Lent 2021 **LOGIC AND SET THEORY—EXAMPLES 2** PAR

1. Write down subsets of the reals that have order-types $\omega + \omega + \omega$, ω^2 and ω^3.

2. Let α, β and γ be ordinals. If $\alpha \leqslant \beta$, must we have $\alpha + \gamma \leqslant \beta + \gamma$? If $\alpha < \beta$, must we have $\alpha + \gamma < \beta + \gamma$?

3. Show that the inductive and synthetic definitions of ordinal multiplication agree.

4. Is there a non-zero ordinal α with $\alpha\omega = \alpha$? What about $\omega\alpha = \alpha$?

5. Let α, β, γ be ordinals. Must we have $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$? Must we have $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$?

6. Find two totally ordered sets such that neither is isomorphic to a subset of the other. Can you find three such sets?

7. Let α, β and γ be ordinals. Must we have $\alpha^{\beta+\gamma} = \alpha^\beta\alpha^\gamma$? Must we have $(\alpha^\beta)^\gamma = \alpha^{\beta\gamma}$? Must we have $(\alpha\beta)^\gamma = \alpha^\gamma\beta^\gamma$?

8. Show that, for every countable ordinal α, there is a subset of $\mathbb{Q}$ of order-type α. Why is there no subset of $\mathbb{R}$ of order-type ω_1?

9. Is ω_1 the supremum of a countable set of countable ordinals?

10. What is the smallest fixed point of $\alpha \mapsto \omega^\alpha$? The next smallest? And the next smallest? Show that the fixed points are unbounded, and explain why this means that we may index the fixed points by the ordinals. Is there a countable ordinal α such that α is the α-th fixed point?

11. Let κ, λ and μ be cardinals. Show that $\kappa^\lambda\kappa^\mu = \kappa^{\lambda+\mu}$.

12. What is the cardinality of the set of all functions from $\mathbb{R}$ to $\mathbb{R}$? What is the cardinality of the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$?

13. Explain why, for each $n \in \mathbb{N}$, there is no surjection from ω_n to ω_{n+1}. Use this fact to show that there is no surjection from ω_ω to $\{f \mid f\colon \omega_0 \to \omega_\omega\}$, the set of functions from ω_0 to ω_ω. Deduce that $2^{\aleph_0} \neq \aleph_\omega$.

14. Show that we must use AC in any proof that, given any two sets, there is an injection from one into the other.

15. Prove, without using AC, that a countable union of countable sets cannot have cardinality $\aleph_2$.