Proposition 6. Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ and let $x \in \mathbb{R}^n$. Then f is differentiable at x iff each $f_i\colon \mathbb{R}^n \to \mathbb{R}$ $(1 \leqslant i \leqslant m)$ is differentiable at x. Moreover, if f is differentiable at x then $(Df|_x)_i = (Df_i)|_x$ for each i, and all partial derivatives of f exist at x with the matrix A of $Df|_x$ being given by $A_{ij} = D_j f_i(x)$.

Proof. Write

$$f(x+h) = f(x) + \alpha(h) + \varepsilon(h)\|h\|$$

where $\alpha \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. Then for each i with $1 \leqslant i \leqslant m$ we have

$$f_i(x+h) = f_i(x) + \alpha_i(h) + \varepsilon_i(h)\|h\|$$

where $\alpha_i \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$. Now, $\varepsilon(h) \to 0$ as $h \to 0$ iff for each i we have $\varepsilon_i(h) \to 0$ as $h \to 0$, establishing the first two of the three claims above.

Finally, suppose f is differentiable at x with $Df|_x = \alpha$. Then, writing $e_1, e_2, \ldots, e_n$ and $e'_1, e'_2, \ldots, e'_m$ for the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, we have for each j that

$$\frac{f(x+te_j) - f(x)}{t} = \frac{\alpha(te_j) + \varepsilon(te_j)\|te_j\|}{t} = \alpha(e_j) \pm \varepsilon(te_j) \to 0$$

as $t \to 0$. Hence $D_j f$ exists for each j and

$$\alpha(e_j) = D_j f(x) = \sum_{i=1}^{m} D_j f_i(x) e'_i$$

as required. $\square$

Proposition 7. Let $m, n \geqslant 1$ and let $\|.\|$, $\|.\|'$ denote the operator norm and the Euclidean norm respectively on $\mathcal{M}_{m\times n}$. Then there exist constants c and d (depending on m and n) such that for all $A \in \mathcal{M}_{m\times n}$ we have $\|A\| \leqslant c\|A\|'$ and $\|A\|' \leqslant d\|A\|$.

Proof. Let $A \in \mathcal{M}_{m\times n}$.

Let $x \in \mathbb{R}^n$ with $\|x\| = 1$. Then

$$\|Ax\|^2 = \sum_{i=1}^{m}((Ax)_i)^2 = \sum_{i=1}^{m}\Big(\sum_{j=1}^{n} A_{ij}x_j\Big)^2 \leqslant \sum_{i=1}^{m}\Big(\sum_{j=1}^{n}\|A\|'\Big)^2 = \sum_{i=1}^{m}(n\|A\|')^2 = mn^2\|A\|'^2$$

and so $\|Ax\| \leqslant n\sqrt{m}\|A\|'$. Thus $\|A\| \leqslant n\sqrt{m}\|A\|'$.

For the other way round, we have

$$\|A\|'^2 = \sum_{i=1}^{m}\sum_{j=1}^{n}(A_{ij})^2 = \sum_{j=1}^{n}\Big(\sum_{i=1}^{m}(A_{ij})^2\Big) = \sum_{j=1}^{n}\|Ae_j\|^2 \leqslant \sum_{j=1}^{n}\|A\|^2 = n\|A\|^2.$$

Thus $\|A\|' \leqslant \sqrt{n}\|A\|$. $\square$