Lent 2021 **LOGIC AND SET THEORY—-EXAMPLES 3** PAR

1. How many different partial orders are there (up to isomorphism) on a set of 4 elements?

2. Let P be the poset $\{A \subset \mathbb{N} \mid A \text{ or } \mathbb{N}\backslash A \text{ is finite}\}$ ordered by inclusion. Does every chain in P have an upper bound? A least upper bound?

3. Let X be a set and let P be the poset

$$\{< \subset X \times X \mid < \text{ is a well-ordering of a subset of } X\}$$

ordered by $<_1 \leqslant <_2$ if $<_1$ is an initial segment of $<_2$ (N.B. *is* not merely *is isomorphic to*). Why is P non-empty? Show that any chain in P has an upper bound and hence that P contains a maximal element. Deduce WOP.

4. Give a direct proof of Zorn (not using ordinals and not using AC) for countable posets.

5. Use Zorn to prove one (or more) of the following (if you understand at least one):
 (i) Every commutative ring with a 1 has a maximal ideal;
 (ii) Every Hilbert space has an orthonormal basis;
 (iii) Let G be an infinite bipartite graph with parts X and Y such that every $x \in X$ has finite degree and, for all $A \subset X$, $|\Gamma(A)| \geqslant |A|$. Show that G contains a matching from X to Y. *(Don't attempt this one unless you know Hall's Theorem.)*

6. Prove the Bourbaki-Witt theorem without using AC.

7. Formulate sets of axioms in suitable languages (to be specified) for the following theories.
 (i) The theory of fields of characteristic 2
 (ii) The theory of posets having no maximal element
 (iii) The theory of bipartite graphs
 (iv) The theory of algebraically closed fields
 (v) The theory of groups of order 60
 (vi) The theory of simple groups of order 60
 (vii) The theory of real vector spaces

8. Write down axioms (in the language of posets) for the theory of total orders that are dense (between any two elements is a third) and have no greatest or least element. Show that every countable model of this theory is isomorphic to $\mathbb{Q}$. Why does it follow that this theory is complete?

9. Show that the theory of fields of positive characteristic is not axiomatizable (in the language of fields), and that the theory of fields of characteristic zero is axiomatizable but not finitely axiomatizable.

10. Write down axioms, in a suitable language, for the theory of groups that have an element of infinite order. Can this be done in the language of groups?

11. Let L be the language consisting of a single function symbol f, of arity 1. Write down a theory T that asserts that f is a bijection with no finite orbits, and describe the countable models of T. Prove that T is a complete theory.

12. Is every countable model of PA isomorphic to $\mathbb{N}$?

13. Is $\text{PA} \cup \{\neg\text{Con(PA)}\}$ consistent?

14. Does PA have witnesses?