Mich. 2016 **NUMBERS AND SETS—EXAMPLES 4** PAR

1. The *symmetric difference* $A\triangle B$ of two sets A and B is defined to be the set of elements x such that x is in precisely one of A and B. Prove that $\triangle$ is associative; that is, prove that if A, B and C are sets then $A\triangle(B\triangle C) = (A\triangle B)\triangle C$.

2. We know that if n has prime factorization $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ then $\phi(n) = (p_1^{\alpha_1} - p_1^{\alpha_1-1}) \dots (p_k^{\alpha_k} - p_k^{\alpha_k-1})$. Use the inclusion-exclusion principle to give an alternative proof of this result.

3. How many subsets of $\{1, 2, \dots, m\}$ have even size?

4. Let A_1, A_2, A_3, ... be sets such that for each n we have $A_1 \cap \dots \cap A_n \neq \emptyset$. Must $A_1 \cap A_2 \cap \dots \neq \emptyset$?

5. Let $f\colon A \to B$ and $g\colon B \to C$. If $g \circ f$ is injective, must g be injective? If $g \circ f$ is injective, must f be injective? What if we replace 'injective' with 'surjective'?

6. Find an injection from $\mathbb{R}^2$ to $\mathbb{R}$. Is there an injection from the set of all sequences of real numbers to $\mathbb{R}$?

7. Define a relation R on $\mathbb{N}$ by setting aRb if $a|b$ or $b|a$. Is R an equivalence relation?

8. Show that there does not exist an unountable set of pairwise disjoint discs in the plane. Is there an uncountable set of pairwise disjoint circles in the plane? Is there an uncountable set of pairwise disjoint snowmen in the plane?

A disc *is the set of points at distance at most r from some point P of the plane, for some $r > 0$. A* circle *is the set of points at distance exactly r from some point P of the plane, for some $r > 0$. A* snowman *is the union of two circles that bound discs with exactly one point in common.*

9. Show that the collection of all finite subsets of $\mathbb{N}$ is countable.

10. A function $f\colon \mathbb{N} \to \mathbb{N}$ is *increasing* if $f(n+1) \geqslant f(n)$ for all n, and *decreasing* if $f(n+1) \leqslant f(n)$ for all n. Is the set of increasing functions countable? What about the set of decreasing functions?

11. Let S be a collection of subsets of $\mathbb{N}$ such that for all A, $B \in S$ we have $A \subset B$ or $B \subset A$. Can S be uncountable?

12. Find a bijection from the rationals to the non-zero rationals. Show futher that there is such a bijection that is order-preserving, i.e. a bijection f such that if $x < y$ then $f(x) < f(y)$.

Suppose now that each rational number has been coloured either blue or red, in such a way that between any two blue numbers there is a red number and between any two red numbers there is a blue number. Can we always find a bijection from the rationals to the non-zero rationals that is both order-preserving and colour-preserving, no matter how this colouring has been done?

13. Each of an infinite sequence of Trappist set theorists is going to a party where each will receive a coloured hat, either red or blue. Each set theorist will be able to see every hat but his own. After all hats are assigned, each set theorist must write down (in silence, obviously) a guess as to his own hat colour. You are asked to supply them with a strategy such that, should they follow it, only finitely many of them will guess wrongly. Can you do it?