Trinity 2020 **GRAPH THEORY** PAR

Throughout the sheet, G is a bipartite graph with vertex classes X and Y.

1. A square sheet of paper of side length n is divided up into n polygons each of area n. A second square sheet of paper of side length n is also divided up into n polygons each of area n. The first sheet of paper is placed on top of the second sheet of paper, with both sheets aligned in the same way so that the first sheet completely covers the second. Show that it is possible to stick n pins through the sheets in such a way that each of the $2n$ polygons has a pin through it.

2. An $n \times n$ *Latin square* is an $n \times n$ square array of numbers with each of the numbers 1, 2, $\ldots$, n appearing precisely once in each row and precisely once in each column. For $r < n$. an $r \times n$ *Latin rectangle* is an $r \times n$ rectangular array of numbers (r rows, n columns) with each of the numbers 1, 2, $\ldots$, n appearing precisely once in each row and at most once in each column. Prove that every $r \times n$ Latin rectangle may be extended to an $n \times n$ Latin square.

3. Suppose $|\Gamma(A)| > |A|$ for all $A \subset X$ except for $A = \emptyset$ and $A = X$. Show that, for every edge e of G, there is a matching from X to Y containing e.

4. A *d-defective matching* from X to Y is a collection of $|X| - d$ edges in G sharing no vertices (so a 0-defective matching is just a matching). Find a necessary condition similar to Hall's condition for G to contain a d-defective matching from X to Y. Can you show that your condition is also sufficient?

5. A *1-to-d matching* from X to Y is a collection of $d|X|$ edges in G sharing no vertices in Y such that each vertex of X is in precisely d of the edges (so a 1-to-1 matching is just a matching). Find a necessary condition similar to Hall's condition for G to contain a 1-to-d matching from X to Y. Can you show that your condition is also sufficient?

6. Suppose that every $x \in X$ has at least one neighbour in Y. Suppose also that, for every edge xy of G with $x \in X$ and $y \in Y$, the vertex x has at least as many neighbours in Y as the vertex y has in X. Show that G contains a matching from X to Y.

$^+$7. Can we do something similar to Hall's Theorem if we drop the requirement for G to be bipartite? That is to say, for a general graph G, can we find an interesting necessary and sufficient condition for G to contain a collection M of edges with each vertex of G appearing in precisely one of the edges in M?