rational vector space of dimension smaller that $\aleph_\omega$, there is a finite colouring without an infinite monochromatic sumset. Note that this establishes the answer for the reals if we assume CH. (It is still unknown if the reals have such a bad colouring if we do not make extra set-theoretic assumptions.) However, they were unable to find a vector space with the positive property (of having no bad colourings).

Our aim in this paper is to show that such a vector space does exist. We show that this is the case for any dimension that is at least $\beth_\omega$ (read 'beth-omega'), which is defined to be $\sup\{\aleph_0, 2^{\aleph_0}, 2^{2^{\aleph_0}}, \dots\}$. Note that if we assume GCH then this is exactly $\aleph_\omega$, which would be best possible in light of the result of [4]. We do not know if the vector space of dimension $\aleph_\omega$ has this property if we do not assume GCH.

We also prove a similar result for multiple sums such as $X + X + X$ and so on. The proof involves a perhaps unexpected use of the Hales-Jewett theorem.

For a finite or infinite cardinal κ, we write $\mathbb{Q}^\kappa$ to denote the vector space of dimension κ over $\mathbb{Q}$. That is, $\mathbb{Q}^\kappa$ is the direct *sum* of κ copies of $\mathbb{Q}$, not the direct product. We shall take $\mathbb{Q}^\kappa$ to come equipped with a basis e_0, e_1, e_2, ... that is well-ordered by the smallest ordinal of cardinality κ.

2 Main Result

Consider $\mathbb{Q}^{\beth_\omega}$, the $\beth_\omega$-dimensional vector space over $\mathbb{Q}$. As remarked above, we shall consider $\mathbb{Q}^{\beth_\omega}$ to come equipped with a well-ordered basis B whose elements we shall denote by e_0, e_1, e_2, ...

Suppose $x \in \mathbb{Q}^{\beth_\omega}$ with $x \neq 0$. We may write x in terms of the basis B and delete all zero entries to obtain a finite list of non-zero rationals. We call this list the *pattern* of x. More formally, given a non-zero $x \in \mathbb{Q}^{\beth_\omega}$, there is a unique way to express x in the form $x = \sum_{i=1}^n x_i e_{\alpha_i}$ where n is a positive integer, each x_i is a non-zero rational and $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ are ordinals. The *pattern* of x is $(x_1, x_2, \dots, x_n)$. We shall often denote the pattern $(x_1, x_2, \dots, x_n)$ simply by $x_1 x_2 \dots x_n$. We say that the pattern $x_1 x_2 \dots x_n$ has *length* n and write $\ell(x_1 x_2 \dots x_n) = n$.

Given a finite colouring of $\mathbb{Q}^{\beth_\omega}$, we seek an infinite set $X \subset \mathbb{Q}^{\beth_\omega}$ with $X + X$ monochromatic. There are two stages to the proof.

We first show (Lemma 1) that, given a finite set Π of patterns, there is a large subspace of $\mathbb{Q}^{\beth_\omega}$ on which the colour of an x with pattern in Π depends only on the pattern. The subspace produced is spanned by a subset of the original basis B of $\mathbb{Q}^{\beth_\omega}$. This part of the proof is a fairly standard application of the Erdős-Rado theorem [1].

The heart of the proof comes in the second stage. The main obstacle to overcome is to determine how we should proceed following the reduction given by Lemma 1. That is to say, which patterns should we consider and how do we force all the elements of $X+X$ to have the desired pattern or patterns? While we are able to work within a subspace spanned by a countable subset $A \subset B$, it is interesting to note that our proof often requires this subset A to have an order-type greater than ω. We therefore ask the subspace produced in Lemma 1 to have dimension $\aleph_1$; this allows us to always find A as required.

We now proceed to the first of the two stages detailed above. First, we recall the Erdős-Rado theorem. As usual, we denote by $\exp_r(\kappa)$ the r-fold exponential of κ, i.e. $\exp_0(\kappa) = \kappa$ and $\exp_{r+1}(\kappa) = 2^{\exp_r(\kappa)}$.

Erdős-Rado theorem ([1]). *Let r be a non-negative integer and let κ be an infinite cardinal. Suppose the $(r+1)$-element subsets of a set of cardinality $\exp_r(\kappa)^+$ are coloured with κ colours. Then there is a subset of cardinality κ^+ all of whose $(r+1)-$element subsets are the same colour.*

In particular, this immediately implies that for every positive integer r, if the r-element subsets of a set of cardinality $\beth_\omega$ are coloured with finitely many colours then there is a subset of cardinality $\aleph_1$ all of whose r-element subsets are the same colour.

Lemma 1. *Let k be a positive integer and suppose $\mathbb{Q}^{\beth_\omega}$ is k-coloured. Let Π be a finite set of patterns. Then there is a subset $A \subset B$ of cardinality $\aleph_1$ such that for each $\pi \in \Pi$ the set*

$$\{x \in \mathbb{Q}^{\beth_\omega} : x \text{ is in the span of } A \text{ and has pattern } \pi\}$$

is monochromatic.

Proof. Let c be the given k-colouring of $\mathbb{Q}^{\beth_\omega}$.

Let r be the length of the longest pattern in Π. Let

$$\Pi' = \{\underbrace{00\ldots0}_{r-\ell(\pi)}\pi : \pi \in \Pi\}.$$

Write $\Pi' = \{\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(n)}\}$. We define n k-colourings c_1, c_2, $\ldots$, c_n of the r-element subsets of B as follows. Given $S \subset B$ with $|S| = r$, write $S = \{e_{\alpha_1}, e_{\alpha_2}, \ldots, e_{\alpha_r}\}$ with $\alpha_1 < \alpha_2 < \cdots < \alpha_r$. Then set

$$c_i(S) = \sum_{j=1}^{r} \pi_j^{(i)} e_{\alpha_j}.$$

Now define a single k^n-colouring c' of the r-element subsets of B by

$$c'(S) = (c_1(S), c_2(S), \ldots, c_n(S)).$$

We apply the Erdős-Rado theorem to this final colouring c' to obtain $A' \subset B$ with all r-subsets of A' the same colour and $|A'| = \aleph_1$. Removing the r least elements of A', we obtain our set A as required. □

We are now ready to proceed to the main part of the proof.

Theorem 2. *Let k be a positive integer, and suppose $\mathbb{Q}^{\beth_\omega}$ is k-coloured. Then there is an infinite set $X \subset \mathbb{Q}^{\beth_\omega}$ such that the sumset $X + X$ is monochromatic.*

Proof. Let c be the given k-colouring of $\mathbb{Q}^{\beth_\omega}$.

For $a = 0, 1, 2, \ldots, k$, let π_a be the pattern

$$\pi_a = \underbrace{22\ldots2}_{a}\underbrace{11\ldots1}_{2(k-a)}$$

and let $\Pi = \{\pi_a : 0 \leqslant a \leqslant k\}$. By Lemma 1, we can find $A \subset B$ with $|A| = \aleph_1$ and colours c_a $(0 \leqslant a \leqslant k)$ such that if x is in the span of A and has pattern π_a then $c(x) = c_a$. By the pigeonhole principle, we must have $c_a = c_b$ for some a and b with $0 \leqslant a < b \leqslant k$.

Let C be a subset of A of order-type $\alpha = \omega(b-a+2)$ and list the elements of C in order as f_0, f_1, f_2, $\ldots$.

Now let $X = \{x_i : i < \omega\}$, where, for each $i < \omega$, we define

$$x_i = \sum_{r=0}^{a-1} f_r + \sum_{r=1}^{b-a} f_{\omega r+i} + \sum_{r=0}^{2(k-b)-1} \frac{1}{2} f_{\omega(b-a+1)+r}.$$

Then for all i, $j \in \mathbb{N}$, we observe that $x_i + x_j$ has pattern π_a or π_b according as $i \neq j$ or $i = j$. Thus $X + X$ is monochromatic, as claimed. □

3 Extensions

There are two obvious directions in which one might seek to extend Theorem 2.

First, what if instead of simply requiring that X be infinite, we seek an X of cardinality $\aleph_1$, say, or of some larger specified cardinality? This is possible if we start with a vector space of sufficiently large cardinality, and requires only a trivial modification to the proof of Theorem 2.

Theorem 3. *Let k be a positive integer and let κ be an infinite cardinal. Then there is an infinite cardinal λ such that whenever the λ-dimensional rational vector space $\mathbb{Q}^\lambda$ is k-coloured, there is a subset $X \subset \mathbb{Q}^\lambda$ with $|X| = \kappa$ and $X + X$ monochromatic.*

Indeed, with a similar application of the Erdős-Rado theorem as above, we may take

$$\lambda = \sup\{\kappa, 2^\kappa, 2^{2^\kappa}, \dots\}.$$

More interestingly, what if rather than simply looking for the sumset $X + X$ we seek a monochromatic sum of many copies of X? For example, define the *triple sumset* of X to be

$$X + X + X = \{x + y + z : x, y, z \in X\}.$$

If we finitely colour $\mathbb{Q}^{\beth_\omega}$, can we always find an infinite $X \subset \mathbb{Q}^{\beth_\omega}$ with $X + X + X$ monochromatic?

Let us first consider informally how one might try to extend the proof of Theorem 2 to deal with this problem. Previously, we split our basis vectors into "stretches" of length ω. Depending on the colouring, we then defined each x_i to either take value $\frac{1}{2}$ or 1 on certain fixed coordinates in the stretch (a "fixed stretch"), or we defined each x_i to take value 1 on coordinate i of the stretch and 0 elsewhere (a "variable stretch"). This resulted in $x_i + x_j$ always having a pattern consisting of 1's and 2's. More precisely, the pattern on a given fixed stretch is always the same, whereas the pattern on a variable stretch could be either 11 or 2.

Now, suppose we consider $x_h + x_i + x_j$ with a similar definition of the x_i. The variable stretches will now have pattern 111 or 21 or 12 or 3. To deal with this, it turns out that we need a somewhat unexpected application of the Hales-Jewett Theorem [2].

Theorem 4. *Let k and t be positive integers and suppose $\mathbb{Q}^{\beth_\omega}$ is k-coloured. Then there is an infinite set $X \subset \mathbb{Q}^{\beth_\omega}$ such that $\underbrace{X + X + \cdots + X}_{t}$ is monochromatic.*

Proof. Let c be the given k-colouring of $\mathbb{Q}^{\beth_\omega}$.

Let Π be the set of all patterns of the form $x_1 x_2 \dots x_n$ where $x_1, x_2, \dots, x_n$ are positive integers summing to t. Note that Π is finite. Let N be a positive integer such that whenever Π^N is k-coloured it contains a monochromatic combinatorial line. (Such N exists by the Hales-Jewett Theorem.) Let Π' be the set of patterns obtained by concatenating N patterns from Π.

By Lemma 1, there exist a subset $A \subset B$ with $|A| = \aleph_1$ and colours c_π ($\pi \in \Pi'$) such that if x is in the span of A and has pattern π then $c(x) = c_\pi$.

We induce a colouring of Π^N by giving $(\pi_1, \ldots, \pi_N) \in \Pi^N$ the colour of any x in the span of A with pattern $\pi_1 \pi_2 \ldots \pi_N$. (Note that this does not depend on the choice of x).

We may now find a monochromatic combinatorial line L in Π^N. Let J be the set of active coordinates of L and, for each $\pi \in \Pi$, let I_π be the set of inactive coordinates where L takes constant value π. (Note that we take our coordinates to range from 0 to $N-1$.)

Let C be a subset of A of order-type ωN and list the elements of C in order as f_0, f_1, f_2, ... Let $X = \{x_i : i < \omega\}$ where

$$x_i = \sum_{r \in J} f_{\omega r + i} + \sum_{\pi \in \Pi} \sum_{r \in I_\pi} \sum_{s=1}^{\ell(\pi)} \frac{\pi_s}{t} f_{\omega r + s}.$$

Then each element of $\underbrace{X + X + \cdots + X}_{t}$ has pattern in L and thus $\underbrace{X + X + \cdots + X}_{t}$ is monochromatic. □

We remark that, exactly as the proof of Theorem 2 was adapted to yield Theorem 3, we may similarly adapt the proof of Theorem 4 to give:

Theorem 5. *Let k and t be positive integers and let κ be an infinite cardinal. Then there is an infinite cardinal λ such that whenever $\mathbb{Q}^\lambda$ is k-coloured there is an infinite set $X \subset \mathbb{Q}^\lambda$ with $|X| = \kappa$ and $\underbrace{X + X + \cdots + X}_{t}$ monochromatic.*

As with Theorem 3, it suffices to take

$$\lambda = \sup\{\kappa, 2^\kappa, 2^{2^\kappa}, \ldots\}.$$

References

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