Transitive Sets in Euclidean Ramsey Theory

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Abstract

A finite set X in some Euclidean space $\mathbb{R}^n$ is called *Ramsey* if for any k there is a d such that whenever $\mathbb{R}^d$ is k-coloured it contains a monochromatic set congruent to X. This notion was introduced by Erdős, Graham, Montgomery, Rothschild, Spencer and Straus, who asked if a set is Ramsey if and only if it is *spherical,* meaning that it lies on the surface of a sphere. This question (made into a conjecture by Graham) has dominated subsequent work in Euclidean Ramsey theory.

In this paper we introduce a new conjecture regarding which sets are Ramsey; this is the first ever 'rival' conjecture to the conjecture above. Calling a finite set *transitive* if its symmetry group acts transitively—in other words, if all points of the set look the same—our conjecture is that the Ramsey sets are precisely the transitive sets, together with their subsets. One appealing feature of this conjecture is that it reduces (in one direction) to a purely combinatorial statement. We give this statement as well as several other related conjectures. We also prove the first non-trivial cases of the statement.

Curiously, it is far from obvious that our new conjecture is genuinely different from the old. We show that they are indeed different by proving that not every spherical set embeds in a transitive set. This result may be of independent interest.

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1 Introduction

Euclidean Ramsey theory originates in the sequence of papers [2], [3] and [4] by Erdős, Graham, Montgomery, Rothschild, Spencer and Straus. A finite set X in some Euclidean space $\mathbb{R}^n$ is said to be *Ramsey* if for every positive integer k there exists a positive integer d such that whenever $\mathbb{R}^d$ is k-coloured it must contain a monochromatic subset congruent to X. For example, it is easy to see that (the set of vertices of) the r-dimensional regular simplex is Ramsey: $\mathbb{R}^{kr}$ contains a collection of $kr+1$ points with each pair at distance 1, and whenever $\mathbb{R}^{kr}$ is k-coloured some $r+1$ of these points must be the same colour. On the other hand, the subset $\{0,1,2\}$ of $\mathbb{R}$ is not Ramsey. To see this, observe that a copy of $\{0,1,2\}$ is simply a collection of three collinear points x, y, $z \in \mathbb{R}^n$ with $\|z-x\| = 2$ and $y = \frac{1}{2}(x+z)$. It follows from the parallelogram law that $\|y\|^2 = \frac{1}{2}(\|x\|^2 + \|z\|^2) - 1$. It is now easy to write down a 4-colouring of $\mathbb{R}^n$ with no monochromatic copy of $\{0,1,2\}$ by colouring each point u according to its distance $\|u\|$ from the origin—for example by $c(u) = \lfloor \|u\|^2 \rfloor \pmod 4$.

So which finite sets are Ramsey? This question was first considered by Erdős, Graham, Montgomery, Rothschild, Spencer and Straus in [2]. They showed that any Ramsey set must be *spherical;* that is, it must be contained in the surface of a sphere. Their proof can be viewed as a (much more difficult) extension of the proof above that $\{0,1,2\}$ is not Ramsey. So the key question remaining was: is *every* spherical set Ramsey? It has been widely believed for some time that this is in fact the case—indeed, Graham conjectured this in [7] and offered \$1000 for its proof.

Various cases of this conjecture have been proved. In the original paper [2] it is shown, by means of a product argument, that if X and Y are Ramsey then so is $X \times Y$. (Here if $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ then we regard $X \times Y$ as a subset of $\mathbb{R}^{n+m}$.) In particular, it follows that any *brick,* meaning the set of vertices of a cuboid in n dimensions, is Ramsey. Further progress has been slow, with each new step requiring significant new ideas. Frankl and Rödl [5] showed that every triangle is Ramsey, and later [6] that every non-degenerate simplex is Ramsey. This left the next interesting case as the regular pentagon, which was shown to be Ramsey by Kříž [11]. Kříž actually showed that any finite set X which is acted on transitively by a soluble group G of isometries is Ramsey and, slightly more generally, that G need not itself be soluble so long as it has a soluble subgroup H whose action on X has at most two orbits. In particular, this implies that all regular polygons are Ramsey and that the Platonic solids in 3 dimensions are Ramsey. In addition, this result was used by Cantwell [1] to show that the 120-cell, the largest regular polytope in 4 dimensions, is Ramsey. However, the conjecture itself is still wide open. Indeed, it is not even known whether or not every cyclic quadrilateral is Ramsey.

Our starting point in this paper is a feature that we have observed to be common to all known proofs that particular sets are Ramsey. In each case, the proof that the set is Ramsey proceeds by first embedding it in a (finite) *transitive* set—a set whose symmetry group acts transitively—and then making some clever combinatorial argument to show that this transitive set has the Ramsey property required.

As an example, let us digress for a moment to see why every triangle embeds into a transitive set. Of course, a right-angled triangle is a subset of a rectangle, and more generally any acute-angled triangle is a subset of a cuboid in 3 dimensions. For a general triangle ABC, consider a variable point D on the perpendicular dropped from C to AB. Choose D such that the angle AOB, where O is the circumcentre of triangle ABD, is a rational multiple of π. It follows that A and B lie on some regular polygon Π (with centre O). Viewing Π as living in the xy-plane, we now form a new copy Σ of Π by translating Π in the z-direction and rotating it about its centre: the resulting 'twisted prism' $\Pi \cup \Sigma$ is transitive and will, if the translation and rotation are chosen correctly, contain an isometric copy of ABC.

We mention that the transitive sets in which Frankl and Rödl [5] embed their triangles are in fact very different. Indeed, the actual machinery for embedding a set into a transitive set and proving this transitive set Ramsey can differ greatly from paper to paper, and the transitive set can have a *much* higher dimension than the original set, but we have noticed that the transitivity is always present. Based on this, and some other facts (discussed below), we are led to the following conjecture which asserts that transitivity is the key property.

Conjecture A. *A finite set $X \subset \mathbb{R}^n$ is Ramsey if and only if it is (congruent to) a subset of a finite transitive set.*

In general, when we say that a set X is transitive we shall assume implicitly that X is finite. For brevity, we shall say that a finite set in $\mathbb{R}^d$ is *subtransitive* if it is congruent to a subset of a transitive set in some $\mathbb{R}^n$. We stress that this transitive set may have higher dimension than the original subtransitive set.

We believe that this is a very natural conjecture for various reasons. To begin with, it turns out that there are several clean statements any of which would imply that all transitive sets are Ramsey. Moreover, some of these statements are purely combinatorial. In the other direction, the transitivity of every Ramsey set would give a clear conceptual reason as to why every Ramsey set must be spherical. Indeed, it is easy to see that the points of any transitive set all lie on the surface of the unique smallest closed ball containing it.

We remark that it is not clear *a priori* that Conjecture A is genuinely different from the old conjecture: could it be that a finite set X is spherical if and only if it is subtransitive? As we remark above, every subtransitive set is spherical, but do there exist spherical sets that are not subtransitive? We show that such sets do indeed exist. This result may be of independent interest. One might hope that the proof of this result would give some insight into showing the existence of a spherical set that is not Ramsey. However, we do not see a way to make this work.

The plan of the paper is as follows. In §2, we consider the 'if' direction of our conjecture: 'every subtransitive set is Ramsey', or, equivalently, 'every transitive set is Ramsey'. Our first step is to remove the geometry, so to say: we show that this direction of Conjecture A would follow from a Hales-Jewett-type statement for groups (Conjecture C). Our next step is to remove the group theory by showing that Conjecture C can be reformulated as another equivalent Hales-Jewett-type statement which is purely combinatorial (Conjecture E). This shows that our conjecture is, in a certain sense, natural—if Conjecture E is true then it will provide a genuine combinatorial reason why every transitive set must be Ramsey. In §3 we prove some of the first non-trivial cases of Conjecture E. This is already enough to yield some new examples of Ramsey sets. In §4, we show the existence of spherical sets that are not subtransitive. More precisely, we show that, for any $k \geqslant 16$, almost every cyclic k-gon is not subtransitive. We remark that our proof is non-constructive: we have no explicit example of such a k-gon. Finally, in §5, we briefly discuss the 'only if' direction of Conjecture A and give some further problems.

For a general overview of Ramsey Theory, see the book of Graham, Rothschild and Spencer [8]. We make use of van der Waerden's theorem [14], and certain formulations of our main conjecture have a similar flavour to the Hales-Jewett theorem [9]; for both of these, see [8]. For further related results and problems, we refer the reader to the original sequence of papers [2], [3] and [4] by Erdős, Graham, Montgomery, Rothschild, Spencer and Straus, and to the later survey [7] of Graham. For previous work on subtransitive sets, see Johnson [10].

Our notation is standard. In particular, for natural numbers m, n with $m \leq n$ we write $[n]$ for the set $\{1, 2, \ldots, n\}$ and $[m, n]$ for the set $\{m, m+1, \ldots n\}$. For any set A, we write $A^{(m)}$ for the collection of subsets of A of order m. If A is a set of integers and j is an integer, we write $A + j$ for the set $\{i + j : i \in A\}$. We write S_n to denote the symmetric group of all $n!$ permutations of $[n]$.

2 Is every transitive set Ramsey?

We now discuss the 'if' direction of Conjecture A: how might we prove that every transitive set is Ramsey?

Refining the notion of Ramsey, we say that a set $Y \subset \mathbb{R}^d$ is *k-Ramsey for X* if any k-colouring of Y yields a subset congruent to X. It follows from a compactness argument that if X is Ramsey then, for each k, there is a finite set Y such that Y is k-Ramsey for X (see [2] or [8]).

As we remarked above, every set known to be Ramsey is proved so by first embedding it in a finite transitive set X. In fact, all such proofs then continue by colouring a large product X^n, or, more precisely, a scaling $\lambda X^n = \{\lambda x : x \in X^n\}$. (For $X \subset \mathbb{R}^m$, we view X^n as a subset of $\mathbb{R}^{mn}$.) This leads us to make the following stronger conjecture.

Conjecture B. *Let $X \subset \mathbb{R}^m$ be a finite transitive set. Then, for any k, there exists an n such that some scaling of X^n is k-Ramsey for X.*

The 'if' direction of Conjecture A would, of course, follow immediately from Conjecture B.

We digress for a moment to comment on a related notion. In their initial paper [2] on Euclidean Ramsey theory, Erdős, Graham, Montgomery, Rothschild, Spencer and Straus define a set X to be *sphere-Ramsey* if for any positive integer k there exists a positive integer d and a positive real number r such that whenever $\{x \in \mathbb{R}^{d+1} : \|x\| = r\}$, the d-dimensional sphere of radius r, is k-coloured it contains a subset congruent to X. It is obvious that any sphere-Ramsey set must be Ramsey. As we observed earlier, it is easy to show that any transitive set is spherical. Hence Conjectures A and B would together imply that a set is sphere-Ramsey if and only if it is Ramsey.

Our aim in the remainder of this section will be to reformulate Conjecture B as an equivalent purely combinatorial statement (Conjecture E). Our first step is to 'remove the geometry' by recasting Conjecture B in terms of the symmetry group of X.

Let G be a group, n a positive integer and $I \subset [n]$. Suppose $\vec{g} = (g_1, g_2, \ldots, g_n) \in G^n$ and $h \in G$. We write $\vec{g} \times_I h$ for the word $(k_1, k_2, \ldots, k_n) \in G^n$ where

$$k_i = \begin{cases} g_i & \text{if } i \notin I \\ g_i h & \text{if } i \in I \end{cases}.$$

As we shall see, Conjecture B is equivalent to the following conjecture.

Conjecture C. *Let G be a finite group. Then for any positive integer k there exist positive integers n and d such that whenever G^n is k-coloured there exists a word $\vec{g} \in G^n$ and a set $I \subset [n]$ with $|I| = d$ such that the set $\{\vec{g} \times_I h : h \in G\}$ is monochromatic.*

The reader familiar with the Hales-Jewett theorem [9] will see some resemblance. Indeed, without the restriction '$|I| = d$', Conjecture C would be an easy consequence of the Hales-Jewett theorem.

We remark that in Conjecture C we cannot insist that $d = 1$. More generally, we cannot even insist that $\vec{g}$ be constant on I. Indeed, to see this, 2-colour G^n by colouring a vector according to whether the number of occurrences of the identity element e lies between 1 and d or between $d+1$ and $2d$ (modulo $2d$).

We next show how Conjecture B may be deduced directly from Conjecture C.

Proposition 2.1. *Conjecture C implies Conjecture B*

Proof. Assume that Conjecture C is true. Let X be a finite transitive set and let k be a positive integer. Let G be the symmetry group of X. By Conjecture C, we may choose n in such a way that whenever G^n is k-coloured there exists a word $\vec{g} \in G^n$ and a set $I \subset [n]$ with $|I| = d$ such that the set $\{\vec{g} \times_I h : h \in G\}$ is monochromatic.

Suppose X^n is k-coloured. We induce a k-colouring of G^n by picking $x \in X$ and giving $(g_1, g_2, \dots, g_n)$ the colour of $(g_1(x), g_2(x), \dots, g_n(x))$. Now choose $I \subset [n]$ with $|I| = d$ and $\vec{g} \in G^n$ such that the set $\{\vec{g} \times_I h : h \in G\}$ is monochromatic. For notational convenience, assume wlog that $I = [d]$, so that the set

$$Y = \{g_1 h(x), \dots, g_d h(x), g_{d+1}(x), \dots, g_n(x) : h \in G\}$$

is monochromatic.

As G acts transitively on X, we have that $X = \{h(x) : h \in G\}$, and so

$$Z = \{(\underbrace{h(x), \dots, h(x)}_{d}, \underbrace{x, \dots, x}_{n-d}) : h \in G\}$$

is a scaling of X by factor $\sqrt{d}$. But $g_1, \dots, g_m$ are isometries and so Y and Z are isometric. Hence $\frac{1}{\sqrt{m}} X^n$ is k-Ramsey for X. □

Conjectures B and C are in fact equivalent, but we have no simple and direct way of deducing Conjecture C from Conjecture B. In the case of a group G which acts as the symmetry group of some transitive set of order $|G|$, Conjecture C may be deduced easily from Conjecture B. But, for example, the cyclic group C_3 does not act as the symmetry group of any 3-point set. The proof of this direction of the equivalence must wait until the end of the current section.

Having successfully removed the geometry, our next task is to 'remove the groups'. One may think of Conjecture C as saying that the varying parts of our m words ($m = |G|$) contain the group table of G, possibly with some columns omitted and some columns repeated. That is, we obtain the m rows of some Latin-square-type pattern. Instead of asking merely for these m rows, we could instead demand all $m!$ permutations of the elements of G. This now gives us a purely combinatorial statement—a 'fixed-block-size' Hales-Jewett-type conjecture.

We first need a preliminary definition. We wish to consider a collection of words of the following form: we fix m blocks, make our words the same as each other outside these blocks, and take each of the $m!$ possible arrangements of 1, 2, ..., m amongst the blocks. Formally, a *block permutation set* in $[m]^n$ is a set B formed in the following way. First, select pairwise disjoint subsets I_1, ..., $I_m \subset [n]$ and elements $a_i \in [m]$ for each $i \notin \bigcup_{j=1}^m I_j$. For each $\pi \in S_m$, define $a^\pi \in [m]^n$ by

$$(a^\pi)_i = \begin{cases} \pi(j) & \text{if } i \in I_j \\ a_i & \text{if } i \notin \bigcup_{j=1}^m I_j \end{cases} .$$

Now set $B = \{a^\pi : \pi \in S_m\} \subset [m]^n$.

If $\sum_{j=1}^m |I_j| = d$ then we say that B is of *degree* d. We sometimes refer to the sets $I_1, I_2, \ldots, I_m$ as *blocks*.

We remark that a block permutation set need not contain precisely $m!$ elements: some of the blocks could be empty.

Conjecture D. *Let m and k be positive integers. Then there exist positive integers n and d such that whenever $[m]^n$ is k-coloured it contains a block permutation set of degree d.*

Again, note that if the degree condition were omitted then this would follow easily from the Hales-Jewett theorem. Indeed, any block permutation set in $[m]^n$ is contained in some m-dimensional combinatorial subspace (m-parameter set). Furthermore, as with Conjecture C, we cannot require each block to have size 1.

It is clear that Conjecture C follows from Conjecture D. While Conjecture D appears much stronger, it is in fact equivalent to Conjecture C, as we now show. (In fact, this follows from later results; we include it here because the proof is concise and direct.)

Proposition 2.2. *Conjectures C and D are equivalent.*

Proof. For the non-trivial direction of the implication, assume Conjecture C is true, and let m and k be positive integers. We apply Conjecture C to the symmetric group S_m and obtain integers n and d as above. Now suppose that $[m]^n$ is k-coloured. We induce a colouring of S_m^n by giving $(\pi_1, \ldots, \pi_n) \in S_m^n$ the colour of $(\pi_1^{-1}(1), \ldots, \pi_n^{-1}(1)) \in [m]^n$. Now choose $\vec{\pi} = (\pi_1, \ldots, \pi_n) \in S_m^n$ and $I \subset [n]$ with $|I| = d$ such that the set $\{\vec{\pi} \times_I \sigma : \sigma \in S_m\}$ is monochromatic. For simplicity of notation, assume wlog that $I = [d]$. Thus the set

$$\{(\sigma^{-1}\pi_1^{-1}(1), \ldots, \sigma^{-1}\pi_d^{-1}(1), \pi_{d+1}^{-1}(1), \ldots, \pi_n^{-1}(1)) : \sigma \in S_m\} \subset [m]^n$$

is monochromatic. We now take $I_j = \{i \in I : \pi_i^{-1}(1) = j\}$ for each $j = 1, 2, \ldots, m$. □

We now give a further equivalent formulation of this conjecture which we hope might be more amenable to proof.

Conjecture D asks for a collection of words of a certain type all of the same colour. We may think of this collection as being represented by the 'pattern' $12\ldots m$. More generally, we could consider an arbitrary pattern. For example, to realise the pattern 11223 we would demand five blocks, and a word for each way of assigning the symbol 1 to two of the blocks, the symbol 2 to another two of the blocks and the symbol 3 to the remaining block. So in this case we would have a total of 30 words (if all of the blocks were non-empty).

Formally, we define a *template* over $[m]$ to be a non-decreasing word $\tau \in [m]^\ell$ for some ℓ.

Next, we define a *block set with template τ*. The reader should bear in mind our earlier definition of a block permutation set, which is a certain special case: it is a block set with template $12\ldots m$.

Suppose that $\tau \in [m]^\ell$ is a template. For each $j \in [m]$, let c_j be the number of times that the symbol j appears in the template τ; that is, $c_j = |\{i \in [\ell] : \tau_i = j\}|$. Note that $\sum_{j=1}^m c_j = \ell$. We define the set S of *rearrangements* of τ by

$$S = \left\{ \pi \in [m]^\ell : \left|\{i \in [\ell] : \pi_i = j\}\right| = c_j \quad \forall j \in [m] \right\}.$$

A *block set with template τ* in $[m]^n$ is a set B formed in the following way. First, select pairwise disjoint subsets $I_1, \ldots, I_\ell \subset [n]$ and elements $a_i \in [m]$ for each $i \notin \bigcup_{j=1}^\ell I_j$. For each $\pi \in S$, define $a^\pi \in [m]^n$ by

$$(a^\pi)_i = \begin{cases} \pi_j & \text{if } i \in I_j \\ a_i & \text{if } i \notin \bigcup_{j=1}^\ell I_j \end{cases}.$$

Now set $B = \{a^\pi : \pi \in S\} \subset [m]^n$.

As before, if $\sum_{j=1}^{\ell} |I_j| = d$ then we say that B is of *degree* d, and we may refer to the sets I_1, I_2, ..., I_ℓ as *blocks.*

We are now ready to state the conjecture.

Conjecture E. *Let* m *and* k *be positive integers and let* τ *be a template over* $[m]$. *Then there exist positive integers* n *and* d *such that whenever* $[m]^n$ *is* k*-coloured it contains a monochromatic block set of degree* d *with template* τ.

It is easy to see that Conjectures D and E are equivalent. Conjecture D is simply the special case of Conjecture E for the template $12\ldots m$. In the other direction, Conjecture E for the template $\tau_1 \ldots \tau_\ell$ on alphabet $[m]$ follows immediately from Conjecture D on alphabet $[\ell]$. Indeed, given a colouring of $[m]^n$ we use the map $(x_1, \ldots, x_n) \mapsto (\tau_{x_1}, \ldots, \tau_{x_n})$ to induce a colouring of $[\ell]^n$ and then apply Conjecture D.

What can we say about the initial cases of Conjecture E? The case $m = 1$ is trivial. In the next case, $m = 2$, the conjecture is true for all templates by an easy application of Ramsey's theorem, as we shall explain in §3. We shall then prove the first non-trivial cases: templates of the form $1\ldots12\ldots23$.

It turns out that, in all of these cases, the block set we produce is *uniform:* that is, all of its blocks have the same size. This suggests the following conjecture, which perhaps appears more natural.

Conjecture F. *Let* m *and* k *be positive integers and let* τ *be a template over* $[m]$. *Then there exist positive integers* n *and* d *such that whenever* $[m]^n$ *is* k*-coloured it contains a monochromatic* uniform *block set of degree* d *with template* τ.

At first sight, Conjecture F may appear rather stronger than Conjecture E—we have imposed an additional constraint on the monochromatic block set that we seek. However, they do in fact turn out to be equivalent. We do not see how to deduce Conjecture F directly from Conjecture E, but instead go via the geometric Conjecture B.

To prove that all of the conjectures of this section (Conjectures B–F) are equivalent, it now suffices to show that we may deduce Conjecture F from Conjecture B. So, how might we use Conjecture B to deduce Conjecture F for the template $1\ldots m$? One approach would be to 'embed' the template into $\mathbb{R}^m$ as follows.

Suppose $\alpha_1, \alpha_2, \ldots, \alpha_m$ are real numbers. Let X be the set in $\mathbb{R}^m$ of all permutations of the vector $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ and let $Y = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}^{nm}$. Note that $X^n \subset Y$. We shall think of Y as the image of $[m]^{nm}$ in the obvious way.

Certainly X is transitive (S_m acts on it), so by Conjecture B we know that (provided n is sufficiently large) whenever Y is k-coloured it contains a monochromatic set congruent to a fixed scaling sX of X. One way such a set could occur is as the image of an s^2-uniform block set, but of course there may be many other ways as well.

The heart of the proof is to ensure that the *only* subsets of Y congruent to sX are the images of s^2-uniform block sets.

Lemma 2.3. *Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be algebraically independent real numbers and define X and Y as above. Then, for $s > 0$, every subset of Y congruent to sX is the image of an s^2-uniform block set.*

Proof. Let $W \subset Y$ be congruent to sX. As there is an isometry from sX to W, we may find a bijection $\theta \colon X \to W$ with $\|\theta(x) - \theta(x')\| = s\|x - x'\|$ for all x, $x' \in X$. For each permutation π, let $x_\pi = (\alpha_{\pi(1)}, \alpha_{\pi(2)}, \ldots, \alpha_{\pi(m)})$ and let $y_\pi = \theta(x_\pi)$. So $W = \{y_\pi : \pi \in S_m\}$. We denote the identity permutation by e.

We first consider the points y_e and $y_{(12)}$. Since $\|x_e - x_{(12)}\|^2 = 2(\alpha_2 - \alpha_1)^2$ we have that $\|y_e - y_{(12)}\|^2 = 2s^2(\alpha_2 - \alpha_1)^2$. Similarly, $\|y_e - y_{(13)}\|^2 = 2s^2(\alpha_3 - \alpha_1)^2$.

For distinct i, $j \in [mn]$, let λ_{ij} denote the number of coordinates in which y_e takes value α_i but $y_{(12)}$ takes value α_j. Similarly, let μ_{ij} denote the number of coordinates in which y_e takes value α_i but $y_{(13)}$ takes value α_j. Then

$$\|y_e - y_{(12)}\|^2 = \sum_{i,j} \lambda_{ij}(\alpha_i - \alpha_j)^2 \qquad \text{and} \qquad \|y_e - y_{(13)}\|^2 = \sum_{i,j} \mu_{ij}(\alpha_i - \alpha_j)^2$$

and so

$$(\alpha_3 - \alpha_1)^2 \sum_{i,j} \lambda_{ij}(\alpha_i - \alpha_j)^2 = (\alpha_2 - \alpha_1)^2 \sum_{i,j} \mu_{ij}(\alpha_i - \alpha_j)^2.$$

By comparing the coefficients of appropriate monomials in $\alpha_1, \ldots, \alpha_m$, it is easy to check that all of the λ_{ij} and μ_{ij} must be zero, except for λ_{12} and μ_{13} which must be equal. (For example considering the coefficient of α_3^4 and using the fact that $\lambda_{ij} \geq 0$ for all i, j reveals that $\lambda_{ij} = 0$ if either i or j is 3; then considering the coefficient of $\alpha_3^2\alpha_i\alpha_j$ gives that $\lambda_{ij} = 0$ unless $(i, j) = (1, 2)$.) In other words, this means that when changing from y_e to $y_{(12)}$, all that happens is that some coordinates change from α_1 to α_2 and some other coordinates change from α_2 to α_1 Moreover, as $\lambda_{12} = 2s^2$, we see that the total number of coordinates that change is $2s^2$.

More generally, the same argument shows that, for any permutation π and transposition (ij), to change from y_π to $y_{\pi(ij)}$ it is only necessary to change some coordinates from $\alpha_{\pi(i)}$ to $\alpha_{\pi(j)}$ and some other coordinates from $\alpha_{\pi(j)}$ to $\alpha_{\pi(i)}$. Let $U_\pi(ij)$ be the set of coordinates which change from $\alpha_{\pi(i)}$ to $\alpha_{\pi(j)}$

when one changes y_π into $y_{\pi(ij)}$. Again as above, we have $|U_\pi(ij)|+|U_\pi(ji)| = 2s^2$ for all π, i and j. As the set $U_\pi(ij)$ is a subset of the coordinates in y_π which equal $\alpha_{\pi(i)}$, we have that, for fixed π and j, the sets $U_\pi(ij)$ are pairwise disjoint as i varies. Moreover, for distinct i and j the sets $U_\pi(ij)$ and $U_\pi(ji)$ are disjoint. Of course, for fixed π and i, the sets $U_\pi(ji)$ could intersect (and, indeed, we shall show that not only do they intersect but, in fact, they are identical).

Now, consider the distance $\|y_{(ij)\pi} - y_{(ik)\pi}\|^2$. The vectors $x_{\pi(ij)}$ and $x_{\pi(ik)}$ are equal in all coordinates except for coordinates i, j and k. In these coordinates the vector $x_{\pi(ij)}$ takes values $\alpha_{\pi(j)}$, $\alpha_{\pi(i)}$ and $\alpha_{\pi(k)}$ respectively; while the vector $x_{\pi(jk)}$ takes the values $\alpha_{\pi(k)}$, $\alpha_{\pi(j)}$, and $\alpha_{\pi(i)}$ respectively. Hence

$$\|x_{\pi(ij)} - x_{\pi(ik)}\|^2 = \left((\alpha_{\pi(i)} - \alpha_{\pi(j)})^2 + (\alpha_{\pi(j)} - \alpha_{\pi(k)})^2 + (\alpha_{\pi(k)} - \alpha_{\pi(i)})^2\right)$$

and so

$$\|y_{\pi(ij)} - y_{\pi(ik)}\|^2 = s^2 \left((\alpha_{\pi(i)} - \alpha_{\pi(j)})^2 + (\alpha_{\pi(j)} - \alpha_{\pi(k)})^2 + (\alpha_{\pi(k)} - \alpha_{\pi(i)})^2\right).$$

An alternative way to calculate this distance is to consider explicitly how the vector $y_{\pi(ij)}$ differs from the vector $y_{\pi(ik)}$. Imagine that we first change $y_{\pi(ij)}$ to y_π and then to $y_\pi(jk)$. When going from $y_{\pi(ij)}$ to y_π the coordinates in $U_\pi(ij)$ change from $\alpha_{\pi(j)}$ to $\alpha_{\pi(i)}$ and the coordinates in $U_\pi(ji)$ change from $\alpha_{\pi(i)}$ to $\alpha_{\pi(j)}$. Then when going from y_π to $y_{\pi(ik)}$ the coordinates in $U_\pi(ik)$ change from $\alpha_{\pi(i)}$ to $\alpha_{\pi(k)}$ and the coordinates in $U_\pi(ki)$ change from $\alpha_{\pi(k)}$ to $\alpha_{\pi(i)}$. Hence

$$\begin{aligned}\|y_{\pi(ij)} - y_{\pi(ik)}\|^2 = {} & |U_\pi(ji)|(\alpha_{\pi(j)} - \alpha_{\pi(i)})^2 + |U_\pi(ki)|(\alpha_{\pi(k)} - \alpha_{\pi(i)})^2 \\ & + |U_\pi(ij) \setminus U_\pi(ik)|(\alpha_{\pi(j)} - \alpha_{\pi(i)})^2 \\ & + |U_\pi(ik) \setminus U_\pi(ij)|(\alpha_{\pi(k)} - \alpha_{\pi(i)})^2 \\ & + |U_\pi(ij) \cap U_\pi(ik)|(\alpha_{\pi(j)} - \alpha_{\pi(k)})^2.\end{aligned}$$

We now have two expressions for $\|y_{\pi(ij)} - y_{\pi(jk)}\|^2$, each of which is a polynomial in $\alpha_1, \ldots, \alpha_m$ with rational coefficients (recall that $2s^2$ is an integer). Comparing coefficients of $\alpha_{\pi(j)}\alpha_{\pi(k)}$ gives that $|U_\pi(ij) \cap U_\pi(ik)| = s^2$. In particular for every π and every i, j we have that $|U_\pi(ij)| \geq s^2$ and thus, as $|U_\pi(ij)| + |U_\pi(ji)| = 2s^2$, that $|U_\pi(ij)| = s^2$ for all π, i and j. Therefore the set $U_\pi(ij)$ is independent of j.

Finally, it follows easily from the definition that $U_\pi(ij) = U_{\pi(ij)}(ij)$. Hence, for any $\ell \neq i$, we have $U_{\pi(i\ell)}(ij) = U_{\pi(i\ell)}(i\ell) = U_\pi(i\ell) = U_\pi(ij)$. But S_n is generated by the transpositions of the form $(i\ell)$ for $\ell \in [m] \setminus \{i\}$, so in fact $U_\pi(ij)$ is independent of π (in addition to being independent of j). We may thus define $I_i = U_\pi(ij)$. It is now clear that $\{y_\pi : \pi \in S_k\}$ is exactly the image of an s^2-uniform block set with blocks $I_1, I_2, \ldots, I_k$. □

Proposition 2.4. *Conjecture B implies Conjecture F.*

Proof. Suppose Conjecture B holds. We shall deduce Conjecture F for the template $1 \dots m$: the full Conjecture F then follows exactly as Conjecture E follows from Conjecture D.

Fix k and let $\alpha_1, \dots, \alpha_m$ be algebraically independent real numbers. Form the set X as in Lemma 2.3. By Conjecture B there are n and s such that X^n is k-Ramsey for sX. Let Y be as defined in Lemma 2.3.

Suppose that $[m]^{mn}$ is k-coloured. This induces a colouring of Y, and Y contains X^n. Hence Y contains a monochromatic copy of sX, and by Lemma 2.3 this is exactly the image of an s^2-uniform block set with template $1 \dots m$. □

3 The first cases of Conjecture E

In this section, we consider some small cases of Conjecture E. As we mentioned in §2, the case $m = 1$ is trivial. In the case $m = 2$ the conjecture is true for all templates by an easy application of Ramsey's theorem, as we now explain.

Consider the template $\underbrace{1 \dots 1}_{r}\underbrace{2 \dots 2}_{s}$ and let $k \in \mathbb{N}$. By Ramsey's theorem, there exists a positive integer n such that whenever $[n]^{(s)}$ is k-coloured it has a monochromatic subset of size $r + s$. Suppose now $[2]^n$ is k-coloured. We induce a k-colouring of $[n]^{(s)}$ by giving $A \in [n]^{(s)}$ the colour of the word $a^A \in [2]^n$ defined by

$$(a^A)_i = \begin{cases} 1 & \text{if } i \notin A \\ 2 & \text{if } i \in A \end{cases} .$$

Let $B \in [n]^{(r+s)}$ be monochromatic. Now take $I_1, I_2, \dots, I_{r+s}$ to be the singleton subsets of B and $a_i = 1$ for all $i \notin B$, giving our monochromatic block set.

We now prove the first non-trivial cases of Conjecture E. In a sense, the first non-trivial case corresponds to the template 123. In fact, we prove the stronger result that the conjecture holds for all templates of the form $1 \dots 12 \dots 23$. Note that in what follows, the proof can be simplified for templates with only one 1, i. e. those of the form $12 \dots 23$: in this case, the application of van der Waerden's theorem is replaced by the pigeonhole principle.

Theorem 3.1. *Conjecture E is true for $m = 3$ and templates of the form* $\underbrace{1 \dots 1}_{r}\underbrace{2 \dots 2}_{s}3$.

Proof. For the sake of definiteness, we begin by fixing the values of certain parameters.

Let $\ell = r + s + 1$.

By van der Waerden's theorem, there exists a positive integer a such that whenever $[0, a-1]$ is k-coloured there exists a monochromatic arithmetic progression or length $r+1$. Let $t = a!$ and $d = \ell t$.

By Ramsey's theorem, there exists a positive integer b such that whenever $[b]^{(t)}$ is k^a-coloured, there exists a monochromatic subset of order d. Let $u = b + a - 1$ and $v = u + tr$

By Ramsey's theorem again, there exists a positive integer n such that whenever $[n]^{(u)}$ is $k^{\binom{u}{t}}$-coloured there exists a monochromatic subset of order v.

Now suppose $[3]^n$ is k-coloured, say by $c_1 \colon [3]^n \to [k]$. We shall consider only the set $A \subset [3]^n$ of those words containing precisely t 3's and $u-t$ 2's.

Let $B \subset [2]^n$ be the set of words of length n containing u 2's and $n-u$ 1's, and let $C \subset \{2,3\}^u$ be the collection of words of length u containing t 3's and $u-t$ 2's. There is an obvious bijection $\theta \colon B \times C \to A$: define $\theta(w,x)$ to be the word obtained by replacing the u 2's in w by the letters of the word x (in order). We may thus induce a $k^{\binom{u}{t}}$-colouring c_2 of B by the complete colouring of $\{(w,x) : x \in C\}$. Formally, let $C = \{x_1, \ldots, x_{\binom{u}{t}}\}$, and define $c_2 \colon B \to [k]^{\binom{u}{t}}$ by $c_2(w) = (c_1(\theta(w,x_1)), \ldots, c_1(\theta(w, x_{\binom{u}{t}})))$.

Similarly to the case $m = 2$, this yields a $k^{\binom{u}{t}}$-colouring c_3 of $[n]^{(u)}$: we give a set $X \in [n]^{(u)}$ the colour of the word $w^X \in B \subset [2]^n$ with

$$(w^X)_i = \begin{cases} 1 & \text{if } i \notin X \\ 2 & \text{if } i \in X \end{cases} .$$

By definition of n, there is a c_3-monochromatic subset $D \subset [n]$ of order v. For notational simplicity, we assume wlog that $D = [v]$.

What we have proved is that for words $w \in A$ where all 2's and 3's are contained within the first v positions, the colour $c_1(w)$ depends only on the relative ordering of the 2's and the 3's: 'the positions of the 1's do not matter'. Thus we may induce a k-colouring c_4 of the subset $E \subset \{2,3\}^u$ of words with precisely t 3's. Formally, we define $c_4(w) = c_1(w')$ where w' is the word consisting of w followed by $n-u$ 1's.

We next induce a k^a-colouring c_5 of $[b]^{(t)}$ by colouring the set $R \in [b]^{(t)}$ according to the colours of the following a words: the word the positions of whose 2's form the set R, the word the positions of whose 2's form the set $R+1$, the word the positions of whose 2's form the set $R+2$, and so on. That is, we define $c_5 \colon [b]^{(t)} \to [k]^a$ by $c_5(R) = (c_4(w^{R,0}), \ldots, c_4(w^{R,a-1}))$

where $w^{R,j} \in E$ is defined by

$$(w^{R,j})_i = \begin{cases} 2 & \text{if } i-j \notin R \\ 3 & \text{if } i-j \in R \end{cases}.$$

By definition of b, we may extract a c_5-monochromatic subset $F \subset [b]$ of order d.

This now induces a k-colouring c_6 of $[0, a-1]$ by $c_6(j) = c_5(R+j)$ where $R \in F^{(t)}$. (Note that this does not depend on the choice of R.) So by definition of a, there is a monochromatic arithmetic progression of length $r+1$, say $p-rq$, $p-(r-1)q$, $p-(r-2)q$, $\ldots$, p.

Write $F = \{i_1, i_2, \ldots, i_d\}$ with $i_1 < i_2 < \cdots < i_d$. For $1 \le j \le \ell$, let

$$I_j = \{p + i_{(\lambda\ell+j-1)q+\mu} + rq\lambda : 0 \le \lambda < \frac{t}{q}, 1 \le \mu \le q\}.$$

Since $t = a!$ and $q < a$, we must have that q is a factor of t, and so $|I_j| = t$ for each j. Furthermore, the largest element appearing in any of the I_j is $p + i_d + r(t-q) < a - 1 + b + rt = v$, so each $I_j \subset [v]$.

For $i \notin \bigcup_{j=1}^{\ell} I_j$, let

$$a_i = \begin{cases} 2 & \text{if } i \le v \\ 1 & \text{if } i > v \end{cases}.$$

Let S be the set of rearrangements of the template $\underbrace{1\ldots1}_{r}\underbrace{2\ldots2}_{s}3$. It remains to check that for each $\pi \in S$ the words a^π all have the same colour.

So let $\pi \in S$. It is clear that a^π contains t 3's and rt 1's, and so $v - tr - t = u - t$ 2's. Moreover, as each $I_j \subset [v]$, all of the 2's and 3's in a^π are contained within the first v positions of a^π. Hence the colour of a^π is determined completely by the relative positions of the 2's and 3's.

Write I for the set of positions of the 3's amongst the 2's and 3's in a^π (so $I \in [u]^{(t)}$). As our template contains only one 3, there is a unique $j \in [\ell]$ such that $\pi_j = 3$. Let h be the number of 1's appearing to the left of the unique 3 in π. Then

$$\begin{aligned} I &= \{p + i_{(\lambda\ell+j-1)q+\mu} + rq\lambda - (r\lambda + h)q : 0 \le \lambda < \frac{t}{q}, 1 \le \mu \le q\} \\ &= \{p + i_{(\lambda\ell+j-1)q+\mu} - hq : 0 \le \lambda < \frac{t}{q}, 1 \le \mu \le q\} \\ &= X_j + p - hq, \end{aligned}$$

where

$$X_j = \{i_{(\lambda\ell+j-1)q+\mu} : 0 \le \lambda < \frac{t}{q}, 1 \le \mu \le q\} \in F^{(t)}$$

and $0 \le h \le r$. The result follows. $\square$

We have now proved Conjecture E for templates of the form $1 \dots 12 \dots 23$. So the first open case is:

Problem G. *Prove Conjecture E with* $m = 3$ *for the template* 112233.

Finally, we remark that the uniformity of the block sets in our proof of Theorem 3.1 is already enough to yield some new examples of Ramsey sets. Indeed, from the template $\underbrace{1 \dots 1}_{r}\underbrace{2 \dots 2}_{s}3$ we obtain that, for any distinct reals α, β and γ, the set $X \subset \mathbb{R}^{r+s+1}$ consisting of all those points x having r coordinates α, s coordinates β and one coordinate γ is Ramsey. In general, the set X does not satisfy the conditions of Kříž's theorem [11]. However, we do not know whether or not it embeds into a larger set that does.

4 Not all spherical sets are subtransitive

In this section, we show that our conjecture is genuinely different from the old conjecture [7], by showing that there exists a finite spherical set that is not subtransitive. Specifically, we show that if $k \geqslant 16$ then there exists a cyclic k-gon that is not subtransitive.

One natural approach is to aim for a non-constructive proof showing that almost no cyclic k-gon is subtransitive. However, the space of cyclic k-gons has k degrees of freedom, whereas the space of (non-isometric) orbits of a fixed group of isometries of $\mathbb{R}^n$ can have many more. This is the main obstacle that we have to overcome.

For convenience, we consider labelled, oriented k-gons, i. e. we label the vertices $1, 2, \dots, k$ in clockwise order and consider two the same if there is an isometry between them preserving the labels. Now suppose that $x_1 \dots x_k$ is a cyclic k-gon with circumcentre x_0 and circumradius r. Then it is uniquely determined by the ordered k-tuple $(r, \angle x_1x_0x_2, \angle x_1x_0x_3, \dots, \angle x_1x_0x_k)$. This allows us to think of the set of cyclic k-gons as a subset $\mathcal{P} \subset \mathbb{R}^k$ of non-zero (k-dimensional Lebesgue) measure.

We show that, for any $k \geqslant 16$, the set of subtransitive cyclic k-gons has measure zero. The reader is cautioned that, throughout what follows, when we refer to 'orthogonal planes' we use the term in the sense of orthogonal affine subspaces of a real vector space. That is, two planes Π_1 and Π_2 are orthogonal if for all $x_1, y_1 \in \Pi_1$ and $x_2, y_2 \in \Pi_2$ we have $(x_1 - y_1) \perp (x_2 - y_2)$. (So, in particular, it is not possible to find two orthogonal planes in $\mathbb{R}^3$.)

The idea of the proof is as follows. We show that every transitive k-gon can be embedded in some $\mathbb{R}^n$ as $g_1(y) \dots g_n(y)$ for some $y \in \mathbb{R}^n$ and one of countably many different k-tuples $(g_1, \dots, g_k)$ of orthogonal transformations of $\mathbb{R}^n$. So we may assume that n and $(g_1, \ldots, g_k)$ are given. We begin by fixing a 'reference' k-gon $g_1(x) \ldots g_k(x)$. Now, for any $y \in \mathbb{R}^n$, the distances $\|g_i(x) - g_i(y)\|$ $(1 \leq i \leq k)$ are all the same. Hence we would like to show that if $x_1 \ldots x_k$ is a k-gon then the set of k-gons $y_1 \ldots y_k$ with all distances $\|x_i - y_i\|$ the same has dimension strictly less than k. Unfortunately, this is not quite true—there are many ways to construct such a $y_1 \ldots y_k$ in a plane orthogonal to the plane of $x_1 \ldots x_k$. But this difficulty is easily surmounted: instead of fixing a single reference k-gon, we start from a (necessarily finite) maximal pairwise-orthogonal family of k-gons in $\mathbb{R}^n$. The heart of the proof is in the following lemma.

Lemma 4.1. *Let $x_1 \ldots x_k$ be a fixed cyclic k-gon in $\mathbb{R}^n$ with $k \geqslant 16$. Let $\mathcal{Q} \subset \mathcal{P}$ be the set of cyclic k-gons which can be embedded in $\mathbb{R}^n$ as $y_1 \ldots y_k$ in such a way that*

(i) $\|x_1 - y_1\| = \|x_2 - y_2\| = \cdots = \|x_k - y_k\|$; and

(ii) the planes of $y_1 \ldots y_k$ and $x_1 \ldots x_k$ are non-orthogonal.

Then $\mathcal{Q}$ has measure zero.

Proof. We may assume wlog that $n = 5$, as any two non-orthogonal planes in $\mathbb{R}^n$ lie in a 5-dimensional affine subspace.

We parameterize the space $\mathcal{P}'$ of cyclic k-gons in $\mathbb{R}^5$ as follows. First, choose points y_1, y_2, y_3 in general position in $\mathbb{R}^5$. These determine a circle γ, with centre y_0, say, and the cyclic k-gon $y_1 \ldots y_k$ is now determined by the angles $\angle y_1 y_0 y_4$, $\angle y_1 y_0 y_5$, ..., $\angle y_1 y_0 y_k$. So we have $\mathcal{P}' \subset \mathbb{R}^{15+(k-3)} = \mathbb{R}^{12+k}$.

Let $\mathcal{Q}' \subset \mathcal{P}'$ be the set of all possible embeddings $y_1 \ldots y_k$ of cyclic k-gons from $\mathcal{Q}$ into $\mathbb{R}^5$ satisfying (i) and (ii). Our aim is to show that the dimension of $\mathcal{Q}'$ is, in fact, much smaller than $12 + k$.

Suppose $y_1 \ldots y_k \in \mathcal{Q}'$. Let $r = \|y_1 - x_1\|$, let γ be the circle through y_1, y_2 and y_3, and, for each $i \geqslant 2$, let S_i be the 4-sphere with centre x_i and radius r. For each $i \geqslant 2$, we must have $y_i \in S_i$. Moreover, for each $i \geqslant 4$ we must also have $y_i \in \gamma$. But for each i, either $\gamma \cap S_i$ finite or $\gamma \subset S_i$.

We shall prove that in fact $\gamma \subset S_i$ for at most 2 distinct values of $i \geqslant 4$. Assume for a contradiction that $\gamma \subset S_{\ell_1} \cap S_{\ell_2} \cap S_{\ell_3}$ for some $4 \leq \ell_1 < \ell_2 < \ell_3 \leq k$. Then for i, $j \in \{1, 2, 3\}$, we have $\|y_j - x_{\ell_i}\| = r$, and so $y_j \cdot x_{\ell_i} = \frac{1}{2}(\|y_i\|^2 + \|x_{\ell_i}\|^2 - r)$. It follows that if i_1, i_2, j_1, $j_2 \in \{1, 2, 3\}$ then $(x_{\ell_{i_1}} - x_{\ell_{i_2}}) \cdot (y_{j_1} - y_{j_2}) = 0$, i.e. $x_{\ell_{i_1}} - x_{\ell_{i_2}}$ is perpendicular to $y_{j_1} - y_{j_2}$. But this implies that the planes of $x_1 \ldots x_k$ and $y_1 \ldots y_k$ are orthogonal, a contradiction.

Hence $\mathcal{Q}'$ is contained in a finite union of 15-dimensional submanifolds of $\mathbb{R}^{k+12}$ $(15 = 5 + 4 + 4 + 1 + 1)$. So, for example by Sard's theorem [13], $\mathcal{Q}$ has measure zero. □

Theorem 4.2. *For each $k \geqslant 16$, the set of subtransitive cyclic k-gons has measure zero. In particular, there exists a cyclic 16-gon which is not subtransitive.*

Proof. Let $\mathcal{S}$ be the set of subtransitive cyclic k-gons. Suppose $P \in \mathcal{S}$. Then P can be embedded into some $\mathbb{R}^n$ as $y_1 \dots y_k$ in such a way that $y_i = g_i(y)$ ($1 \leq i \leq k$) for some $y \in \mathbb{R}^n$ and $g_1, g_2, \dots, g_k$ elements of the orthogonal group $O(n)$ with $\langle g_1, \dots, g_k \rangle$ finite.

Fix such n and $g_1, g_2, \dots, g_k$. For $y \in \mathbb{R}^n$, write $\vec{g}(y)$ for the k-gon $g_1(y) \dots g_k(y)$. Let $\mathcal{S}'$ be the set of $P \in \mathcal{S}$ which can be embedded into $\mathbb{R}^n$ as $\vec{g}(y) = g_1(y) \dots g_k(y)$. Let $\vec{g}(x_1), \dots, \vec{g}(x_p)$ be a (necessarily finite) maximal family of pairwise-orthogonal embeddings of polygons from $\mathcal{S}'$ into $\mathbb{R}^n$. Then if $\vec{g}(y)$ is any embedding of a polygon from $\mathcal{S}'$ into $\mathbb{R}^n$ there must be some i such that $\vec{g}(y)$ and $\vec{g}(x_i)$ are not orthogonal. Write $x = x_i$. As each g_i is an orthogonal map, we have

$$\|g_1(x) - g_1(y)\| = \|g_2(x) - g_2(y)\| = \dots = \|g_k(x) - g_k(y)\|.$$

So by Lemma 4.1, $\mathcal{S}'$ is a union of finitely many sets each of measure zero, and hence $\mathcal{S}'$ has measure zero.

Now, a finite group G has only countably many orthogonal representations, up to conjugation by an orthogonal transformation. Indeed, every orthogonal representation is a direct sum of irreducible representations. and the group G has only finitely many inequivelant irreducible linear representations. Moreover, two irreducible orthogonal representations which are isomorphic by some linear map are, in fact, isomorphic by an orthogonal map (because, for example, by Schur's Lemma there is a unique G-invariant inner product on $\mathbb{R}^n$ up to multiplication by a scalar—for more details, see e. g. Lemma 4.7.1 of [15]).

Hence the orthogonal groups $O(n)$ have only countably many distinct finite subgroups (up to conjugation), and given a finite subgroup of $O(n)$ there are only finitely many ways to select from it a sequence of k elements. Thus $\mathcal{S}$ is a countable union of sets of measure zero and so itself has measure zero. □

While our proof shows that almost every cyclic 16-gon is not subtransitive, it does not provide an explicit construction. We hope that such an explicit construction of a polygon P might provide some insight into proving that P is not Ramsey. We are therefore interested in a solution to the following problem.

Problem H. *Give an explicit construction of a cyclic polygon that is not subtransitive.*

We also find it unlikely that it is necessary to go as far as 16-gons:

Conjecture I. *There exists a cyclic quadrilateral that is not subtransitive.*

Indeed, we believe that almost no cyclic quadrilateral should be subtransitive.

We remark that it is easy to check that all trapezia are subtransitive. In fact, Kříž [12] showed that all trapezia are Ramsey. This may also be deduced from his general result in [11].

5 Are Ramsey sets subtransitive?

For this direction of our conjecture, we have no results at all—so in this section we only mention a few heuristic ideas.

Given a Ramsey set X, why might there be a transitive set containing it? There are certainly some 'structured' sets containing X—namely the sets that are k-Ramsey (for some k) for X. Of course, it is impossible that every such set is transitive, as we may always add points to a set that is k-Ramsey for X to destroy any symmetry that is present. So one should focus on the *minimal* k-Ramsey sets for X.

To fix our ideas, let us consider the simplest possible case, when X is the set $\{0, 1\}$. What are the minimal 2-Ramsey sets for X? For a finite set S in $\mathbb{R}^n$, define the *graph* of S to be the graph on vertex-set S in which we join two points if they are at unit distance. Then the minimal 2-Ramsey sets for X are precisely those sets whose graph is an odd cycle. Now, such a set can be very far from transitive: indeed, it might have no isometries at all. However, there are two key points. The first is that, for any such set, we can transform it, preserving unit distances, to obtain a transitive set. The second, perhaps more important, is that a *minimum-sized* such set has to be transitive, as it has to be an equilateral triangle. And similarly for the sets that are k-Ramsey for X: minimal such sets are sets whose graphs are $(k+1)$-critical (meaning that they have chromatic number $k+1$ but the removal of any vertex decreases the chromatic number), and the unique minimum-sized such set is the regular simplex on $k+1$ vertices.

Similar phenomena seem to be present in other examples. We cannot expect in general to focus only on sets that are 2-Ramsey for a given set X: it is certainly possible to find X that is not Ramsey, but such that there does exist a set Y that is 2-Ramsey for X (for example, $X = \{0, 1, 2\}$—see [2]). But this is perhaps not surprising, as of prime importance will be how the copies of X 'fit together' inside the set S, and one may need more colours to 'encode' this information. For example, it may be that one should look at the k-Ramsey sets for X, where k is $2^{|X|}$—the intuitive idea being that the colouring of S obtained by, for a point s of S, listing those points of X can map to s in an embedding of X into S, might give key information about the structure of S. At any rate, we wonder if the following is true.

Problem J. *Let X be a Ramsey set. Must there exist a k such that every minimum-sized set that is k-Ramsey for X is transitive?*

As an alternative, we suggest the following modification of this question which is perhaps more approachable. In place of a minimum-sized set that is k-Ramsey for X, we might instead consider a minimal set k-Ramsey for X with the further property that it cannot be transformed to contain additional copies of X whilst retaining all those already present.

Finally, we consider an algorithmic question. It is easy to see that one can determine in finite time whether or not a given set is spherical. However, it is not clear that it is possible to determine in finite time whether or not a given set is subtransitive. So if our conjecture were true, it would leave open the problem of finding an algorithm to determine whether or not a given set is Ramsey. So we ask:

Problem K. *Is there an algorithm for testing in finite time whether or not a given set is subtransitive?*

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