Consistency for Partition Regular Equations

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August 28, 2003

Abstract

It is easy to deduce from Ramsey's Theorem that, given positive integers $a_1, a_2, \dots, a_m$ and a finite colouring of the set $\mathbb{N}$ of positive integers, there exists an injective sequence $(x_i)_{i=1}^\infty$ with all sums of the form $\sum_{i=1}^m a_i x_{r_i}$ $(r_1 < r_2 < \cdots < r_m)$ lying in the same colour class. The consistency version of this result, namely that, given positive integers $a_1, a_2, \dots, a_m$ and $b_1, b_2, \dots b_n$, and a finite colouring of $\mathbb{N}$, there exist injective sequences $(x_i)_{i=1}^\infty$ and $(y_i)_{i=1}^\infty$ with all sums of the form $\sum_{i=1}^m a_i x_{r_i}$ $(r_1 < r_2 < \cdots < r_m)$ *and* all sums of the form $\sum_{i=1}^n b_i y_{r_i}$ $(r_1 < r_2 < \cdots < r_n)$ in the *same* colour class, was open for some time, being recently proved by Hindman, Leader and Strauss. The proof is long and relies heavily on the structure of the semigroup $\beta\mathbb{N}$ of ultrafilters on $\mathbb{N}$. Our aim in this note is to present a short proof of this result which does not use properties of $\beta\mathbb{N}$. Our proof also gives various results not obtainable by the previous method of proof.

1 Introduction

A (finite or infinite) matrix A with integer entries is said to be *image partition regular,* or simply *partition regular,* if, whenever the set $\mathbb{N} = \{1, 2, 3, \dots\}$ of positive integers is finitely coloured, there exists a vector x of positive integers with all the elements of Ax contained in the same colour class in $\mathbb{N}$. Equivalently, we may speak of the 'system' Ax as being partition regular. Many natural theorems of Ramsey Theory, such as those of Schur [7] and van der Waerden [9], can be formulated as the statement that a certain matrix is partition regular. Those finite matrices which are partition regular have been characterized by Hindman and Leader [3], building on work of Rado [6].

The situation for infinite matrices is less well understood. In this case, the simplest known examples of infinite partition regularity come directly from Ramsey's Theorem: it is easy to show that, given positive integers $a_1, a_2, \dots, a_m$ and a finite colouring of $\mathbb{N}$, there exists an infinite sequence $x_1 < x_2 < \cdots$ such that the set

$$S = \left\{ \sum_{i=1}^m a_i x_{r_i} : r_1 < r_2 < \cdots < r_m \right\}$$

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is monochromatic. Indeed, we may simply colour the m-sets from $\mathbb{N}$ by giving the set $\{r_1, r_2, \ldots, r_m\}$ $(r_1 < r_2 < \cdots < r_m)$ the colour of $\sum_{i=1}^m a_i x_{r_i}$ and now Ramsey's Theorem guarantees the existence of an infinite set $M \subset \mathbb{N}$ all of whose m-subsets have the same colour. We refer to this system as the 'Ramsey' system $R(a_1, a_2, \ldots, a_m)$.

We remark in passing that the reader may be worried that, since we have a condition $x_1 < x_2 < \cdots$, our Ramsey systems are not of the general form given above. However, it is always possible to convert to that form, for example by replacing x_1, x_2, x_3, ... with new variables y_1, $y_1 + y_2$, $y_1 + y_2 + y_3$, ... We urge the reader to ignore this minor detail.

One might say that these Ramsey systems were 'trivially' partition regular; the first non-trivial examples of infinite partition regular matrices were given by Hindman [2] and by Milliken [5] and Taylor [8]. However, in this paper we do not assume familiarity with these matrices.

One of the most important notions in partition regularity is that of 'consistency'. We say that two partition regular matrices A and B are *consistent* if the matrix $\binom{A\ O}{O\ B}$ is also partition regular; in other words, A and B are consistent if, given any finite colouring of $\mathbb{N}$, we can find vectors x and y of positive integers such that all the entries of Ax and all the entries of By lie in the same colour class. In the finite case, it follows from the characterization of partition regular matrices that any pair of partition regular matrices is consistent. However, consistency fails in the infinite case: it was shown in [1] that two infinite partition regular matrices need not be consistent.

This left outstanding the question of whether the Ramsey systems defined above were consistent. This question was eventually answered affirmatively by Hindman, Leader and Strauss [4]. However, their proof is long and relies heavily on the structure of the semigroup $\beta\mathbb{N}$ of ultrafilters on $\mathbb{N}$. Various results from logic show that the existence of a proof of this result in ZFC implies the existence of a proof in ZF, so a short proof not using properties of $\beta\mathbb{N}$ was wanted. We present such a proof here. Our proof also gives various results not obtainable by the methods of [4].

2 Proof of main result

Theorem 1. *Let a_1, a_2, ..., a_m and b_1, b_2, ..., b_n be positive integers. Then whenever $\mathbb{N}$ is finitely coloured, there exists a pair of sequences $x_1 < x_2 < \cdots$ and $y_1 < y_2 < \cdots$ such that the set*

$$\left\{\sum_{i=1}^m a_i x_{r_i} : r_1 < r_2 < \cdots < r_m\right\} \cup \left\{\sum_{i=1}^n b_i y_{r_i} : r_1 < r_2 < \cdots < r_n\right\}$$

is monochromatic.

Proof. Given a finite colouring of $\mathbb{N}$, we induce a finite colouring of the mn-sets from $\mathbb{N}$ by giving the set $\{r_{ij} : 1 \le i \le m, 1 \le j \le n\}$, where $r_{i_1 j_1} < r_{i_2 j_2}$ if $i_1 < i_2$ or if $i_1 = i_2$ and $j_1 < j_2$, the colour of $\sum_{i=1}^m \sum_{j=1}^n a_i b_j r_{ij}$. By Ramsey's Theorem, there is an infinite monochromatic set for this colouring; in other words, there is a sequence $z_1 < z_2 < \cdots$ such that all $\sum_{i=1}^m \sum_{j=1}^n a_i b_j z_{r_{ij}}$ $(r_{i_1 j_1} < r_{i_2 j_2}$ if $i_1 < i_2$ or if $i_1 = i_2$ and $j_1 < j_2)$ have the same colour.

The choice of the x_i is now clear: we may take

$$x_i = \sum_{j=1}^{n} b_j z_{ni+j}.$$

Our idea for choosing the y_i is to make them share some fixed 'common start'. First, fix some $z_{r_{ij}}$ $(1 \leq i \leq m-1,\ 1 \leq j \leq n)$, with $r_{i_1 j_1} < r_{i_2 j_2}$ if $i_1 < i_2$ or if $i_1 = i_2$ and $j_1 < j_2$, and with all the $z_{r_{ij}}$ congruent mod $\sum_{k=1}^{n} b_k$. We can now take

$$y_i \ = \ a_m z_{i+r_{m-1,n}} \ + \ \frac{\sum_{j=1}^{m-1} \sum_{k=1}^{n} a_j b_k z_{r_{jk}}}{\sum_{k=1}^{n} b_k}.$$

Then for $s_1 < s_2 < \cdots < s_n$, we have

$$\sum_{i=1}^{n} b_i y_{s_i} = \sum_{i=1}^{m-1} \sum_{j=1}^{n} a_i b_j z_{r_{ij}} + \sum_{j=1}^{n} a_m b_j z_{j+r_{m-1,n}},$$

and we are done. □

We remark that the above proof extends by induction to deal with consistency for any finite collection of Ramsey systems.

Let us also remark that the result extends easily to give consistency for infinite collections of Ramsey systems. Indeed, suppose we had an infinite sequence R_1, R_2, ... of Ramsey systems ($R_i = R(a_1^{(i)}, a_2^{(i)}, \ldots, a_{N_i}^{(i)})$) which were not consistent. Then (reordering our sequence if necessary) we would be able to find a partition $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_k$ of $\mathbb{N}$ such that for all i, $1 \leq i \leq k$, there was no sequence $x_1 < x_2 < \cdots$ of positive integers with all sums of the form $\sum_{j=1}^{N_i} a_j^{(i)} x_{r_j}$ $(r_1 < r_2 < \ldots < r_{N_i})$ lying in C_i. But then the finite collection R_1, R_2, ..., R_k would be inconsistent, contradicting the finite result above.

The definition of the Ramsey system $R(a_1, a_2, \ldots, a_m)$ can be extended by removing the restriction that all of the integers a_1, a_2, ..., a_m must be positive; indeed, it is still easy to show that, for any non-zero integers a_1, a_2, ..., a_m with $a_m > 0$, the system $R(a_1, a_2, \ldots, a_m)$ is partition regular. In [4] it is shown that two such systems $R(a_1, a_2, \ldots, a_m)$ and $R(b_1, b_2, \ldots, b_n)$ are consistent as long as $\sum_{i=1}^{m} a_i$ and $\sum_{i=1}^{n} b_i$ are both non-zero. However, the methods of [4] were not able to deal with the case where one of the sums is allowed to be zero but the other is non-zero. This is as far as it is possible to go, as if both sums are zero then the two matrices need not be consistent: Hindman, Leader and Strauss [4] provide $R(1,-1,-1,1)$ and $R(-1,1,-1,1)$ as an example of a pair of inconsistent Ramsey systems.

Interestingly, our method of proof does cover the case where one of the sums is allowed to be zero but the other is non-zero.

Theorem 2. *Let a_1, a_2, ..., a_m and b_1, b_2, ..., b_n be non-zero integers with $a_m, b_n > 0$ and $\sum_{i=1}^{n} b_i \neq 0$. Then whenever $\mathbb{N}$ is finitely coloured, there exists a pair of sequences $x_1 < x_2 < \cdots$ and $y_1 < y_2 < \cdots$ such that the set*

$$\left\{ \sum_{i=1}^{m} a_i x_{r_i} : r_1 < r_2 < \cdots < r_m \right\} \cup \left\{ \sum_{i=1}^{n} b_i y_{r_i} : r_1 < r_2 < \cdots < r_n \right\}$$

is (contained in $\mathbb{N}$ and is) monochromatic.

Proof. The proof is exactly the same as that of Theorem 1 except that instead of colouring $\mathbb{N}^{(mn)}$, we colour $N^{(mn)}$ for an infinite set $N \subset \mathbb{N}$ chosen so that all the sums we need to work with are positive. To be more precise, we take $N = \{w_1, w_2, \ldots\}$ where, having chosen w_1, w_2, ..., w_{p-1}, we choose $w_p > w_{p-1}$ sufficiently large that all expressions of the form $\sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j w_{r_{ij}}$ ($r_{i_1 j_1} < r_{i_2 j_2}$ if $i_1 < i_2$ or if $i_1 = i_2$ and $j_1 < j_2$, $r_{mn} = p$), $\sum_{j=1}^{n} b_j w_{r_j}$ ($r_1 < r_2 < \cdots < r_n = p$) or $a_m w_p + \frac{\sum_{j=1}^{m-1} \sum_{k=1}^{n} a_j b_k w_{r_{jk}}}{\sum_{k=1}^{n} b_k}$ ($r_{i_1 j_1} < r_{i_2 j_2}$ if $i_1 < i_2$ or if $i_1 = i_2$ and $j_1 < j_2$) are positive. □

We remark that, exactly as in the positive case, Theorem 2 can be extended to any finite or infinite collection R_1, R_2, ... of Ramsey systems ($R_i = R(a_1^{(i)}, a_2^{(i)}, \ldots, a_{N_i}^{(i)})$) with at most one of the sums $\sum_{j=1}^{N_i} a_j^{(i)}$ ($i = 1$, 2, ...) being zero.

In summary, we know that two Ramsey systems $R(a_1, a_2, \ldots, a_m)$ and $R(b_1, b_2, \ldots, b_n)$ are consistent if at most one of the sums $\sum_{i=1}^{m} a_i$ and $\sum_{i=1}^{n} b_i$ is zero, but that if both sums are zero then they need not be consistent. This leaves open the following question:

Question 3. *Precisely which pairs of Ramsey systems are consistent?*

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