Topological Spaces

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https://www.dpmms.cam.ac.uk/∼or257/teaching/notes/TopSp.pdf

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1 Topological spaces

In Part IB Analysis II you have encountered the notion of a metric space (X, d), and what is means for a function $f : (X, d) \to (X', d')$ between metric spaces to be continuous. Namely, that given an $x_0 \in X$ and $\epsilon > 0$, there exists a $\delta > 0$ for which $d(x, x_0) < \delta$ implies $d'(f(x), f(x_0)) < \epsilon$. Looking at this definition, we see that whether a given function $f : X \to X'$ is continuous or not depends on the metrics d and d' in a very loose way: f will be continuous for many choices of metrics on X and X'. In other words, continuity of f does not really depend on the metrics involved, but only on some shadow of the notion of a metric. This shadow is the notion of a topological space. Once we describe this notion we will see that it is not necessary to have a metric to discuss continuity, and that for many purposes it is an irrelevance whether a given topological space comes from a metric space or not.

1.1 Language

The generalisation from metric to topological spaces is shift the focus from *distance* between points to *openness* of sets: we simply axiomatise how open sets behave, and declare anything that behaves in that way to be called "open sets". Recall that the *power set* $P(X)$ of a set X is the set of all subsets of X.

Definition. Let X be a set. A *topology* on X is a collection $\mathcal{T} \subseteq P(X)$ of subsets of X satisfying:

(i) $\emptyset, X \in \mathcal{T}$,

(ii) if $\{U_\alpha\}_{\alpha \in I}$ is a collection of elements of $\mathcal{T}$ then $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$,

(iii) if $\{U_\alpha\}_{\alpha \in I}$ is a *finite* collection of elements of $\mathcal{T}$ then $\bigcap_{\alpha \in I} U_\alpha \in \mathcal{T}$.

A set X with a chosen topology $\mathcal{T}$ is called a *topological space* (or simply *space*). Elements of the set X are called *points* and elements of $\mathcal{T}$ are called *open sets*. If $x \in U \in \mathcal{T}$ then we say that U is an *open neighbourhood* of x.

It is usually safe to leave $\mathcal{T}$ implicit and say "X is a topological space" to mean "X is a set with a chosen topology $\mathcal{T}_X$", and say "U is open in (or an open subset of) X" to mean "$U \in \mathcal{T}_X$". We generally do this, except when e.g. talking about two topologies on the same set in which case we diligently give names to the topologies.

Whenever one introduces a kind of mathematical object, one should immediately introduce the appropriate kind of "morphisms" between such objects. In the case of topological spaces, these are the continuous functions:

Definition. A function $f : X \to Y$ between topological spaces is called *continuous* if whenever V is an open subset of Y, the preimage $f^{-1}(V) := \{x \in X : f(x) \in V\}$ is an open subset of X.

Once we have a notion of "morphism" between things, we can say what it means for two things to be the same, i.e. to be "isomorphic". For some reason, in the subject of Topological Spaces it is given another name:

Definition. A function $f : X \to Y$ between topological spaces is a *homeomorphism* if it is continuous and has a continuous inverse. In more detail, this is the same as asking that

(i) f is a bijection, and

(ii) f and f^{-1} are both continuous.

If there exists a homeomorphism between spaces X and Y we say that they are *homeomorphic*, and write $X \cong Y$.

The topologies on a given set can be ordered by containment.

Definition. If $\mathcal{T} \subseteq \mathcal{T}'$ are topologies on a set X then we say that $\mathcal{T}$ is *coarser* than $\mathcal{T}'$, and that $\mathcal{T}'$ is *finer* than $\mathcal{T}$.

This is precisely the same as saying that the function $\mathrm{id} : (X, \mathcal{T}') \to (X, \mathcal{T})$ is continuous.

1.1.1 Topologies from metrics

To check that we are not just playing with words, let us show that the metric space notions of "open" and "continuous" fit into the framework developed so far. Recall from Part IB Analysis II that $U \subseteq X$ is open with respect to the metric d if for each $x \in V$ there exists an $\epsilon > 0$ such that the ball $B_\epsilon(x) := \{y \in Y \,:\, d(x, y) < \epsilon\}$ lies within U.

Proposition. Let (X, d) be a metric space, and let $\mathcal{T}_d$ denote the collection of subsets of X which are open with respect to the metric d. Then $\mathcal{T}_d$ is a topology on X. We call it the topology on X *induced by the metric* d.

Proof. The set $\emptyset \subseteq X$ is open vacuously, and $X \subseteq X$ is too because any ball in X lies in X, so these are both in $\mathcal{T}_d$.

Suppose $\{U_\alpha\}_{\alpha \in I}$ is a collection of subsets of X in $\mathcal{T}_d$, and let $x \in \bigcup_{\alpha \in I} U_\alpha$. By definition of the union we must have $x \in U_\beta$ for some $\beta \in I$, and as U_β is open with respect to the metric d there exists an $\epsilon > 0$ such that $B_\epsilon(x) \subset U_\beta \subset \bigcup_{\alpha \in I} U_\alpha$. This shows that $\bigcup_{\alpha \in I} U_\alpha$ is in $\mathcal{T}_d$ as required.

Suppose $\{U_\alpha\}_{\alpha \in I}$ is a finite collection of subsets of X in $\mathcal{T}_d$, and let $x \in \bigcap_{\alpha \in I} U_\alpha$. Then for each $\alpha \in I$ we have $x \in U_\alpha$, and as U_α is open with respect to the metric d there exists an $\epsilon_\alpha > 0$ such that $B_{\epsilon_\alpha}(x) \subset U_\alpha$. Now $\epsilon := \inf_{\alpha \in I}(\epsilon_\alpha) > 0$ as the set I is finite so this infimum is actually the minimum of the ϵ_α's. Then $B_\epsilon(x) \subset B_{\epsilon_\alpha}(x) \subset U_\alpha$ for each $\alpha \in I$, so $B_\epsilon(x) \subseteq \bigcap_{\alpha \in I} U_\alpha$. This shows that $\bigcap_{\alpha \in I} U_\alpha$ is in $\mathcal{T}_d$ as required. □

That is, a metric space defines a topological space whose open sets are the sets that were called "open" in Part IB Analysis II.

Example. The Euclidean metric $d(x, y) := ||x - y||$ defines a topology on $\mathbb{R}^n$, the *Euclidean topology.* The induced metric defines a topology on any subset of $\mathbb{R}^n$. △

When we discuss sets like $\mathbb{R}^n$, $[0,1]^n$, $[0,1)$, and the like we usually consider them with this topology, unless we make a point to emphasise otherwise.

Proposition. If (X, d_X) and (Y, d_Y) are metric spaces, which can be considered as topological spaces by the previous Proposition, then a function $f : X \to Y$ is continuous in the sense of topological spaces if and only if it is continuous in the sense of metric spaces.

Proof. Suppose first that $f : X \to Y$ is continuous in the sense of metric spaces, let $V \subseteq Y$ be open, and in order to show that $f^{-1}(V)$ is open choose a $x_0 \in f^{-1}(V)$: we need to find a ball inside $f^{-1}(V)$ centred at x_0 As V is open, there is an $\epsilon > 0$ such that $B_\epsilon(f(x_0)) \subseteq V$. As f is continuous in the sense of metric spaces, there is a $\delta > 0$ such that $d(x, x_0) < \delta$ implies $d'(f(x), f(x_0)) < \epsilon$. In other words, $f(B_\delta(x_0)) \subseteq B_\epsilon(f(x_0))$, or equivalently $B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$. But then $B_\delta(x_0)$ is contained in $f^{-1}(V)$, as required.

Suppose now that $f : X \to Y$ is continuous in the sense of topological spaces, let $x_0 \in X$ and $\epsilon > 0$ be given: we need to find a $\delta > 0$ such that $d(x, x_0) < \delta$ implies $d'(f(x), f(x_0)) < \epsilon$. As $B_\epsilon(f(x_0))$ is open in Y, the set $f^{-1}(B_\epsilon(f(x_0)))$ is open in X and contains x_0. By the definition of openness in the previous Proposition, there is a $\delta > 0$ for which $B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$. Spelling this out it precisely says that if $d(x, x_0) < \delta$ then $d'(f(x), f(x_0)) < \epsilon$. □

We can also characterise convergence of sequences in topological, rather than metric, terms.

Definition. Let $(X, \mathcal{T})$ be a topological space, $x_1, x_2, x_3, \ldots \in X$ be a sequence in X. We say the sequence *converges to* $x_\infty \in X$ if for every open neighbourhood $U \ni x_\infty$ there is an N such that $x_n \in U$ for all $n \geq N$. Equivalently, only finitely-many terms in the sequence lie outside of U.

Proposition. Let (X, d) be a metric space, giving the topological space $(X, \mathcal{T}_d)$. A sequence $\{x_n\}$ in X converges to x_∞ in the sense of metric spaces if an only if it converges to $\{x_n\}$ in the sense of topological spaces.

Proof. Suppose first that $\{x_n\}$ converges to x_∞ in the metric sense. If $U \ni x_\infty$ is an open neighbourhood then there is an $\epsilon > 0$ such that $B_\epsilon(x_\infty) \subseteq U$. By metric convergence there is an N such that $d(x_n, x_\infty) < \epsilon$ for all $n \geq N$, i.e. such that $x_n \in B_\epsilon(x_\infty) \subseteq U$ for all $n \geq N$. Thus the sequence converges to x_∞ in the topological sense.

Suppose now that $\{x_n\}$ converges to x_∞ in the topological sense, and let $\epsilon > 0$. Then $B_\epsilon(x_\infty) \ni x_\infty$ is an open neighbourhood, so there is an N such that $x_n \in B_\epsilon(x_\infty)$ for all $n \geq N$, i.e. $d(x_n, x_\infty) < \epsilon$ for all $n \geq N$. Thus the sequence converges to x_∞ in the metric sense. □

That is, whether a sequence in (X, d) converges (to x_∞) does not actually depend on the metric d, but only on the topology $\mathcal{T}_d$.

At this point we can start asking ourselves what properties of a metric space (X, d) in fact depend on the metric, and what properties only depend on the topological space $(X, \mathcal{T}_d)$. The former of course means "up to isometry", whereas the latter means "up to homeomorphism". We have just discussed that continuity and convergence are topological properties: let's see a non-topological property.

Example. Consider $\mathbb{R}$ and $(0, 1)$ with their Euclidean topologies. The function

$$x \longmapsto \tan(\pi(x - 1/2)) : (0, 1) \longrightarrow \mathbb{R}$$

is a bijection, and is continuous (in the sense of metric spaces and so also in the topological sense). Its inverse is also continuous, by the-inverse-of-a-continuous-strictly-monotonic-function-is-continuous from Part IA Analysis I. It therefore defines a homeomorphism $\mathbb{R} \cong (0, 1)$.

But the metric spaces $(\mathbb{R}, d)$ and $((0, 1), d)$, with the Euclidean metrics, are not isometric. For example, the metric space $(\mathbb{R}, d)$ is complete, but $((0, 1), d)$ is not as the sequence $\{\frac{1}{n}\}$ in $(0, 1)$ is Cauchy but has no limit. So "being complete" is a property of metric spaces that does not only depend on the induced topological space. △

1.1.2 Further examples

There are many basic examples of topological spaces which do not obviously come from metrics, lots of which cannot possibly come from metrics as we will see.

Example. Let X be a set. The finest topology on X is $\mathcal{T}_{\text{disc}} := P(X)$, consisting of all subsets. It is called the *discrete topology.* Any function from $(X, \mathcal{T}_{\text{disc}})$ to any topological space is continuous.

This topology is induced by the metric

$$d(x, y) := \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}.$$

To see this first observe that $B_{1/2}(x) = \{x\}$ so the single points are open in the topology induced by d, and by taking unions it follows that all subsets are open in this topology, so it is $\mathcal{T}_{\text{disc}}$. △

Example. Let X be a set. The coarsest topology on X is $\mathcal{T}_{\text{indisc}} := \{\emptyset, X\}$. It is called the *indiscrete topology.* Any function from any topological space to $(X, \mathcal{T}_{\text{indisc}})$ is continuous.

Suppose that this topology arises from a metric d on X. Then for any point x the set $B_\epsilon(x)$ is open, and it is not empty as it contains x so must be the whole set X. But then for any $x, y \in X$ we have $d(x, y) < \epsilon$ for all $\epsilon > 0$, so in fact $d(x, y) = 0$ by taking the infimum. By the axioms of a metric it follows that $x = y$, so X has (at most) one element.

Let $\{x_n\}$ be a sequence in $(X, \mathcal{T}_{\text{indisc}})$, and $x_\infty \in X$ be any element. This point only has one open neighbourhood, $X \ni x_\infty$, which contains all x_n's. According to our definition the sequence $\{x_n\}$ converges to x_∞ in the space $(X, \mathcal{T}_{\text{indisc}})$: *any sequence converges to any limit in this topological space!* Clearly we will have to give up on some of the intuition we have about metric spaces. △

Example. Let X be a set. The collection

$$\mathcal{T}_{\text{cofinite}} := \{X \setminus F \,:\, F \subseteq X \text{ is finite}\} \cup \{\emptyset\}$$

defines the *cofinite topology* on X. You will verify that this is a topology on Example Sheet 1. △

Example. Let $X = \{o, c\}$, and $\mathcal{T} := \{\emptyset, \{o\}, \{o, c\}\}$. This is the *Sierpiński space.* △

Example. Let $X = \mathbb{R}$. The *(right) order topology* is $\mathcal{T}_{\text{ord}} := \{(a, \infty) \,:\, a \in [-\infty, \infty]\}$. If $\{(a_\alpha, \infty)\}_{\alpha \in I}$ is a collection of elements of $\mathcal{T}_{\text{ord}}$ then

$$\bigcup_{\alpha \in I} (a_\alpha, \infty) = (\inf_{\alpha \in I} a_\alpha, \infty) \in \mathcal{T}_{\text{ord}}$$

and, if the set I is finite,

$$\bigcap_{\alpha \in I} (a_\alpha, \infty) = (\max_{\alpha \in I} a_\alpha, \infty) \in \mathcal{T}_{\text{ord}}.$$

△

1.1.3 (Sub)bases

It is sometimes convenient to describe a topology by giving less data that the full collection of open sets: something like a generating collection of open sets. There are two related ways of doing this:

Definition. Let $\mathcal{T}$ be a topology on X.

(i) A *basis* for $\mathcal{T}$ is a subset $\mathcal{B} \subseteq \mathcal{T}$ such that every element of $\mathcal{T}$ is a union of elements of $\mathcal{B}$.

(ii) A *subbasis* for $\mathcal{T}$ is a subset $\mathcal{S} \subseteq \mathcal{T}$ such that every element of $\mathcal{T}$ is obtained by taking a union of sets obtained as finite intersections of elements of $\mathcal{S}$. In other words, $\mathcal{T}$ is the smallest topology containing $\mathcal{S}$.

If $\mathcal{S} \subseteq P(X)$ is *any* collection of subsets of X which cover X, then there is a unique topology $\mathcal{T}$ on X for which $\mathcal{S}$ is a subbasis: it consists of the sets obtained by taking a union of sets obtained as finite intersections of elements of $\mathcal{S}$. You will verify that this is indeed a topology on Example Sheet 1.

(Sub)bases for a topology are convenient because we can test continuity of functions on subbasis elements rather than arbitrary open sets.

Lemma. Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a function, and $\mathcal{S} \subseteq \mathcal{T}_Y$ be a subbasis. If $f^{-1}(U) \in \mathcal{T}_X$ for each $U \in \mathcal{S}$, then f is continuous.

Proof. If V is open in Y then by definition of a subbasis we have $V = \bigcup_{\alpha \in I} V_\alpha$ for a set I where $V_\alpha = \bigcap_{\beta \in J_\alpha} U_{\alpha,\beta}$ for finite sets J_α and subbasis elements $U_{\alpha,\beta} \in \mathcal{S}$. Then

$$f^{-1}(V) = \bigcup_{\alpha \in I} f^{-1}(V_\alpha) = \bigcup_{\alpha \in I} \left(\bigcap_{\beta \in J_\alpha} f^{-1}(U_{\alpha,\beta}) \right)$$

is open, as each $f^{-1}(U_{\alpha,\beta})$ is open by assumption, and the intersections are finite. □

Example. Consider $\mathbb{R}^n$ with the topology induced by the Euclidean metric $d(x, y) = ||x - y||$. The collection $\mathcal{B}$ of *balls*

$$B_r(x) := \{y \in \mathbb{R}^n : d(x, y) < r\}, \quad x \in \mathbb{R}^n, r > 0,$$

is a basis for the topology on $\mathbb{R}^n$, by definition of the metric topology.

The collection $\mathcal{C}$ of *cubes*

$$(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n) := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : a_i < x_i < b_i\}$$

is also a basis for the topology on $\mathbb{R}^n$. To see this, it suffices to show that for each $y = (y_1, \ldots, y_n) \in B_r(x)$, there is a $V \in \mathcal{C}$ with $y \in V \subseteq B_r(x)$. This can be achieved by taking

$$V = (y_1 - \epsilon, y_1 + \epsilon) \times \cdots \times (y_n - \epsilon, y_n + \epsilon)$$

with $0 < \epsilon < \frac{r - d(x,y)}{\sqrt{n}}$, as some elementary geometry will show.

The collection $\mathcal{QB}$ of *rationally-centred balls with rational radius*

$$B_q(x) := \{y \in \mathbb{R}^n : d(x, y) < q\}, \quad x \in \mathbb{Q}^n, q \in \mathbb{Q}_{>0}$$

is also a basis for the topology on $\mathbb{R}^n$, as you can check. This means that the topology on $\mathbb{R}^n$ is determined by countably-many open sets. This is known as being *second-countable*, and is a convenient property for a topology to have (though we will not see why in this course). △

1.1.4 Closed sets, closure, and interior

Definition. Let X be a topological space. We say that a set $C \subseteq X$ is *closed* if $X \setminus C$ is open.

By taking complements, the defining properties of a collection of open sets defining a topology give corresponding properties of the collection of closed sets.

Proposition. Let $(X, \mathcal{T})$ be a topological space, and $\mathcal{F} := \{C \subseteq X \,:\, X \setminus C \in \mathcal{T}\}$ be the collection of closed subsets. Then

(i) $\emptyset, X \in \mathcal{F}$,

(ii) if $\{C_\alpha\}_{\alpha \in I}$ is a collection of elements of $\mathcal{F}$ then $\bigcap_{\alpha \in I} C_\alpha \in \mathcal{F}$,

(iii) if $\{C_\alpha\}_{\alpha \in I}$ is a *finite* collection of elements of $\mathcal{F}$ then $\bigcup_{\alpha \in I} C_\alpha \in \mathcal{F}$. □

(In case it is not obvious, the intersection and union properties are reversed compared to open sets.) We can also used closed sets to discuss continuity

Proposition. A function between topological spaces is continuous if and only if the preimage of each closed set is closed. □

You will prove these on Example Sheet 1.

Definition. Let $(X, \mathcal{T})$ be a topological space.

(i) The *closure* $\overline{A}$ of a subset $A \subseteq X$ is the smallest closed subset of X containing A. There is one because it is given by the following formula

$$\overline{A} := \bigcap_{\text{closed sets } C \text{ containing } A} C.$$

As arbitrary intersections of closed sets are closed, this is indeed a closed set.

We say that A is *dense in* X if $\overline{A} = X$.

(ii) The *interior* $\mathring{A}$ of a subset $A \subseteq X$ is the largest open subset of X contained in A, i.e.

$$\mathring{A} := \bigcup_{\text{open sets } U \text{ contained in } A} U.$$

As arbitrary intersections of open sets are open, this is indeed an open set.

This sort of definition looks very clean, but makes it difficult to answer basic questions like: is some point $x \in X$ in the closure (or interior) of A? One source of points in the closure of a set A are its limit points:

Definition. Let X be a topological space and $A \subseteq X$. A point $x \in X$ is a *limit point* of A if there is a sequence $\{x_n\}$ in A converging to x.

We have seen that in general sequences can converge to more than one point, which suggests we should treat this concept a bit cautiously.

Proposition. If $C \subseteq X$ is a closed subset of a topological space, then its limit points are in C.

Proof. Let $\{x_n\}$ be a sequence in C converging to $x \in X$. If $x \notin C$ then $x \in X \setminus C$ is an open neighbourhood of x, so by the definition of convergence $x_n \in X \setminus C$ for all large enough n, a contradiction. □

This means that every limit point of a subset $A \subseteq X$ lies in its closure $\overline{A}$.

Example. The subset $\mathbb{Q} \subset \mathbb{R}$ has $\overline{\mathbb{Q}} = \mathbb{R}$, because every real number is the limit of a sequence of rational numbers. That is, $\mathbb{Q}$ is dense in $\mathbb{R}$. △

Example. The closure of the subset $(0,1) \subset \mathbb{R}$ is $[0,1]$. To see this, observe that 0 is the limit of the sequence $\{\frac{1}{n+1}\}$ in $(0,1)$, and 1 is the limit of the sequence $\{1 - \frac{1}{n+1}\}$ in $(0,1)$, so $[0,1] \subseteq \overline{(0,1)}$. But $[0,1]$ is a closed set containing $(0,1)$, so also $\overline{(0,1)} \subseteq [0,1]$, and hence they are equal. △

In a metric space you have seen in Part IB Analysis II that the closed sets are exactly those which contain all their limit points. This is not the case in general.

Example. Consider the *cocountable topology* on $\mathbb{R}$, i.e.

$$\mathcal{T}_{\text{cocountable}} := \{\mathbb{R} \setminus F \,:\, F \subseteq \mathbb{R} \text{ is countable}\} \cup \{\emptyset\}.$$

You can check that this is indeed a topology, similar to the cofinite topology. The closed sets in this space are precisely $\mathbb{R}$ and the countable sets.

If $\{x_n\}$ is a sequence in this space, and $x \in \mathbb{R}$ is a point, then

$$\{x\} \cup (\mathbb{R} \setminus \{x_n \,:\, n \in \mathbb{N}\})$$

is an open neighbourhood of x. This contains all but finitely-many elements of the sequence x_n if and only if the sequence is eventually constantly x. Thus the only convergent sequences in $(\mathbb{R}, \mathcal{T}_{\text{cocountable}})$ are the eventually constant sequences, and they converge to their eventually constant value. It follows that the limit points of a set A are just the points in A. But if A is not countable then its closure must be $\mathbb{R}$. △

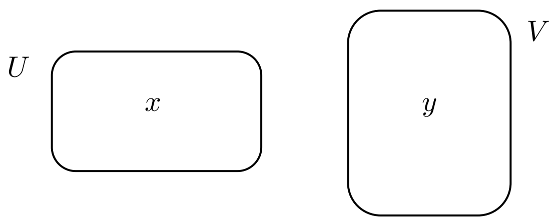
The overall lesson here is that although sequences and their limits tell us much about metric spaces, they are not a refined enough notion to probe general topological spaces. We will therefore stop discussing them.

1.1.5 Hausdorffness

We have seen that in the topological space $(X, \mathcal{T}_{\text{indisc}})$ a sequence can and does have more than one limit point; the same is true for some of the other topologies we have defined. This is not so in a metric space: why not?

Definition. A topological space $(X, \mathcal{T})$ is *Hausdorff* if for each pair $x, y \in X$ of distinct points, there are open neighbourhoods $U \ni x$ and $V \ni y$ which are disjoint.

It is obligatory to include the following instructive picture:



Hausdorffness is the most important "separation condition" on topological spaces, and is the only one we will discuss in this course. Using your intuition from metric spaces it seems like it should obviously be true: Felix Hausdorff certainly thought so, and included it in his original (1914) definition of a topological space. These days we do not include it as an axiom, but as a very convenient property.

Proposition. A topological space arising from a metric space is Hausdorff.

Proof. Let d be a metric on X giving a topology $\mathcal{T}$. If $x, y \in X$ are distinct then $\delta := d(x, y) > 0$, so consider the open neighbourhoods $B_{\delta/2}(x) \ni x$ and $B_{\delta/2}(y) \ni y$. Suppose for a contradiction that the open neighbourhoods $B_{\delta/2}(x)$ and $B_{\delta/2}(y)$ are not disjoint, and choose a $z \in B_{\delta/2}(x) \cap B_{\delta/2}(y)$ then $d(x, z) < \delta/2$ and $d(z, y) < \delta/2$ so by the triangle inequality $d(x, y) \leq d(x, z) + d(z, y) < \delta$, a contradiction. □

Proposition. A sequence in a Hausdorff topological space has at most one limit.

Proof. Suppose that x_∞ and x'_∞ are both limits of the sequence $\{x_n\}$, and are distinct. As the space is Hausdorff, there are disjoint open neighbourhoods $U \ni x_\infty$ and $V \ni x'_\infty$. As $\{x_n\}$ converges to x_∞, only finitely-many x_n's lie outside of U, but then only finitely-many can lie inside of V, so the sequence cannot in fact converge to x'_∞. □

Proposition. In a Hausdorff topological space the one-point sets are closed.

Proof. Let X be a Hausdorff space and $x \in X$. For each $y \neq x$ we can find disjoint open neighbourhoods $U \ni y$ and $V_y \ni x$, and in particular $X \setminus U$ is a closed set containing x and not containing y. Thus y is not in the closure $\overline{\{x\}}$: this goes for all $y \neq x$, so $\overline{\{x\}} = \{x\}$ is closed. □

Most of the non-metric topologies we have discussed are not Hausdorff. The indiscrete topology clearly is not (unless the space only has one point).

Example. Consider the cofinite topology $(X, \mathcal{T}_{\text{cofinite}})$. An open neighbourhood of $x \in X$ is a set of the form $X \setminus F$ with F finite and $x \notin F$. If $y \in X$ is another point, and $X \setminus G$ is an open neighbourhood of y (so G is finite and $y \notin G$), then

$$(X \setminus F) \cap (X \setminus G) = X \setminus (F \cup G).$$

As F and G are finite, if X is infinite then this set must be non-empty, so if X is infinite then $(X, \mathcal{T}_{\text{cofinite}})$ is not Hausdorff as any open neighbourhoods of the distinct points x and y must intersect.

(If X is finite one checks that $(X, \mathcal{T}_{\text{cofinite}}) = (X, \mathcal{T}_{\text{disc}})$; we have seen that this topology is induced by a metric, so is Hausdorff.) △

1.2 Generating new topological spaces

It is relatively infrequent that we describe topological spaces by saying what all the open sets are. More commonly, we use constructions that make new topological spaces out of old.

1.2.1 The subspace topology

Definition. Let $(X, \mathcal{T}_X)$ be a topological space, and $Y \subseteq X$ be a subset. The *subspace topology* on Y is the set

$$\mathcal{T}|_Y := \{U \cap Y \,:\, U \in \mathcal{T}_X\}.$$

A *subspace* of $(X, \mathcal{T}_X)$ is a subset of X equipped with the subspace topology.

When we are leaving topologies implicit, writing $Y \subseteq X$ means "Y is a subspace of X".

Proposition. This is indeed a topology.

Proof. It contains the sets $\emptyset = \emptyset \cap Y$ and $Y = X \cap Y$. If $\{U_\alpha \cap Y\}_{\alpha \in I}$ is a collection of elements of $\mathcal{T}|_Y$ then

$$\bigcup_{\alpha \in I} (U_\alpha \cap Y) = \left(\bigcup_{\alpha \in I} U_\alpha \right) \cap Y$$

which lies in $\mathcal{T}|_Y$ as $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}_X$. If the set I is finite then

$$\bigcap_{\alpha \in I} (U_\alpha \cap Y) = \left(\bigcap_{\alpha \in I} U_\alpha \right) \cap Y$$

which lies in $\mathcal{T}|_Y$ as $\bigcap_{\alpha \in I} U_\alpha \in \mathcal{T}_X$. □

Proposition. The inclusion $i : (Y, \mathcal{T}|_Y) \to (X, \mathcal{T}_X)$ is continuous. In fact, $\mathcal{T}|_Y$ is the coarsest topology on Y for which i is continuous.

Proof. If U is open in $(X, \mathcal{T}_X)$ then $i^{-1}(U) = U \cap Y$ is open in $(Y, \mathcal{T}|_Y)$ by definition, so i is indeed continuous.

If $\mathcal{T}' \subset \mathcal{T}|_Y$ is a proper subset, then there is a set of the form $U \cap Y$, with U open in X, which is not in $\mathcal{T}'$. But then $i : (Y, \mathcal{T}') \to (X, \mathcal{T}_X)$ is not continuous, as $i^{-1}(U) = U \cap Y$ is not open in $(Y, \mathcal{T}')$. So i is not continuous in any strictly coarser topology than $\mathcal{T}|_Y$. □

We could have instead defined the subspace topology as: the coarsest topology on Y for which the inclusion $i : Y \to X$ is continuous. Why does this sentence actually define anything? There certainly exists some topology on Y for which i is continuous (the discrete topology), and if there are two topologies on Y for which i is continuous then their intersection is also a topology for which i is continuous: thus there exists a smallest (=coarsest) topology for which i is continuous. With this definition it becomes an exercise to work out that the open sets in this coarsest topology must be exactly those of the form $U \cap Y$ for U open in X.

This is perhaps a different way of making definitions than you are used to. It is a definition via the desirable properties that the thing is supposed to satisfy (followed by an argument that such a thing exists), rather than by giving a construction of it (followed by checking that the construction has desirable properties). This way of thinking requires some getting used to, but is powerful.

Here is another characterisation of the subspace topology, along similar lines. For any topological space Z, a function $f : (Z, \mathcal{T}_Z) \to (Y, \mathcal{T}|_Y)$ is continuous if and only if the composition

$$(Z, \mathcal{T}_Z) \xrightarrow{f} (Y, \mathcal{T}|_Y) \xrightarrow{i} (X, \mathcal{T}_X)$$

is continuous. We can phrase this as the following "universal property":

To give a continuous function f from a space Z *into* the subspace Y is exactly the same as giving a continuous function $\tilde{f} : Z \to X$ such that $\tilde{f} = i \circ f$, i.e. such that $\tilde{f}$ happens to land in Y.

There is an elementary observation about the subspace topology which is used frequently and often implicitly. If $(X, \mathcal{T}_X)$ is a topological space, and $U \subseteq X$ is an open subset with the subspace topology, then open subsets of U are also open in X. This is because an open subset of U is precisely $U \cap V$ for some open subset of V, so is an intersection of open subsets of X and hence is open. Similarly, if $C \subseteq X$ is a closed subset with the subspace topology, then closed subsets of C are also closed in X.

The following, especially part (ii), is often a convenient way of checking that functions are continuous, by checking that they are on certain subspaces.

Lemma (Gluing Lemma)**.** Let $f : X \to Y$ be a function between topological spaces.

(i) If there is a cover $\{U_\alpha\}_{\alpha \in I}$ of X by open sets and the restrictions $f|_{U_\alpha} : U_\alpha \to Y$ are continuous for each $\alpha \in I$, then f is continuous.

(ii) If there is a *finite* cover $\{C_\alpha\}_{\alpha \in I}$ of X by closed sets and the restrictions $f|_{C_\alpha} : C_\alpha \to Y$ are continuous for each $\alpha \in I$, then f is continuous.

Proof. Let $V \subseteq Y$ be open. As the U_α cover we have

$$f^{-1}(V) = \bigcup_{\alpha \in I} f^{-1}(V) \cap U_\alpha = \bigcup_{\alpha \in I} f|_{U_\alpha}^{-1}(V),$$

and as each $f|_{U_\alpha}$ is continuous $f|_{U_\alpha}^{-1}(V)$ is open in U_α. As U_α is open, $f|_{U_\alpha}^{-1}(V)$ is also open in X, so the above shows that $f^{-1}(V)$ is a union of open sets and so open.

The closed set version is analogous, using that *finite* unions of closed sets are closed. □

Example. The subset

$$S^n := \{x \in \mathbb{R}^{n+1} \; : \; ||x|| = 1\} \subset \mathbb{R}^{n+1}$$

is called the *n-dimensional sphere* or *n-sphere*, and is given the subspace topology from the Euclidean topology on $\mathbb{R}^{n+1}$. Equivalently, the Euclidean metric on $\mathbb{R}^{n+1}$ induces a metric on the subset S^n and we can give it the metric topology: these are the same. When $n = 1$ this is the unit circle in the plane.

The subset

$$D^n := \{x \in \mathbb{R}^n \; : \; ||x|| \leq 1\} \subset \mathbb{R}^n$$

is called the *n-dimensional disc* or *n-disc*, again with the subspace topology. It contains S^{n-1} as a subspace. △

1.2.2 The quotient topology

Definition. Let $(X, \mathcal{T}_X)$ be a topological space, and $\sim$ be an equivalence relation on X with set of equivalence classes $X_{/\sim}$, and quotient map $\pi : X \to X_{/\sim}$. The *quotient topology* on $X_{/\sim}$ is

$$\mathcal{T}_{X/\sim} := \{U \subseteq X_{/\sim} \; : \; \pi^{-1}(U) \in \mathcal{T}_X\}.$$

Proposition. This is indeed a topology.

Proof. As $\pi^{-1}(\emptyset) = \emptyset \in \mathcal{T}_X$ and $\pi^{-1}(X_{/\sim}) = X \in \mathcal{T}_X$, $\mathcal{T}_{X/\sim}$ contains $\emptyset$ and $X_{/\sim}$.

If $\{U_\alpha\}$ is a collection of elements of $\mathcal{T}_{X/\sim}$, then

$$\pi^{-1}\left(\bigcup_{\alpha\in I} U_\alpha\right) = \bigcup_{\alpha\in I} \pi^{-1}(U_\alpha)$$

is a union of elements in $\mathcal{T}_X$ so is in $\mathcal{T}_X$, and hence $\bigcup_{\alpha\in I} U_\alpha \in \mathcal{T}_{X_{/\sim}}$.

If $\{U_\alpha\}$ is a finite collection of elements of $\mathcal{T}_{X/\sim}$, then

$$\pi^{-1}\left(\bigcap_{\alpha\in I} U_\alpha\right) = \bigcap_{\alpha\in I} \pi^{-1}(U_\alpha)$$

is a finite intersection of elements in $\mathcal{T}_X$ so is in $\mathcal{T}_X$, and hence $\bigcap_{\alpha\in I} U_\alpha \in \mathcal{T}_{X_{/\sim}}$. □

Proposition. The quotient map $\pi : (X, \mathcal{T}_X) \to (X_{/\sim}, \mathcal{T}_{X_{/\sim}})$ is continuous. In fact $\mathcal{T}_{X_{/\sim}}$ is the finest topology on $X_{/\sim}$ for which π is continuous.

Proof. This is a tautology: we defined the open sets in $X_{/\sim}$ to be exactly those whose preimage is open. □

This discussion is completely dual to the subspace topology, with the injection $i : Y \to X$ replaced with the surjection $\pi : X \to X_{/\sim}$ and "coarsest" replaced by "finest". In fact mentioning equivalence relations is needlessly complicated: we could have taken the input data to be a topological space X and a surjection $\pi : X \to Z$, and given Z the finest topology for which π is continuous. A surjection $\pi : X \to Z$ tautologically defines an equivalence relation on X by $x \sim x' \Leftrightarrow \pi(x) = \pi(x')$, and there is a canonical induced bijection $[x] \mapsto \pi(x) : X_{/\sim} \to Z$, so this is not really any different to what we have done but it makes the analogy with the subspace topology clearer.

There is an alternative characterisation the quotient topology, analogous to what we said about the subspace topology. For any topological space $(Z, \mathcal{T}_Z)$, a function $f : (X_{/\sim}, \mathcal{T}_{X_{/\sim}}) \to (Z, \mathcal{T}_Z)$ is continuous if and only if the composition

$$(X, \mathcal{T}_X) \xrightarrow{\pi} (X_{/\sim}, \mathcal{T}_{X_{/\sim}}) \xrightarrow{f} (Z, \mathcal{T}_Z)$$

is continuous. This is tremendously useful, and is the (only) way one should think about quotient spaces. Namely, its "universal property" is:

> To give a continuous function f *out* of the quotient space $X_{/\sim}$ to a space Z is exactly the same as giving a continuous function $\tilde{f} : X \to Z$ such that $\tilde{f}(x) = \tilde{f}(x')$ whenever $x \sim x'$.

The condition "$\tilde{f}(x) = \tilde{f}(x')$ whenever $x \sim x'$" is exactly saying that $\tilde{f} = f \circ \pi$ for a (unique) function $f : X_{/\sim} \to Z$, and the quotient topology is precisely designed so that f is continuous whenever $\tilde{f}$ is.

This gives the following convenient way to identify quotient spaces.

Definition. A continuous map $f : X \to Y$ is *a quotient map* if it is surjective and

$$U \in \mathcal{T}_Y \iff f^{-1}(U) \in \mathcal{T}_X.$$

If $f : X \to Y$ is a quotient map, and $\sim$ is the equivalence relation on X given by

$$x \sim x' \iff f(x) = f(x'),$$

then the above discussion gives a continuous function $\tilde{f} : X_{/\sim} \to Y$. As well as being continuous this is surjective (as f was surjective), injective (as $\tilde{f}([x]) = \tilde{f}([x'])$ means $f(x) = f(x')$ so means $[x] = [x']$), and in fact a homeomorphism as if $U \subseteq X_{sim}$ is open then $\tilde{f}(U) = f(\pi^{-1}(U))$ is open in Y because $\pi^{-1}(U)$ is open in X and f was a quotient map.

Example. Consider the equivalence relation on $\mathbb{R}$ given by $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$. The function

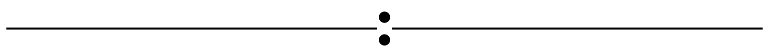
$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto (\sin(2\pi x), \cos(2\pi x)) \end{aligned}$$

is continuous by Part IA Analysis I. This function has image in the subspace $S^1 \subset \mathbb{R}^2$, to by the defining property of the subspace topology it is also continuous when considered as a function $f' : \mathbb{R} \to S^1$. If $x \sim y$ then $f'(x) = f'(y)$, by the periodicity of these trigonometric functions, so f' induces a continuous function $f'' : \mathbb{R}_{/\sim} \to S^1$. This is a bijection. One can laboriously check that it is a homeomorphism, using the explicit description of the quotient topology, but later in the course we will see a trick that immediately implies that f'' is a homeomorphism. △

Example. Consider the subspace $X = \mathbb{R} \times \{0, 1\} \subset \mathbb{R}^2$, and the equivalence relation

$$(x, i) \sim (y, j) \Leftrightarrow (x, i) = (y, j) \text{ or } x = y \neq 0.$$

The quotient space $L := X_{/\sim}$ is obtained by identifying all non-zero numbers on two copies of the real line, and is called the *line with two origins*. It is undrawable, but perhaps we can think of it as something like:



The points $[(0, 0)]$ and $[(0, 1)]$ are distinct. An open neighbourhood $[(0, 0)] \in U \subseteq L$ is a set such that $\pi^{-1}(U) \subseteq \mathbb{R} \times \{0, 1\}$ is open, and contains $(0, 0) \in X$. Thus $\pi^{-1}(U)$ must contain $(-\epsilon, \epsilon) \times \{0\}$ for some $\epsilon > 0$, but it must be a union of equivalences classes for $\sim$ so must also contain $((-\epsilon, 0) \cup (0, \epsilon)) \times \{1\}$. Similarly if $[(0, 1)] \in V$ is an open neighborhood then $\pi^{-1}(V)$ must contain $(-\delta, \delta) \times \{1\}$ for some $\delta > 0$, and so must intersect $((-\epsilon, 0) \cup (0, \epsilon)) \times \{1\}$. It follows that U and V must intersect, i.e. L is not Hausdorff.

As a subspace of a Hausdorff space is again Hausdorff, L cannot be realised as a subspace of any Hausdorff space: in particular it cannot be realised as a subspace of any metric space. △

1.2.3 The product topology

Recall that the *product* of two sets X and Y is

$$X \times Y := \{(x, y) \,:\, x \in X, y \in Y\},$$

and there are *projection functions*

$$\begin{aligned} \pi_X : X \times Y &\longrightarrow X \\ (x, y) &\longmapsto x \end{aligned} \qquad\qquad \begin{aligned} \pi_Y : X \times Y &\longrightarrow Y \\ (x, y) &\longmapsto y. \end{aligned}$$

Definition. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. The *product topology* $\mathcal{T}_{X\times Y}$ on $X \times Y$ consists those subsets $U \subseteq X \times Y$ such that for each $(x, y) \in U$ there are open neighbourhoods $x \in V \in \mathcal{T}_X$ and $y \in W \in \mathcal{T}_Y$ such that $V \times W \subseteq U$.

Proposition. This is indeed a topology. The sets $V \times W$ with $V \in \mathcal{T}_X$ and $W \in \mathcal{T}_Y$ form a basis for the topology $\mathcal{T}_{X\times Y}$.

Proof. We have $\emptyset \in \mathcal{T}_{X\times Y}$ as there is nothing to check, and $X \times Y \in \mathcal{T}_{X\times Y}$ as $X \in \mathcal{T}_X$ and $Y \in \mathcal{T}_Y$.

Let $\{U_\alpha\}_{\alpha\in I}$ be a collection of elements of $\mathcal{T}_{X\times Y}$, and $(x, y) \in \bigcup_{\alpha\in I} U_\alpha$. Then $(x, y) \in U_\beta$ for some $\beta \in I$, and so there exist open neighbourhood $x \in V \in \mathcal{T}_X$ and $y \in W \in T_Y$ such that $(x, y) \in V \times W \subset U_\beta \subset \bigcup_{\alpha\in I} U_\alpha$. Thus $\bigcup_{\alpha\in I} U_\alpha \in \mathcal{T}_{X\times Y}$.

Let $\{U_\alpha\}_{\alpha\in I}$ be a *finite* collection of elements of $\mathcal{T}_{X\times Y}$, and $(x, y) \in \bigcap_{\alpha\in I} U_\alpha$. Then for each $\alpha \in I$ there are open neighbourhoods $x \in V_\alpha \in \mathcal{T}_X$ and $y \in W_\alpha \in \mathcal{T}_Y$ such that $(x, y) \in V_\alpha \times W_\alpha \subseteq U_\alpha$. But then $(x, y) \in (\bigcap_{\alpha\in I} V_\alpha) \times (\bigcap_{\alpha\in I} W_\alpha) \subset \bigcap_{\alpha\in I} U_\alpha$ too, with $\bigcap_{\alpha\in I} V_\alpha \in \mathcal{T}_X$ and $\bigcap_{\alpha\in I} W_\alpha \in \mathcal{T}_Y$ because they are finite intersections. Thus $\bigcap_{\alpha\in I} U_\alpha \in \mathcal{T}_{X\times Y}$.

The sets $V \times W$ are a basis for the product topology by definition. □

Proposition. The projection maps

$$\pi_X : (X \times Y, \mathcal{T}_{X\times Y}) \longrightarrow (X, \mathcal{T}_X) \qquad \pi_Y : (X \times Y, \mathcal{T}_{X\times Y}) \longrightarrow (Y, \mathcal{T}_Y)$$

are both continuous. In fact, $\mathcal{T}_{X\times Y}$ is the coarsest topology on $X \times Y$ for which they are both continuous.

Proof. For $U \in \mathcal{T}_X$ we have $\pi_X^{-1}(U) = U \times Y$ which is open in the product topology, so π_X is indeed continuous; similarly with π_Y.

Let $\mathcal{T}'_{X\times Y}$ be some topology for which the two projection maps are continuous. For $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$ the sets $U \times Y = \pi_X^{-1}(U)$ and $X \times V = \pi_Y^{-1}(V)$ are in $\mathcal{T}'_{X\times Y}$, and hence so is their intersection $U \times V$. All unions of such sets are then also in $\mathcal{T}'_{X\times Y}$, so $\mathcal{T}'_{X\times Y}$ contains $\mathcal{T}_{X\times Y}$ as required. □

Just like the subspace and quotient topologies, there is an alternative characterisation of the product topology. For any topological space $(Z, \mathcal{T}_Z)$, a function $f : (Z, \mathcal{T}_Z) \to (X \times Y, \mathcal{T}_{X\times Y})$ is continuous if and only if the two compositions

$$(Z, \mathcal{T}_Z) \xrightarrow{f} (X \times Y, \mathcal{T}_{X\times Y}) \xrightarrow{\pi_X} (X, \mathcal{T}_X) \qquad (Z, \mathcal{T}_Z) \xrightarrow{f} (X \times Y, \mathcal{T}_{X\times Y}) \xrightarrow{\pi_Y} (Y, \mathcal{T}_Y)$$

are continuous. In other words its "universal property" is:

> To give a continuous function f from a space Z *into* the product space $X \times Y$ is exactly the same as giving a pair of continuous functions $f_X : Z \to X$ and $f_Y : Z \to Y$.

Example. The Euclidean topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the same as the product topology coming from the Euclidean topology on $\mathbb{R}$ twice. △

2 Connectedness and components

The intermediate value theorem from Part IA Analysis I says that if $f : [a, b] \to \mathbb{R}$ is a continuous function such that $f(a) < 0 < f(b)$, then there is an $x \in [a, b]$ such that $f(x) = 0$. This is presented as a theorem about continuous functions, but in a sense it isn't: if $f : [a, \frac{a+b}{2}) \cup (\frac{a+b}{2}, b] \to \mathbb{R}$ is a continuous function with the same properties then the conclusion need not hold. Instead, it can be considered as theorem about the interval $[a, b]$. Namely, it is the theorem that $[a, b]$ is *connected.*

2.1 Connectedness

Definition. A topological space X is *disconnected* if it can be written as $X = U \cup V$ where U and V are disjoint non-empty open subsets. It is *connected* if it is not disconnected.

Example. A space with the coarse topology is connected. △

Example. A discrete space with more than one point is disconnected. △

Example. The subspace $X = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ of $\mathbb{R}$ is disconnected, as the sets $[0, \frac{1}{2})$ and $(\frac{1}{2}, 1]$ are disjoint, non-empty, and open in X (as they are $X \cap (-\infty, \frac{1}{2})$ and $X \cap (\frac{1}{2}, \infty)$). △

Shortly we will show that $[0, 1]$ *is* connected, but first let us discuss another way of describing (dis)connectivity, in terms of continuous functions instead of open sets.

Proposition. A space X is disconnected if and only if there is a surjective continuous function $f : X \to \{0, 1\}$, where the latter has the discrete topology.

Proof. If X is disconnected with $X = U \cup V$ a decomposition into disjoint non-empty open subsets, define the function

$$f(x) = \begin{cases} 0 & x \in U \\ 1 & x \in V. \end{cases}$$

The open subsets of $\{0, 1\}$ are $\emptyset, \{0\}, \{1\}, \{0, 1\}$, and their preimages are $\emptyset, U, V, U \cup V$ which are all open, so f is continuous. It is surjective as U and V are non-empty.

Conversely, if such an f exists then $U := f^{-1}(0)$ and $f^{-1}(1)$ gives a decomposition of X into disjoint non-empty open subsets. □

Theorem. The spaces $[0, 1]$, $[0, 1)$, and $(0, 1)$ are connected.

Proof. For simplicity we consider $[0, 1]$, but the same applies to the other cases. If it were disconnected, there would be a continuous surjective function $f : [0, 1] \to \{0, 1\}$. Compose this with the inclusion $i : \{0, 1\} \to \mathbb{R}$ as a subspace, to get a continuous function $i \circ f : [0, 1] \to \mathbb{R}$ taking the values 0 and 1. By the intermediate value theorem there is a $x \in [0, 1]$ such that $i \circ f(x) = \frac{1}{2}$, but this contradicts the fact that f only takes values 0 and 1. □

Here we have used the intermediate value theorem to prove connectedness, but this is not surprising because connectedness quickly leads to a generalised form of the intermediate value theorem as follows.

Theorem (Intermediate Value Theorem)**.** Let X be connected and $f : X \to \mathbb{R}$ be a continuous function. If there are points $x_0, x_1 \in X$ such that $f(x_0) < 0 < f(x_1)$ then there is an $x \in X$ with $f(x) = 0$.

Proof. Consider the open sets $U := f^{-1}(-\infty, 0)$ and $V := f^{-1}(0, \infty)$, which are disjoint and non-empty (as $x_0 \in U$ and $x_1 \in V$). If there were no $x \in X$ such that $f(x) = 0$, then U and V cover X and so X is disconnected: this is not the case, so there must exist an $x \in X$ with $f(x) = 0$. □

In fact, connectedness is equivalent to "satisfies the conclusion of the Intermediate Value Theorem": if X is not connected, and $f : X \to \{0, 1\}$ is a continuous surjection, then $f - \frac{1}{2} : X \to \mathbb{R}$ is a continuous function which takes the values $\pm\frac{1}{2}$ but does not take the value 0.

To finish we show that the continuous image of a connected space is connected:

Proposition. If $f : X \to Y$ is a continuous surjection and X is connected, then Y is connected.

Proof. We prove the contrapositive. If Y is disconnected let $g : Y \to \{0, 1\}$ be a continuous surjection: then $g \circ f : X \to \{0, 1\}$ is a continuous surjection, so X is disconnected. □

Corollary. If $f : X \to Y$ is a continuous function and X is connected, then $\mathrm{Im}(f)$ is connected.

Proof. The function f induces a continuous surjection $x \mapsto f(x) : X \to \mathrm{Im}(f)$, to which we apply the previous Proposition. □

Connectedness is an intrinsic property of a topological space, which we can sometimes use to show that spaces are not homeomorphic to each other. Sometimes this involves a little trickery, as follows.

Lemma. If $f : X \to Y$ is a homeomorphism and $Z \subseteq X$ is a subspace, then $f|_Z : Z \to f(Z)$ is a homeomorphism.

Proof. The function $f|_Z : Z \to f(Z)$ is continuous by the defining property of the subspace topology. Let $g : Y \to X$ be the inverse to f. Then g sends $f(Z)$ into Z, so have a function $g|_{f(Z)} : f(Z) \to Z$ which is also continuous: it is inverse to $f|_Z$. □

Example. The spaces $[0, 1]$ and $(0, 1)$ are not homeomorphic. If $f : [0, 1] \to (0, 1)$ was a homeomorphism, then restricting to $[0, 1)$ gives a homeomorphism $f|_{[0,1)} : [0, 1) \to f([0, 1)) = (0, 1) \setminus f(1)$. But $[0, 1)$ is connected, and $(0, 1) \setminus f(1)$ is not connected.

By similar games, no two of $[0, 1]$, $[0, 1)$, $(0, 1)$ are homeomorphic. △

Example. The circle $S^1 \subset \mathbb{C}$ is not homeomorphic to $\mathbb{R}$. To see this, we notice that $S^1 \setminus \{1\}$ is homeomorphic to $(0, 1)$ via $t \mapsto e^{2\pi i t}$, so is connected. But if we remove any point from $\mathbb{R}$ then it becomes disconnected. △

The following is often convenient for checking connectedness:

Proposition. If $\{X_\alpha \subseteq X : \alpha \in I\}$ is a collection of subspaces of X such that each X_α is connected and $\bigcap_{\alpha \in I} X_\alpha$ is non-empty, then $\bigcup_{\alpha \in I} X_\alpha$ is connected.

Proof. We may as well suppose $X = \bigcup_{\alpha \in I} X_\alpha$, so let $X = U \cup V$ be a decomposition into disjoint open subsets. To show that X is connected we must show that one of U or V is empty. Each X_α is connected, so one of $X_\alpha \cap U$ and $X_\alpha \cap V$ must be empty, i.e. each X_α must lie in V or in U. But the X_α's have a point in common, so either they all lie in U or they all lie in V. But then one of U or V is empty. □

As an application, and a final source of examples, products of connected spaces are connected.

Corollary. If X and Y are connected spaces then $X \times Y$ is connected.

Proof. If either X or Y is empty so is the product, so it is connected, and hence we suppose both are non-empty. Fix an $x \in X$, and consider the sets $C_y := X \times \{y\} \cup \{x\} \times Y$. This is connected by the Proposition, as $X \times \{y\} \cong X$ and $\{x\} \times Y \cong Y$ are connected, and their intersection is $\{(x, y)\}$. Now $\{C_y : y \in Y\}$ is a collection of connected spaces which cover $X \times Y$, and $\bigcap_{y \in Y} C_y = \{x\} \times Y$ is non-empty by assumption, so the Proposition applies to show that $X \times Y$ is connected. □

2.2 Path-connectedness

A different notion of "connectedness" is: we can get from any point of X to any other point by a continuous path. This is perhaps closer to our intuition of what "connected" should mean, but is a little odd in that it gives the interval $[0, 1]$ a special status. We shall see that this notion is not quite the same as connectedness, but in some situations is an appropriate replacement. Let us first make it precise.

Definition. If X is a topological space and $x_0, x_1 \in X$, then a *path* from x_0 to x_1 is a continuous function $\gamma : [0, 1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

Definition. A topological space X is *path-connected* if for each pair $x_0, x_1 \in X$ there is a path from x_0 to x_1.

Example. The spaces $(0, 1)$, $(0, 1]$, $[0, 1)$, $[0, 1]$, $\mathbb{R}$ are all path-connected. To see this, for points x_0, x_1 in any of these spaces we can take the linear interpolation $\gamma(t) = (1-t) \cdot x_0 + t \cdot x_1$ which gives a path from x_0 to x_1.

For the same reason $\mathbb{R}^n$ is path-connected, as is any convex subset $X \subset \mathbb{R}^n$. (This is because the rigorous definition of *convex* is that $(1 - t) \cdot x_0 + t \cdot x_1 \in X$ whenever $x_0, x_1 \in X$ and $t \in [0, 1]$.) △

Example. $\mathbb{R}^2 \setminus \{0\}$ is path-connected. In this case we cannot wrote down a general formula for the path between any two points, but have to consider cases. If x_0 and x_1 do not lie on a line through the origin, then we may linearly interpolate between them as above. If they do, then we should use a bent line in order to miss the origin. △

Proposition. If X is path-connected then it is connected.

Proof. Let X be path-connected, and suppose for a contradiction that it is disconnected: let $f : X \to \{0, 1\}$ be a continuous surjective function, with $f(x_0) = 0$ and $f(x_1) = 1$. Let $\gamma : [0, 1] \to X$ be a path from x_0 to x_1. Then $f \circ \gamma : [0, 1] \to \{0, 1\}$ is continuous and surjective (as it sends 0 to 0 and 1 to 1), so $[0, 1]$ is disconnected. This is a contradiction. □

Example. $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}$. If it were, then removing a point shows that $\mathbb{R}^2 \setminus \{0\}$ is homeomorphic to $\mathbb{R} \setminus \{t\}$ for some t, but $\mathbb{R} \setminus \{t\}$ is disconnected and so not path-connected, and $\mathbb{R}^2 \setminus \{0\}$ is path-connected by the previous Example. △

It is easy to see that the product of two path-connected spaces is path-connected.

2.3 Components

If a space is not (path-)connected then it can be decomposed into disjoint subspaces which are.

2.3.1 Path components

Let X be a topological space. Define a relation $\sim$ on the set X by saying that $x \sim y$ whenever there is a path from x to y.

Lemma. $\sim$ is an equivalence relation.

Proof. We verify the three axioms of an equivalence relation.

(i) (Reflexivity) For $x \in X$ we have the path $\gamma(t) = x$ in X, which shows that $x \sim x$.

(ii) (Symmetry) If $x \sim y$, meaning that there is a path $\gamma : [0, 1] \to X$ from x to y, then $\gamma^{-1}(t) := \gamma(1 - t)$ is a path from y to x, so $y \sim x$.

(iii) (Transitivity) If $x \sim y$ and $y \sim z$ then there are paths γ from x to y and γ' from y to z. Then $\gamma \cdot \gamma'$ defined by

$$(\gamma \cdot \gamma')(t) := \begin{cases} \gamma(2t) & 0 \leq t \leq 1/2 \\ \gamma'(2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

is a path from x to z. (It is continuous by an application of the Gluing Lemma to the closed cover $\{[0, 1/2], [1/2, 1]\}$ of $[0, 1]$.) Thus $x \sim z$. □

Definition. The *path components* of X are the equivalence classes of $\sim$.

It is tautological that each path component of X is path-connected.

2.3.2 Connected components

We wish to mimic the above, so we replace (image of a) path by connected subspace: define a relation $\approx$ on X by $x \approx y$ whenever there is a connected subspace $C \subseteq X$ containing both x and y.

Lemma. $\approx$ is an equivalence relation.

Proof. We verify the three axioms of an equivalence relation. Reflexivity and Symmetry are built into the definition. For Transitivity, suppose that $x \approx y$ and $y \approx z$, that C_1 is a connected subspace containing x and y, and that C_2 is a connected subspace containing y and z. Then $C_1 \cup C_2$ is a connected subspace (it is the union of connected spaces with non-empty intersection) and contains x and z, so $x \approx z$. □

Definition. The *connected components* of X are the equivalence classes of $\approx$.

Proposition. Each connected component of X is indeed connected.

Proof. Let $C \subset X$ be a connected component, i.e. an equivalence class with respect to $\approx$. For a contradiction, suppose that $f : C \to \{0, 1\}$ is a continuous surjection, and let $f(x) = 0$ and $f(y) = 1$. As $x \approx y$, there is a connected subspace D such that $x, y \in D \subseteq X$. But now each $d \in D$ has $d \approx x$ too, as x and d both lie in the connected subspace D, so $d \in C$ and hence $D \subseteq C$. But then $f|_D : D \to \{0, 1\}$ is still a continuous surjection, which is a contradiction as D is connected. □

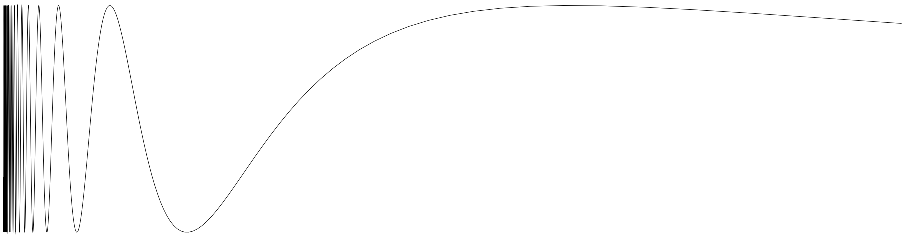
Example. The space $X = (-\infty, 0) \cup (0, \infty)$ has two connected components, $(-\infty, 0)$ and $(0, \infty)$, which are also the path components. △

Example. The subspace $X = \mathbb{Q} \subset \mathbb{R}$ has connected components the one-point sets $\{x\}$. △

As a more sophisticated example, where one sees the difference between connectivity and path-connectivity, we have the following.

Example (Topologists' sine curve)**.** Let

$$S := \{(x, \sin(\tfrac{1}{x})) \in \mathbb{R}^2 : 0 < x \leq 1\}.$$



This is not a closed set, and its closure is the set

$$\overline{S} = S \cup (\{0\} \times [-1, 1]).$$

As S is the continuous image of $(0, 1]$, it is path-connected and so connected. By the Lemma below it follows that $\overline{S}$ is connected. We will show that $\overline{S}$ is not path-connected.

It suffices to show that there is no path $\gamma : [0, 1] \to \overline{S}$ with $\gamma(0) \in \{0\} \times [-1, 1]$ and $\gamma(1) \in S$. Suppose γ is such a path. As $\{0\} \times [-1, 1]$ is closed, $\gamma^{-1}(\{0\} \times [-1, 1])$ is closed in $[0, 1]$ so has a largest element t. Then $\gamma|_{[t,1]} : [t, 1] \to \overline{S}$ is a path which starts in $\{0\} \times [-1, 1]$ and sends all other points into S. Identifying $[t, 1] \cong [0, 1]$ we may suppose that γ has this property.

Let us write $\gamma(s) = (x(s), y(s))$, so that $x(0) = 0$ and for $s > 0$ we have $y(s) = \sin(1/x(s))$. We will find a sequence $s_n \searrow 0$ in $[0, 1]$ so that $y(s_n) = (-1)^n$, which contradicts the existence of a continuous function y defined on $[0, 1]$. To do so, for each n choose a $0 = x(0) < u < x(1/n)$ such that $\sin(1/u) = (-1)^n$. Then, by the Intermediate Value Theorem, there is a $0 < s_n < 1/n$ such that $x(s_n) = u$, and hence $y(s_n) = \sin(1/x(s_n)) = \sin(1/u) = (-1)^n$. △

Lemma. The closure of a connected subspace is connected.

Proof. Let $C \subset X$ be a connected subspace, $C \subset \overline{C}$ its closure. Let $\overline{C} = U \cup V$ be a decomposition into two disjoint open non-empty subsets. Then $C \cap U$ and $C \cap V$ are two disjoint open subsets of C, and as C is connected one of them must be empty. Without loss of generality we may suppose that $C \cap V = \emptyset$, so $C \subset U$. But U and V are also closed in $\overline{C}$, so are closed in X, and hence $C \subset U \subsetneq \overline{C}$ is a properly smaller closed set containing C, a contradiction. □

3 Compactness

The concepts introduced so far have largely been generalisations of concepts for which you already had some intuition from Part IA Analysis I and Part IB Analysis II ("topological spaces" generalise "metric spaces", "continuity" generalises "continuity", "connectedness" generalises "the Intermediate Value Theorem is true", ...). In Part IB Analysis II you have seen the notion of sequential compactness: a metric space is *sequentially compact* if every sequence has a convergent subsequence. We have by now discussed sequences and their convergence in any topological space, so we may repeat this definition verbatim to say what it means for a topological space to be sequentially compact. However, we have also seen that sequences can be inadequate to probe topological spaces in general (recall the cocountable topology on $\mathbb{R}$). The solution to this is the unqualified notion of compactness, which we will now introduce. It is much less intuitive than sequential compactness, and will take some getting used to.

Definition. A collection $\mathcal{X} \subset P(X)$ of subsets of a topological space X *covers* (or *is a cover*) if $\bigcup_{S \in \mathcal{X}} S = X$, i.e. every point of X lies in at least one element of $\mathcal{X}$. An *open cover* $\mathcal{U}$ of X is a cover consisting of open subsets of X. A *subcover* of $\mathcal{X}$ is a $\mathcal{X}' \subseteq \mathcal{X}$ which is again a cover.

Definition. A topological space X is *compact* if every open cover of X has a finite subcover, i.e. if $\mathcal{U}$ is an open cover of X then there is a finite subset $\mathcal{U}' \subseteq \mathcal{U}$ which is again a cover.

Example. $\mathbb{R}$ (with the Euclidean topology) is not compact. To see this note that the (infinitely many!) open sets $\{(n-1, n+1) : n \in \mathbb{Z}\}$ cover $\mathbb{R}$, but the integer $m \in \mathbb{Z} \subset \mathbb{R}$ lies only in the set $(m-1, m+1)$, so no proper subcollection of these open sets covers $\mathbb{R}$. △

Example. If $(X, \mathcal{T})$ is a topological space such that X is finite then it is compact. To see this let $\mathcal{U}$ be an open cover. For each $x \in X$ we have $x \in U_x$ for some $U_x \in \mathcal{U}$. Then $\mathcal{U}' := \{U_x : x \in X\}$, which is finite as it has at most $|X|$ elements, is a finite subcover. △

Example. The subspace

$$X := \{0\} \cup \{\tfrac{1}{n} : n = 1, 2, 3, \ldots\} \subset \mathbb{R}$$

is compact. To see this, suppose $\mathcal{U}$ is an open cover of X. The point $0 \in X$ must lie in some $U_0 \in \mathcal{U}$, and as U_0 is an open set it follows by definition of the subspace topology that $X \cap B_\epsilon(0) \subset U_0$ for some $\epsilon > 0$. Thus $\frac{1}{n} \in U_0$ as long as $\frac{1}{n} < \epsilon$, i.e. as long as $n > \frac{1}{\epsilon}$. This leaves finitely-many points $\frac{1}{m}$ with $1 \leq m \leq \frac{1}{\epsilon}$ perhaps uncovered by U_0, so we choose sets $U_1, U_2, \ldots, U_{\lfloor 1/\epsilon \rfloor} \in \mathcal{U}$ with $\frac{1}{m} \in U_m$ for $1 \leq m \leq \frac{1}{\epsilon}$. Then $\{U_0, U_1, \ldots, U_{\lfloor 1/\epsilon \rfloor}\} \subset \mathcal{U}$ is a finite subcover.

Conversely, the subspace

$$Y := \{\tfrac{1}{n} : n = 1, 2, 3, \ldots\} \subset \mathbb{R}$$

is not compact. In this space every 1-point subset is open, as $\{\frac{1}{n}\} = Y \cap (\frac{1}{n+1}, \frac{1}{n-1})$, so every set is open: it has the discrete topology. But then it has an (infinite!) open cover $\{\{\frac{1}{n}\} : n = 1, 2, 3, \ldots\}$ which has no proper subcover at all. △

The following exploits a similar principle to the (first part of the) previous example: namely that $[0, 1] \subset \mathbb{R}$ is bounded and closed (=contains all its limit points). We will see later that the same principle extends to all bounded and closed subspaces of Euclidean spaces.

Theorem. The space $[0, 1]$ is compact.

Proof. Let $\mathcal{U}$ be an open cover of $[0, 1]$. Consider

$$A := \{a \in [0, 1] : \text{there is a finite } \mathcal{U}' \subseteq \mathcal{U} \text{ whose union contains } [0, a]\}.$$

First note that if $0 \le a \le b$ and $b \in A$ then $a \in A$. Second note that $0 \in A$, as there is some $0 \in U_0 \in \mathcal{U}$ so we may take $\mathcal{U}' := \{U_0\}$. As the set of real numbers A is non-empty, it has a supremum $\alpha \in [0, 1]$. We wish to show that $\alpha = 1$.

Let $\alpha \in U_\alpha \in \mathcal{U}$. Supposing that $\alpha < 1$ then there is an $\epsilon > 0$ such that $\alpha \in (\alpha - \epsilon, \alpha + \epsilon) \subset U_\alpha$. But $\alpha - \epsilon < \alpha$ so by definition of supremum there exists an $\alpha - \epsilon < a \le \alpha$ with $a \in A$. But then $\alpha - \epsilon \in A$ too, so there is a finite collection $U^1, \ldots, U^n \in \mathcal{U}$ such that $U^1 \cup \cdots \cup U^n \supset [0, \alpha - \epsilon]$. Adding U_α to this collection, we have

$$U^1 \cup \cdots \cup U^n \cup U_\alpha \supseteq [0, \alpha - \epsilon] \cup (\alpha - \epsilon, \alpha + \epsilon) = [0, \alpha + \epsilon) \supseteq [0, \alpha + \epsilon/2].$$

Thus $\alpha + \epsilon/2 \in A$, but α was the supremum of A, giving a contradiction. Thus $\alpha = 1$. □

3.1 Elementary properties of compactness

Proposition. If X is a compact topological space and $C \subseteq X$ is a closed subspace, then C is compact.

Proof. Open sets in C have the form $C \cap U$ for U open in X. So suppose $\{C \cap U_\alpha : \alpha \in I\}$ is an open cover of C: we shall produce a finite subcover. Now $X \setminus C$ is open, because C is closed, and

$$\{X \setminus C\} \cup \{U_\alpha : \alpha \in I\}$$

is an open cover of X, because $\bigcup_{\alpha \in I} U_\alpha \supseteq \bigcup_{\alpha \in I} U_\alpha \cap C = C$. As X is compact this has a finite subcover, which we may as well suppose has the form

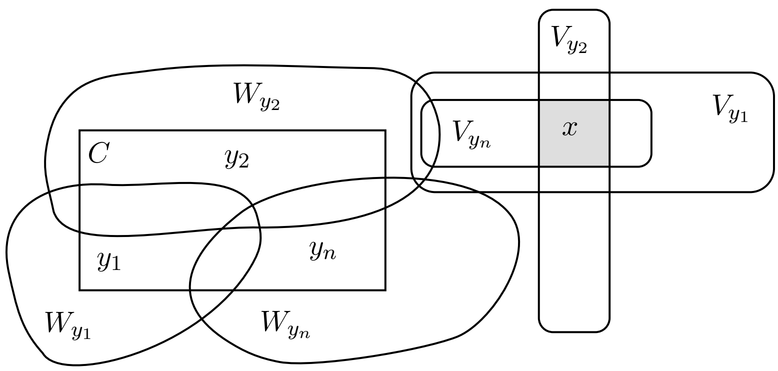
$$\{X \setminus C\} \cup \{U_\alpha : \alpha \in I'\}$$

for $I' \subseteq I$ finite (if $X \setminus C$ is not part of the subcover we may as well add it). Intersecting with C it follows that $\{C \cap U_\alpha : \alpha \in I'\}$ is a finite subcover of $\{C \cap U_\alpha : \alpha \in I\}$. □

Proposition. If X is a Hausdorff topological space and $C \subseteq X$ is a compact subspace, then it is closed.

Proof. We will show that $U := X \setminus C$ is open, by finding for each $x \in U$ an open neighbourhood $x \in U_x \subseteq U$.

To do so, fix such an $x \in U$ then for each $y \in C$ we may use the Hausdorff property to find disjoint open sets $V_y \ni x$ and $W_y \ni y$. Then the collection $\{C \cap W_y : y \in C\}$ is an open cover of C, so by compactness of C it has a finite subcover: let $y_1, \ldots y_n \in C$ be such that $\{C \cap W_{y_1}, \ldots, C \cap W_{y_n}\}$ is such a finite subcover.



Then

$$x \in \bigcap_{i=1}^{n} V_{y_i} =: U_x$$

is a *finite* intersection of open neighbourhoods of x, so is an open neighbourhood of x. It is disjoint from each W_{y_j} (because V_{y_j} is), so is disjoint from $\bigcup_{j=1}^{n} W_{y_j}$ and hence from $C = \bigcup_{j=1}^{n} C \cap W_{y_j}$ too. Thus it lies in $U = X \setminus C$. □

Proposition. If X is a compact topological space and $f : X \to Y$ is a continuous function, then $f(X)$ is a compact subspace of Y.

Proof. Without loss of generality we may replace Y by $f(X)$ and hence suppose that f is surjective. Let $\mathcal{U}$ be an open cover of $Y = f(X)$, and set

$$f^{-1}\mathcal{U} := \{f^{-1}(U) : U \in \mathcal{U}\}.$$

This is an open cover of X. As X is compact it has a finite subcover, so suppose that the sets $f^{-1}(U_1), \ldots, f^{-1}(U_n)$ cover X. Then $U_1 = f(f^{-1}(U_1)), \ldots, U_n = f(f^{-1}(U_n))$ cover $f(X) = Y$ and is a finite subcover of $\mathcal{U}$. □

Corollary. If $f : X \to Y$ is a continuous bijection from a compact topological space X to a Hausdorff topological space Y, then f is a homeomorphism.

Proof. To show that the function $f^{-1} : Y \to X$, which exists because f is a bijection, is continuous, we may use the closed set description f continuity. So we must show that $(f^{-1})^{-1}(C)$ is closed in Y whenever C is closed in X, i.e. that the image under f of a closed set is closed.

If $C \subseteq X$ is closed then it is compact, using the first Proposition above and the fact that X is compact. Thus $f(C) \subseteq Y$ is compact by the third Proposition above, and hence is closed by the second Proposition above and the fact that Y is Hausdorff. □

3.2 Compactness, sequential compactness, and the compact subsets of $\mathbb{R}^d$

We compare compactness with the notion of sequential compactness from Part IB Analysis II (in the case of metric spaces). In the following we suppose that (X, d) is a metric space, which gives a topological space $(X, \mathcal{T}_d)$.

Lemma (Lebesgue Number Lemma)**.** Let (X, d) be sequentially compact, and $\mathcal{U} \subset \mathcal{T}_d$ be an open cover. Then there is a $\delta > 0$ such that each ball $B_\delta(x)$ lies inside some $U_x \in \mathcal{U}$.

Proof. Suppose for a contradiction that no such $\delta > 0$ exists. Then in particular for each $n = 1, 2, 3, \ldots$ there is an $x_n \in X$ such that the ball $B_{1/n}(x_n)$ is not contained in any element of $\mathcal{U}$. By sequential compactness the sequence (x_n) has a convergent subsequence, say $x_{n_i} \to x_\infty$. Let $x_\infty \in U \in \mathcal{U}$, and $\epsilon > 0$ be such that $B_\epsilon(x_\infty) \subseteq U$. For all $i \gg 0$ we have $x_{n_i} \in B_{\epsilon/2}(x_\infty)$ and $\frac{1}{n_i} < \frac{\epsilon}{2}$. By the triangle inequality we then have $B_{1/n_i}(x_{n_i}) \subseteq B_\epsilon(x_\infty) \subseteq U \in \mathcal{U}$, a contradiction. □

Such a $\delta > 0$ is called a *Lebesgue number* for the open cover $\mathcal{U}$: the lemma is saying that every open cover of a sequentially compact metric space has a Lebesgue number.

Theorem. (X, d) is sequentially compact if and only if $(X, \mathcal{T}_d)$ is compact.

Proof. Suppose first that (X, d) is not sequentially compact. Then there is a sequence (t_n) in X without a convergent subsequence. Thus for each $x \in X$ there is an open neighbourhood $U_x \ni x$ which contains only finitely-many t_i's. (Otherwise the sequence of non-negative real numbers $(d(x, t_n))$ would have a subsequence converging to zero, i.e. (t_n) would have a subsequence converging to x.) Then $\mathcal{U} := \{U_x : x \in X\}$ is an open cover of X. If it had a finite subcover then as each U_x contains only finitely-many t_i's, it would follow that the sequence (t_n) only takes finitely-many values. But this is impossible as then it would take one of those values infinitely often and so would have a convergent subsequence. Thus $\mathcal{U}$ does not have a finite subcover, so $(X, \mathcal{T}_d)$ is not compact.

Now suppose that (X, d) is sequentially compact, and let $\mathcal{U}$ be an open cover of $(X, \mathcal{T}_d)$. Let $\delta > 0$ be a Lebesgue number for this cover, which exists by the Lebesgue Number Lemma. We will first show that there exists a finite set $A \subseteq X$ such that[1]

$$X = \bigcup_{a \in A} B_\delta(a).$$

If not, then for every finite set $A \subseteq X$ there exists a $x \in X$ with $d(x, a) \geq \delta$ for all $a \in A$. If this were the case then we could construct a sequence $x_1, x_2, \ldots$ in X inductively by choosing x_i to have distance $\geq \delta$ from each element of the finite set $\{x_1, \ldots, x_{i-1}\}$. This sequence has $d(x_i, x_j) \geq \delta$ for all $i \neq j$, and so has no convergent subsequence, a contradiction. So such a finite set A does exist.

As δ is a Lebesgue number for the open cover $\mathcal{U}$, each $B_\delta(a)$ lies inside a $U_a \in \mathcal{U}$. Then $\{U_a : a \in A\} \subseteq \mathcal{U}$ is a finite subcover. Thus $(X, \mathcal{T}_d)$ is compact. □

Corollary (Heine–Borel(=Bolzano–Weierstrass))**.** A subspace $X \subseteq \mathbb{R}^d$ is compact if and only if it is closed and bounded.

Proof. In Part IB Analysis II this is proved with "compact" replaced by "sequentially compact", but by the above Theorem these notions are the same for metric spaces. □

Corollary (Extreme Value Theorem)**.** If X is a compact topological space and $f : X \to \mathbb{R}$ is a continuous function, then there are elements $a, b \in X$ such that $f(a) \leq f(x) \leq f(b)$ for all $x \in X$.

Proof. The set $f(X) \subseteq \mathbb{R}$ is compact by an earlier Proposition, so is closed and bounded. In particular it contains its supremum B and its infimum A, and $f(x) \subseteq [A, B]$. Choose $a \in X$ such that $f(a) = A$ and $b \in X$ such that $f(b) = B$: then $A = f(a) \leq f(x) \leq f(b) = B$ for all $x \in X$. □

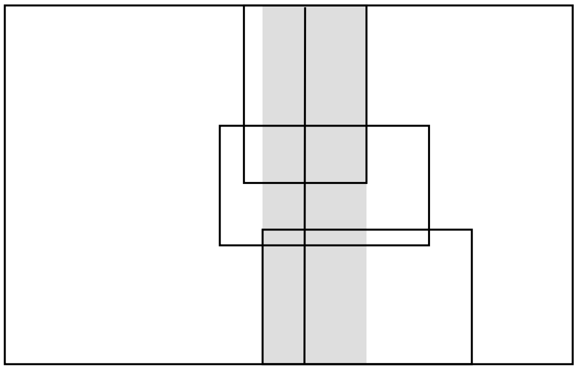
3.3 Compactness of products

Theorem. If X and Y are compact topological spaces, then the product space $X \times Y$ is compact.

Proof. We first prove the following, which only uses that Y is compact:

Claim. If $x_0 \in X$ and $\{x_0\} \times Y \subseteq W$ is an open set, then there is an open neighbourhood $U_{x_0} \ni x_0$ in X such that $U_{x_0} \times Y \subset W$.

[1] This is the definition of a metric space being "totally bounded" from Part IB Analysis II, so we are repeating(?) part of the proof of "sequentially compact = complete and totally bounded".



Proof of Claim. We may cover $\{x_0\} \times Y$ by (its intersection with) open sets of the form $U \times V \subset W$ for U open in X and V open in Y, as such sets are a basis for the product topology on $X \times Y$. As $\{x_0\} \times Y$ is compact this open cover has a finite subcover, so we may find finitely-many such basis elements

$$U_1 \times V_1, \ldots, U_n \times V_n$$

inside W whose union contains $\{x_0\} \times Y$. Let $U_{x_0} := U_1 \cap \cdots \cap U_n$, an intersection of finitely-many open neighbourhood of $x_0 \in X$, so again an open neighbourhood of x_0. Now if $(x, y) \in U_{x_0} \times Y$ then consider $(x_0, y) \in \{x_0\} \times Y$ which lies inside $U_i \times V_i$ for some i. In particular $y \in V_i$. But also $x \in U_{x_0} \subseteq U_i$, so $(x, y) \in U_i \times V_i \subseteq \bigcup_{j=1}^n U_j \times V_j \subseteq W$. Thus $U_{x_0} \times Y \subseteq W$, as required. □

Now suppose that $\mathcal{W}$ is an open cover of $X \times Y$. For $x_0 \in X$ fixed, $\{x_0\} \times Y$ is compact so may be covered by (its intersection with) finitely many $W_1, \ldots, W_n \in \mathcal{W}$, so $\{x_0\} \times Y \subseteq \bigcup_{i=1}^n W_i$ is an open neighbourhood. By the Claim there is a neighbourhood $U_{x_0} \ni x_0$ such that $U_{x_0} \times Y \subseteq \bigcup_{i=1}^n W_i$, i.e. $U_{x_0} \times Y$ may be covered by finitely-many elements of $\mathcal{W}$.

Doing the above for each point $x_0 \in X$, we obtain an open cover $\{U_{x_0} : x_0 \in X\}$ of X. As X is compact, this has a finite subcover $\{U_{x_1}, \ldots, U_{x_m}\}$. Now the finitely-many sets

$$U_{x_1} \times Y, \ldots, U_{x_m} \times Y$$

cover $X \times Y$, and each of them may be covered by finitely-many elements of $\mathcal{W}$. Thus $\mathcal{W}$ has a finite subcover. □

Corollary. $[0, 1]^d$ is compact.

Proof. We have seen that $[0, 1]$ is compact, and the Theorem implies (inductively) that finite products of compact spaces are compact. □

Corollary (Heine–Borel again, topologically)**.** A subspace $X \subseteq \mathbb{R}^d$ is compact if and only if it is closed and bounded.

Proof. Suppose that $X \subseteq \mathbb{R}^d$ is closed and bounded. Then for some $N \gg 0$ it is contained in $[-N, N]^d$. This is homeomorphic to $[0, 1]^d$ so is compact, and we have seen that a closed subspace of a compact space is compact.

Suppose that $X \subseteq \mathbb{R}^d$ is compact. Then it is closed as $\mathbb{R}^d$ is Hausdorff and we have seen that compact subsets of a Hausdorff space are closed. To see that it is bounded, we consider the open cover

$$\{X \cap (-N, N)^d : N \in \mathbb{N}\}$$

of X. As X is compact this must have a finite subcover

$$X \cap (-N_1, N_1)^d, \ldots X \cap (-N_n, N_n)^d$$

with $N_1 \leq \cdots \leq N_n$. But these sets are all contained in $X \cap (-N_n, N_n)^d$, so $X \subset (-N_n, N_n)^d$. □

4 Topology of manifolds

4.1 What is a manifold?

Definition. A *topological manifold* of dimension d is a topological space X which is Hausdorff, and such that every point of X has an open neighbourhood which is homeomorphic to an open subset of $\mathbb{R}^d$. The latter property is abbreviated to *locally Euclidean*.

As a point p in an open subset $U \subseteq \mathbb{R}^d$ is contained in some ball $B_\epsilon(p) \subseteq U$, which is homeomorphic to $\mathbb{R}^d$, every point in a topological manifold has a neighbourhood homeomorphic to $\mathbb{R}^d$ itself. We call this a *Euclidean neighbourhood*.

Remark. In some sources a further axiom is imposed: X is also required to be "*second-countable*", meaning that its topology should have a countable basis. We shall not impose this axiom, though a topological manifold (in our sense) which is compact is indeed second countable: see Example Sheet 3.

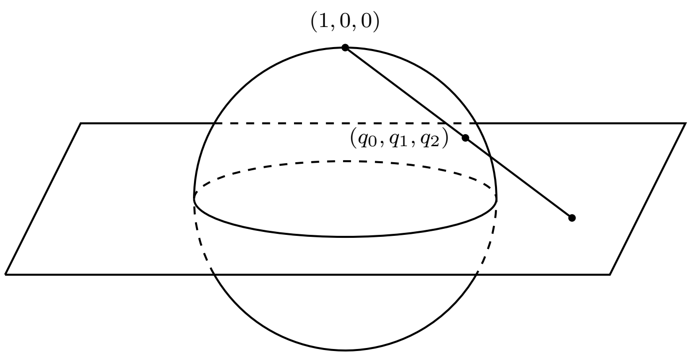
Example. An open subset $U \subseteq \mathbb{R}^d$ is a topological manifold of dimension d. It is Hausdorff as $\mathbb{R}^d$ is, and is locally Euclidean as any point of U has U as an open neighbourhood, which is (homeomorphic to) an open subset of $\mathbb{R}^d$. △

Example. The sphere $S^d \subset \mathbb{R}^{d+1}$ is a topological manifold of dimension d. It is Hausdorff as $\mathbb{R}^{d+1}$ is. To see it is locally Euclidean, we will show that for any point $p \in S^d$ the open neighbourhood $p \in S^d \setminus \{-p\}$ is homeomorphic to $\mathbb{R}^d$. By rotating, we may suppose without loss of generality that $p = (-1, , 0 \ldots, 0)$. Then *stereographic projection*

$$\begin{aligned} S^d \setminus \{(1, 0, \ldots, 0)\} &\longrightarrow \mathbb{R}^d \\ q &\longmapsto \text{the intersection of the line } (-p)q \text{ with the plane } \{0\} \times \mathbb{R}^d \\ (q_0, \ldots, q_d) &\longmapsto \frac{1}{1 - q_0}(q_1, \ldots, q_d) \end{aligned}$$

is a continuous bijection, with continuous inverse

$$(e_1, \ldots, e_d) \longmapsto \frac{1}{\sum e_i^2 + 1}\left(\sum e_i^2 - 1, 2e_1, \ldots, 2e_d\right).$$



△

Example. The *circle* S^1 is the special case $d = 1$ of the above, but this topological manifold also has several other useful descriptions.

There is an equivalence relation $\sim$ on $\mathbb{R}$ given by

$$x \sim y \iff x - y \in \mathbb{Z}.$$

It is usual to write $\mathbb{R}/\mathbb{Z}$ for the quotient space: it is the quotient by the action of the group $(\mathbb{Z}, +, 0)$ on the set $\mathbb{R}$ by translation, with the quotient topology. We have seen earlier how to identify this quotient: The function

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto (\sin(2\pi x), \cos(2\pi x)) \end{aligned}$$

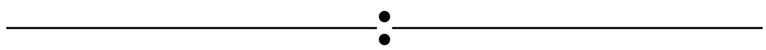
is continuous and $f(x) = f(x')$ if and only if $x \sim x'$, so the universal property of the quotient topology therefore gives a continuous bijection

$$e : \mathbb{R}/\mathbb{Z} \longrightarrow S^1.$$

Every equivalence class for $\sim$ has a representative in $[0, 1] \subset \mathbb{R}$, so $\mathbb{R}/\mathbb{Z}$ is also the quotient of $[0, 1]$ by the induced equivalence relation. In particular it is compact, and S^1 is Hausdorff, so e is a homeomorphism.

Similarly, S^1 can also be described as the quotient space $[0, 1]/0 \sim 1$ △

Example. Recall from earlier in the course the line with two origins $L = \mathbb{R} \times \{0, 1\}/\sim$ where $(x, 0) \sim (x, 1)$ for all $x \neq 0$, visualised as:



Away from the origins it is clearly locally homeomorphic to $\mathbb{R}$. Writing $\pi : \mathbb{R} \times \{0, 1\} \to L$ for the quotient map, the "top origin" $[0, 1]$ is contained in the set $U := \pi((-1, 1) \times \{1\})$, which is open as

$$\pi^{-1}(U) = (-1, 1) \times \{1\} \cup (-1, 0) \times \{0\} \cup (0, 1) \times \{0\}$$

is open in $\mathbb{R} \times \{0, 1\}$. The map $\pi|_{(-1,1)\times\{1\}} : (-1, 1) \times \{1\} \to U$ is a continuous bijection. If $W \subseteq (-1, 1)$ is open then

$$\pi^{-1}(\pi|_{(-1,1)\times\{1\}}(\{1\} \times W)) = W \times \{1\} \cup (W \setminus \{0\}) \times \{0\}$$

is open in $\mathbb{R} \times \{0, 1\}$, so $\pi|_{(-1,1)\times\{1\}}(W \times \{1\})$ is open. Thus the inverse function $\pi|^{-1}_{(-1,1)\times\{1\}} : U \to (-1, 1) \times \{1\}$ is also continuous, i.e. $\pi|_{(-1,1)\times\{1\}} : (-1, 1) \times \{1\} \to U$ is a homeomorphism, giving an open neighbourhood of the "top origin" $[0, 1]$ which is homeomorphic to an open subset of $\mathbb{R}$. A similar argument shows that the "bottom origin" $[0, 0]$ has an open neighbourhood homeomorphic to an open subset of $\mathbb{R}$.

But L is *not a manifold*, as it is not Hausdorff: any open neighbourhoods of $[1, 0]$ and $[0, 0]$ intersect. △

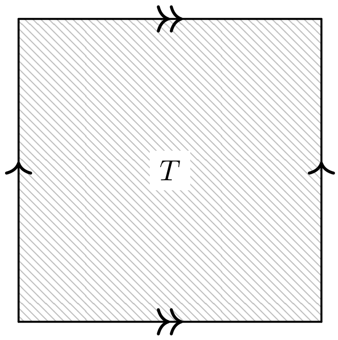
Example. If M and N are topological manifolds, then so is their product $M \times N$. It is easy to see that a product of two Hausdorff spaces is Hausdorff, using the basis for the product topology. If $(m, n) \in M \times N$, let $m \in U \subset M$ be an open neighbourhood homeomorphic to an open subset of $\mathbb{R}^m$, and $n \in V \subset N$ be an open neighbourhood homeomorphic to an open subset of $\mathbb{R}^n$, then $(m, n) \in U \times V \subseteq M \times N$ is an open neighbourhood homeomorphic to an open subset of $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$. Its dimension is the sum of the dimensions of M and N. △

Example. As an example of the above, the *torus* $T := S^1 \times S^1$ is a topological manifold of dimension 2. Using the description $S^1 \cong \mathbb{R}/\mathbb{Z}$ we have another description of T as $\mathbb{R}^2/\mathbb{Z}^2$, the quotient of the $\mathbb{Z}^2$-action on $\mathbb{R}^2$ by translation.

Using the description $S^1 = [0,1]/0 \sim 1$ gives another description of the torus, as $[0,1] \times [0,1]/\approx$ where the equivalence relation is generated by

$$(0,t) \approx (1,t), \quad (s,0) \approx (s,1) \quad \text{for all } s,t \in [0,1].$$

We can depict the identification made by the equivalence relation as follows:

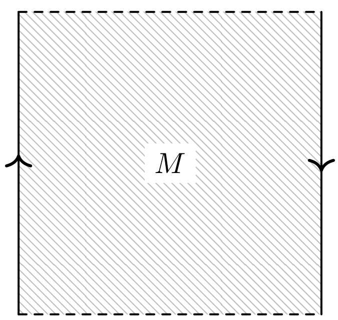


△

Example. Consider the equivalence relation $\sim$ on $[0,1] \times \mathbb{R}$ generated by

$$(1,v) \sim (0,-v),$$

with quotient space M. This is the *Möbius band*. A depiction analogous to that above is:



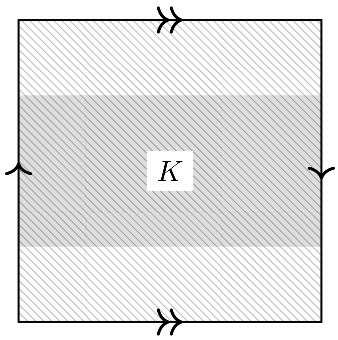
This is a 2-dimensional topological manifold. To see this, consider the subspaces U_t obtained by removing $L_t := \pi(\{t\} \times \mathbb{R})$ from M with $0 < t < 1$, which is the induced quotient of $([0,t) \cup (t,1]) \times \mathbb{R}$. By applying the homeomorphism $(x,v) \mapsto (x,-v)$ to $(t,1] \times \mathbb{R}$, we see that

$$U_t = M \setminus L_t \cong ((t,1] \cup [0,t)) \times \mathbb{R}/(1,v) \sim (0,v) \cong (-1,1) \times \mathbb{R}.$$

The final identification is analogous to Q10 on Example Sheet 1. The open sets U_t cover M and are Euclidean, and any two points in M lie in a single U_t so it follows that M is Hausdorff. △

Example. Combining the last two examples, consider the equivalence relation on $[0,1] \times [0,1]$ given by

$$(0,t) \approx (1,t), \quad (s,0) \approx (1-s,1) \quad \text{for all } s,t \in [0,1].$$



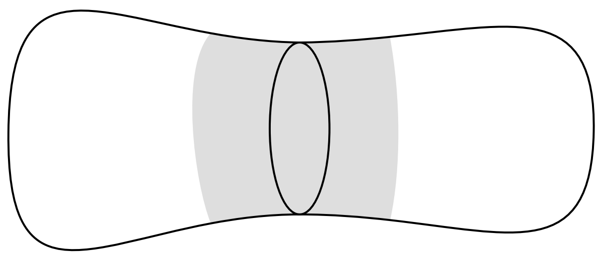
One can check, similarly to the above, that the quotient K is a 2-dimensional topological manifold, called the *Klein bottle*. It visibly contains the Möbius band as a subspace, depicted in grey. △

Example. If M is a d-dimensional topological manifold and $m \in M$ is a point, then we may find a Euclidean neighbourhood $U \ni m$ with $\varphi : U \overset{\cong}{\to} \mathbb{R}^d$. If N is another d-dimensional topological manifold and $n \in V \subseteq N$ is a Euclidean neighbourhood with $\psi : V \overset{\cong}{\to} \mathbb{R}^d$, then the[2] *connected-sum* is the quotient space

$$M\#N := \big((M \setminus \varphi^{-1}(B_1(0))) \sqcup (N \setminus \psi^{-1}(B_1(0)))\big) / \sim,$$

where the equivalence relation $\sim$ is generated by

$$\varphi^{-1}(v) \sim \psi^{-1}(v) \text{ whenever } v \in \mathbb{R}^d \text{ has } |v| = 1.$$



This is covered by three open sets. Firstly we have $M \setminus \varphi^{-1}(\overline{B_1(0)})$ and $N \setminus \psi^{-1}(\overline{B_1(0)})$, which are open subsets of topological manifolds so are themselves topological manifolds. Secondly we have

$$\big((U \setminus \varphi^{-1}(B_1(0))) \sqcup (V \setminus \psi^{-1}(B_1(0)))\big) / \sim.$$

Using the induced homeomorphisms

$$U \setminus \varphi^{-1}(B_1(0)) \cong_\varphi \mathbb{R}^d \setminus B_1(0) \cong S^{d-1} \times (-\infty, 0]$$
$$V \setminus \psi^{-1}(B_1(0)) \cong_\psi \mathbb{R}^d \setminus B_1(0) \cong S^{d-1} \times [0, \infty)$$

[2] As we have described it, "the" connected-sum $M\#N$ involves many choices: of points in M and N, and of Euclidean neighbourhoods of those points. In principle the resulting space depends on these choices, so one should be wary of thinking about $-\#-$ as some kind of binary operation on d-dimensional topological manifolds: better think of "a" connected-sum. In fact the dependence on these choices is only slight, but it is very difficult to prove this: it uses the Annulus Theorem, whose proof (in the final case $d = 4$) was only completed in 1982.

we see that this quotient is homeomorphic to $S^{d-1} \times \mathbb{R}$, so is also a topological manifold. It follows that $M \# N$ is locally Euclidean, and by considering cases it can be checked to be Hausdorff, so it is a topological manifold.

$\triangle$

Example. The real projective space $\mathbb{RP}^d$ is the set of 1-dimensional linear subspaces of $\mathbb{R}^{d+1}$. We can give it a topology by considering it as the quotient of $\mathbb{R}^{d+1} \setminus \{0\}$ by the equivalence relation $x \sim \lambda x$ for λ a non-zero real number. We write $[x_0 : x_1 : \cdots : x_d]$ for the equivalence class of the point $(x_0, \ldots, x_d)$, which is a non-zero vector so some x_i must be non-zero.

The subspaces

$$A_i := \{[x_0 : x_1 : \cdots : x_d] \in \mathbb{RP}^d : x_i \neq 0\}$$

therefore cover $\mathbb{RP}^d$. Their preimages under the quotient map are $\{(x_0, \ldots, x_d) \in \mathbb{R}^{d+1} \setminus 0 : x_i \neq 0\}$ which are open, so the A_i are open in $\mathbb{RP}^d$. Furthermore the function

$$\pi|_{P_i} : P_i := \{(x_0, \ldots, x_d) \in \mathbb{R}^{d+1} \setminus 0 : x_i = 1\} \longrightarrow A_i$$

is a continuous bijection, and is easily checked to be a homeomorphism. As P_i is homeomorphic to $\mathbb{R}^d$ (via $(x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_d) \leftrightarrow (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d)$) it follows that $\mathbb{RP}^d$ is locally Euclidean.

To see that $\mathbb{RP}^d$ is Hausdorff it is convenient to observe that $GL_{d+1}(\mathbb{R})$ acts on it by homeomorphisms (as it acts on $\mathbb{R}^{d+1}$ by homeomorphisms). As every pair of linearly-independent vectors in $\mathbb{R}^{d+1}$ extends to a basis, this action can be used to move any two distinct points in $\mathbb{RP}^d$ to any other two: so without loss of generality we may suppose that the points we wish to separate are $[1 : 0 : \cdots : 0]$ and $[1 : 1 : 0 : \cdots : 0]$. These both lie in the open Euclidean neighbourhood $A_1 \cong \mathbb{R}^d$, so these two points can indeed be separated by open sets.[3]

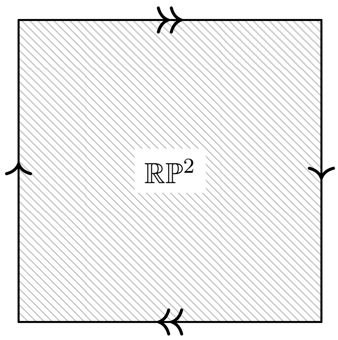
The above perspective on $\mathbb{RP}^d$ is based on the observation that every 1-dimensional linear subspace of $\mathbb{R}^{d+1}$ contains a non-zero vector, which is well-defined up to non-zero scaling. Alternatively, we may use that every 1-dimensional linear subspace contains a unit vector, which is well-defined up to sign. This gives another description of the topological space $\mathbb{RP}^d$ as the quotient space $S^d/\sim$ where $x \sim -x$. You may verify that the two points of view give the same topology on the set $\mathbb{RP}^d$. (This leads to a different proof of Hausdorffness, using that S^d is Hausdorff and that $\sim$ only identifies distinct points which are far apart.)

We can simplify the description just given by observing that every equivalence class for the relation $x \sim -x$ on S^d has a representative in the upper hemisphere $D^d \subset S^d$, and the induced equivalence relation on D^d is given by $x \approx -x$ when x is on the boundary of D^2. In particular the *projective plane* $\mathbb{RP}^2$, a 2-dimensional topological manifold, can be obtained from D^2 by identifying antipodal points on the boundary of the disc. Similarly to the pictures for the torus and Klein bottle, we may depict this as follows:

[3] The points $[1 : 0 : \cdots : 0]$ and $[0 : 1 : 0 : \cdots : 0]$, for example, don't both lie in any single A_i, so it is a little annoying to describe how to separate them directly. I have chosen to use the double transitivity of the $GL_{d+1}(\mathbb{R})$-action is used to get out of this degenerate situation. The only thing that is special about this situation is that we have used the coordinate directions in $\mathbb{R}^d$ to describe the sets A_i, which gives points like these a seemingly spacial status. But this is artificial: instead of using only the coordinate directions we could have used the sets

$$A_\ell := \{[x_0 : x_1 : \cdots : x_d] \in \mathbb{RP}^d : \ell(x_0, \ldots, x_d) \neq 0\}$$

indexed by all linear maps $\ell : \mathbb{R}^{d+1} \to \mathbb{R}$, to cover $\mathbb{RP}^d$. These are easily shown to be open and Euclidean too, any it is true that any two points in $\mathbb{RP}^d$ lie in some A_ℓ so can be separated easily.



△

The first non-trivial result we can prove about topological manifolds is that connectivity and path-connectivity agree for them. This applies in particular to an open subset $U \subset \mathbb{R}^d$.

Theorem. A topological manifold is connected if and only if it is path-connected.

Proof. We already know that path-connected spaces are connected. Suppose then that X is a connected topological manifold. The path-components of X partition it into disjoint subsets, so to show that there is only path-component it suffices to show that the path-components are open sets: if there was more than one of them this would show that X is not connected, but it is.

To show that the path-components are open we must show the following: for each $p \in X$ there is an open neighbourhood of p consisting of points which can be connected to p by a path. But as X is locally Euclidean, and Euclidean space is path-connected, a Euclidean neighbourhood of p has this property. □

This kind of argument is quite powerful. Note that we didn't explicitly say *how* to find a path between any two points, but did show that there must be one. We will use the same principle again in the following section.

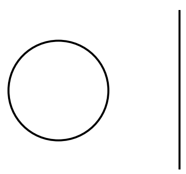
4.2 Connected manifolds are homogeneous

If X is a topological manifold of dimension d, then by definition every point of X has a Euclidean neighbourhood and so any two points have open neighbourhoods which are homeomorphic to each other. But the line-with-two-origins is also locally Euclidean, and yet the two origins are somehow different in nature to the other points: not all points "look like" all other points. However in a topological manifold all points do "look like" all other points, it the following precise sense:

Theorem. If X is a connected topological manifold then for any two points $p, q \in X$ there is a homeomorphism $\varphi : X \xrightarrow{\cong} X$ such that $\varphi(p) = q$.

In other words, the symmetry group of X—which is the group of its self-homeomorphisms—acts transitively on X. The connectivity assumption is essential:

Example. Consider the 1-dimensional topological manifold $X = S^1 \sqcup \mathbb{R}$.



No self-homeomorphism of X can take a point in the S^1 to a point in the $\mathbb{R}$. This is because removing a point in the S^1 gives a topological space with two path-components, but removing a point in the $\mathbb{R}$ gives a topological space with three path-components. If there were a homeomorphism φ sending a point of S^1 to a point of $\mathbb{R}$, it would give a homeomorphism between the complements of these points.

$\triangle$

To prove the Theorem, we start with the following lemma which gives us a supply of homeomorphisms of $\mathbb{R}^d$.

Lemma. If $p, q \in B_1(0) \subset \mathbb{R}^d$, there is a homeomorphism φ of $\mathbb{R}^d$ which sends p to q, and which is the identity outside of $B_1(0)$.

Proof. We may suppose without loss of generality that $p = 0$, as if there is such a homeomorphism φ sending 0 to q, and another ψ sending 0 to p, then $\varphi \circ \psi^{-1}$ sends p to q.

Consider first the homeomorphism

$$\begin{aligned} \rho : \mathbb{R}^d &\longrightarrow B_1(0) \\ x &\longmapsto \tfrac{x}{1+|x|}, \end{aligned}$$

which has inverse given by $\rho^{-1}(y) = \frac{y}{1-|y|}$. There is homeomorphism $x \mapsto x + \rho^{-1}(q)$ of $\mathbb{R}^d$ given by translation, and when conjugated by ρ this gives a homeomorphism

$$\begin{aligned} \varphi'' : B_1(0) &\longrightarrow B_1(0) \\ y &\longmapsto \rho(\rho^{-1}(y) + \rho^{-1}(q)), \end{aligned}$$

which sends 0 to q. This extends to a function

$$\begin{aligned} \varphi' : \overline{B_1(0)} &\longrightarrow \overline{B_1(0)} \\ y &\longmapsto \begin{cases} \varphi''(y) & |y| < 1 \\ y & |y| = 1, \end{cases} \end{aligned}$$

which we will show is continuous. We do so using the sequential characterisation of continuity, as these are metric spaces. If (y_n) is a sequence in $B_1(0)$ converging to $y_\infty \in \overline{B_1(0)}$ on the boundary, then $|y_n| \to 1$ so $1 - |y_n| \searrow 0$. Then

$$\varphi'(y_n) = \varphi''(y_n) = \rho(\rho^{-1}(y_n) + \rho^{-1}(q)) = \frac{\frac{y_n}{1-|y_n|} + \frac{q}{1-|q|}}{1 + |\frac{y_n}{1-|y_n|} + \frac{q}{1-|q|}|} = \frac{y_n + \frac{1-|y_n|}{1-|q|} q}{1 - |y_n| + |y_n + \frac{1-|y_n|}{1-|q|} q|} \to y_\infty$$

by standard estimates. Thus φ' is indeed continuous, and it is therefore a homeomorphism, as it is a continuous bijection from a compact to a Hausdorff space. Then the function

$$\begin{aligned} \varphi : \mathbb{R}^d &\longrightarrow \mathbb{R}^d \\ y &\longmapsto \begin{cases} \varphi'(y) & |y| \leq 1 \\ y & |y| \geq 1 \end{cases} \end{aligned}$$

is continuous by the Gluing Lemma, as is its inverse. $\square$

Proof of the Theorem. Consider the equivalence relation $\sim$ on X given by

$$p \sim q \iff \text{there is a homeomorphism } \varphi \text{ sending } p \text{ to } q.$$

The equivalence classes partition X into disjoint subsets, so to show that there is only one equivalence class it suffices to show that the equivalence classes are open sets: if there was more than one equivalence class this would show that X is not connected, but it is.

To show the equivalence classes are open we must show the following: that for each $p \in X$ there is an open neighbourhood of p consisting of points which can be reached from p by applying a homeomorphism.

Let $U \ni p$ be a Euclidean neighbourhood, with $\tau : U \to \mathbb{R}^d$ sending p to 0. Set $C := \tau^{-1}\overline{B_1(0)} \subseteq U$. For each $q \in \mathring{C} = \tau^{-1}B_1(0)$ the Lemma shows there is a homeomorphism φ_q of $\mathbb{R}^d$ which sends 0 to $\tau(q)$ and which is the identity outside of $B_1(0)$. Then $\tau^{-1} \circ \varphi_q \circ \tau$ is a homeomorphism of U sending p to q and which is the identity outside of C. Define a function

$$\begin{aligned} \psi_q : X &\longrightarrow X \\ x &\longmapsto \begin{cases} x & x \notin C \\ \tau^{-1} \circ \varphi_q \circ \tau(x) & x \in U, \end{cases} \end{aligned}$$

which sends p to q, and which we shall show is a homeomorphism. It is well-defined and a bijection, as $\tau^{-1} \circ \varphi_q \circ \tau$ is the identity outside of C. It is continuous when restricted to the open set U, and is also continuous when restricted to the set $X \setminus C$. Crucially, *the set* $X \setminus C$ *is open*, because X is Hausdorff and C is homeomorphic to $\overline{B_1(0)}$ so is compact and hence closed.[4] By the Gluing Lemma ψ_q is continuous, and the same reasoning shows that ψ_q^{-1} is continuous. Thus ψ is a homeomorphism sending p to q.

This holds for all $q \in \mathring{C} = \tau^{-1}B_1(0)$, so the latter is an open neighbourhood of p consisting of points which can be reached from p by applying a homeomorphism, as required. □

4.3 Compact manifolds embed into Euclidean space

Some of the examples of topological manifolds we have seen arise naturally inside some Euclidean space, or else can be put inside some Euclidean space without much difficulty: S^n and the torus can; if two manifolds can then so can their product; with a little work one can see that if two manifolds can then so can their connected-sum. But in fact all compact manifolds can be put inside some Euclidean space.

Theorem. Let X be a compact d-dimensional topological manifold. Then there is an $N \gg 0$ and a subspace $Y \subseteq \mathbb{R}^N$ homeomorphic to X.

Before embarking on the proof, we consider the following construction. There is a function

$$\begin{aligned} \sigma : \mathbb{R}^d &\longrightarrow [0, \infty) \\ x &\longmapsto \begin{cases} 1 & |x| \leq 1 \\ 1 - |x| & 1 \leq |x| \leq 2 \\ 0 & |x| \geq 2, \end{cases} \end{aligned}$$

[4] This is a subtle point. Using the homeomorphism τ we know that C is closed in U, but we need to know it is closed in X. This is false without the Hausdorff assumption, as you will see on Example Sheet 3.

which is continuous by the Gluing Lemma. Its important features are that it is 1 on the unit ball $B_1(0)$ and is zero outside of a compact subset (namely $\overline{B_2(0)}$).

Proof. For each $x \in X$ there is a Euclidean neighbourhood $U_x \ni x$, with $\tau_x : U_x \xrightarrow{\cong} \mathbb{R}^d$. Let the open set $V_x \subset U_x$ correspond under τ_x to $B_1(0) \subset \mathbb{R}^d$, and let $\sigma_x := \sigma \circ \tau_x : U_x \to [0, \infty)$. There is a function

$$\begin{aligned} b_x : X &\longrightarrow \mathbb{R}^d \\ y &\longmapsto \begin{cases} \sigma_x(y)\tau_x(y) & y \in U_x \\ 0 & y \notin U_x. \end{cases} \end{aligned}$$

On the open[5] set $X \setminus \tau_x^{-1}\overline{B_2(0)}$ this function is identically 0 so is continuous, and it is also continuous on the open set U_x, so by the Gluing Lemma it is continuous. Similarly we may consider σ_x as a continuous function $X \to [0, \infty)$ by setting it to be 0 outside of U_x.

Now $\{V_x : x \in X\}$ is an open cover of X, so there is a finite subcover $V_{x_1}, \ldots, V_{x_n}$. Consider the function

$$\begin{aligned} e : X &\longrightarrow (\mathbb{R}^d \times \mathbb{R})^n \\ y &\longmapsto (b_{x_1}(y), \sigma_{x_1}(y); \ldots; b_{x_n}(y), \sigma_{x_n}(y)), \end{aligned}$$

which is continuous by the universal property of the product topology. If $e(y) = e(y')$ then suppose that $y \in V_{x_i}$ (as the V's cover) so that $\sigma_{x_i}(y) = 1$ and hence $\sigma_{x_i}(y') = 1$ too. But then

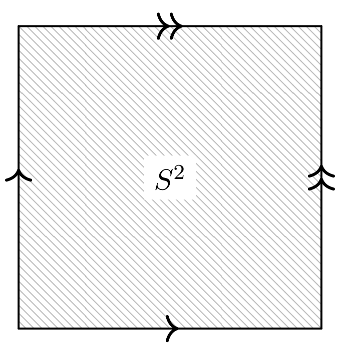
$$\tau_{x_i}(y) = b_{x_i}(y) = b_{x_i}(y') = \tau_{x_i}(y')$$

so $y = y'$ as τ_{x_i} is a homeomorphism. Thus e is injective. Writing $Y := e(X)$, the universal property of the subspace topology shows that $e : X \to Y$ is continuous. It is also a bijection from a compact to a Hausdorff space, so is a homeomorphism as required. $\square$

Example. We have seen that $\mathbb{RP}^2$ can be covered by 3 open sets each homeomorphic to $\mathbb{R}^2$, so the argument above gives a homeomorphism to a subspace of $\mathbb{R}^9$. It is fact homeomorphic to a subspace of $\mathbb{R}^4$ (this can be done explicitly) and is *not* homeomorphic to a subspace of $\mathbb{R}^3$ (this can be proved using ideas in Part III Algebraic Topology). $\triangle$

4.4 Surfaces by gluing polygons

A *(topological) surface* is the typical name for a 2-dimensional topological manifold. Example of surfaces we have seen are the sphere S^2, the torus T, the Klein bottle K, and the projective plane $\mathbb{RP}^2$. We have seen how to obtain the last three by gluing the sides of a square together in pairs, and for completeness we may obtain S^2 in this way too:

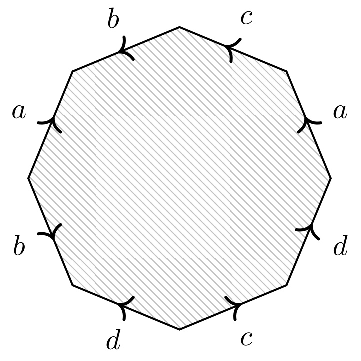


[5]We again use Hausdorffness of X to see that the complement of a compact set is open.

More generally, if $P \subset \mathbb{R}^2$ is a convex polygon with an even number $2n$ of edges, and we label each edge by $a, b, c, d \ldots$ so that any label which is used is used precisely two times, and we furthermore choose a direction of each edge which we depict by drawing an arrow on it, then we can define an equivalence relation $\sim$ on P to be generated by the following rule: If the two edges labelled α have vertices u, v, with direction pointing from u to v, and x, y, with direction pointing from x to y, then

$$t \cdot u + (1-t) \cdot v \sim t \cdot x + (1-t) \cdot y \text{ for each } t \in [0,1].$$

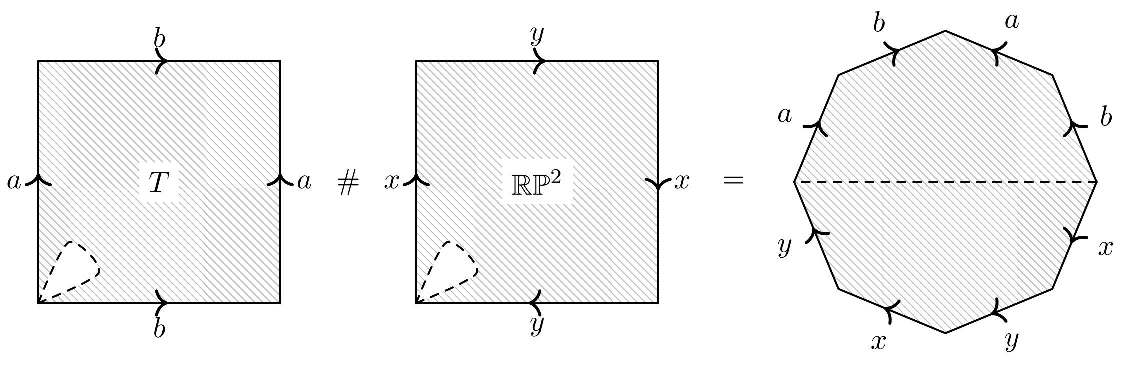
More informally, we glue the two edges labelled α together so that the arrows match up.



The following is proved by treating several cases: points coming from the interior of P, points coming from the interior of an edge of P, and points coming from vertices of P. It is laborious—so we omit it—but is elementary and you will have no difficulty in filling in the details.

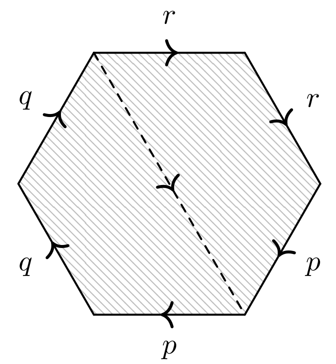
Theorem. $P/\sim$ is a compact topological surface. □

We can describe a connected-sum of two surfaces given by polygons, by choosing carefully the disc which is cut out to form the connected-sum. This is best described by a picture as in the following example:

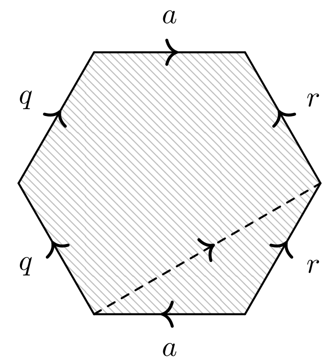


This 8-gon can be replaced by a 6-gon without changing the topological surface it defines, by consolidating the contiguous edges xy with a single edge, say c.

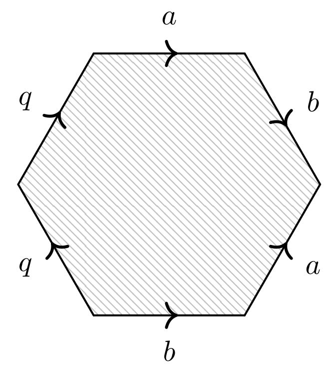
Example. In a similar way, we can obtain a 12-gon giving a connected-sum $\mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$ whose edges are all oriented clockwise and which read $xyxyststuvuv$. We may consolidate each of the contiguous edges xy, st, and uv with single edges p, q, and r to see that $\mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$ is obtained from the 6-gon



Cutting this open along the indicated dashed line a, and then gluing along p, changes this into the 6-gon



which must then still give $\mathbb{RP}^2\#\mathbb{RP}^2\#\mathbb{RP}^2$. Cutting this open along the indicated dashed line b, and then gluing along r, changes this into the 6-gon



which again must give $\mathbb{RP}^2\#\mathbb{RP}^2\#\mathbb{RP}^2$. But after renaming q to be c, this is the 6-gon that we explained above describes a connected-sum $T\#\mathbb{RP}^2$, so we conclude that there is a homeomorphism

$$\mathbb{RP}^2\#\mathbb{RP}^2\#\mathbb{RP}^2 \cong T\#\mathbb{RP}^2.$$

$\triangle$

4.5 Triangulable surfaces

Every surface we have discussed can be given a triangulation in the following sense.

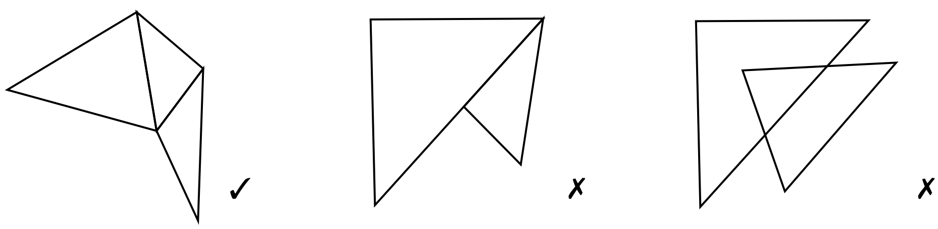
Definition.

(i) A *triangle* in $\mathbb{R}^N$ with vertices non-collinear points $p, q, r \in \mathbb{R}^N$ is the space

$$\{t_0 \cdot p + t_1 \cdot q + t_2 \cdot r \in \mathbb{R}^N : t_0 + t_1 + t_2 = 1, t_i \geq 0\},$$

i.e. the convex hull of, or smallest convex set containing, the vertices p, q, r. Its three *edges* are the convex hulls of $\{p, q\}$, $\{p, r\}$, $\{q, r\}$ respectively.

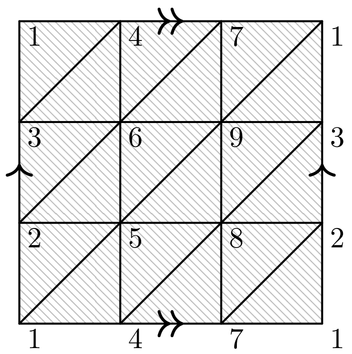
(ii) A collection $K = \{\Delta_1, \ldots, \Delta_n\}$ of triangles in $\mathbb{R}^N$ form a (2-dimensional) *polyhedron* if each intersection $\Delta_i \cap \Delta_j$ with $i \neq j$ is either empty, a common vertex of both triangles, or a common edge of both triangles.



We set $|K| := \bigcup_{i=1}^{n} \Delta_i$, a subspace of $\mathbb{R}^N$.

(iii) A *triangulation* of a topological surface S is the data of a polyhedron K together with a homeomorphism $\phi : |K| \xrightarrow{\cong} S$.

To triangulate the torus, for example, we consider the picture



which gives a homeomorphism to a polyhedron in $\mathbb{R}^9$ by assigning each of the 9 vertices to a basis vector, and each edge or triangle to the convex hull of the basis vectors corresponding to its vertices. Some checking shows that this is indeed a polyhedron (the reason for taking quite so many triangles is to ensure that each edge or triangle in the picture is *uniquely determined by its vertices*: this is what is needed to verify that the triangles in $\mathbb{R}^9$ form a polyhedron).

In this section we will discuss topological surfaces which admit a triangulation. But:

Theorem (Radó)**.** Every compact topological surface admits a triangulation. □

However, this is a quite involved to prove, and is beyond the scope of this course.

Our goal will now be to explain how to tell whether two (triangulable) compact topological surfaces are homeomorphic. We have seen that $\mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$ is homeomorphic to $T \# \mathbb{RP}^2$, so certainly surfaces described in different ways can turn out to be homeomorphic. What we want are *invariants*: some quantities or properties associated to surfaces that we can measure, so that if they differ then the surfaces are definitely not homeomorphic; in fact they will be *complete invariants*, meaning that if they do not differ then the surfaces are homeomorphic. This will give us a way to know that surfaces are homeomorphic without actually having to find a homeomorphism.

Definition. The *Euler characteristic* of a triangulated surface $\phi : |K| \xrightarrow{\cong} S$ is the integer

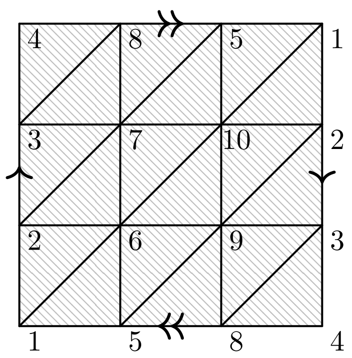
$$\chi(K) := \#\text{vertices of } |K| - \#\text{edges of } |K| + \#\text{triangles of } |K|.$$

In Part II Algebraic Topology you will see that if $\phi' : |K'| \xrightarrow{\cong} S$ is another triangulation of S then $\chi(K') = \chi(K)$, so it is reasonable to call this $\chi(S)$, the Euler characteristic of S.

Example. S^2 is homeomorphic to the tetrahedron, which is a polyhedron having 4 vertices, 6 edges, and 4 faces, so $\chi(S^2) = 4 - 6 + 4 = 2$. △

Example. The triangulation of the torus T drawn above has 9 vertices, 27 edges, and 18 triangles, so $\chi(T) = 9 - 27 + 18 = 0$. △

Example. Taking the picture of the triangulation of the torus above and changing the side identifications gives the following triangulation of the projective plane $\mathbb{RP}^2$,

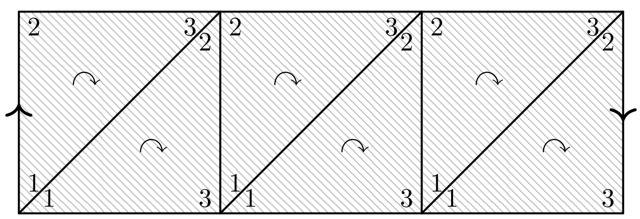


having 10 vertices, 27 edges, and 18 triangles, so $\chi(\mathbb{RP}^2) = 10 - 27 + 18 = 1$. △

Definition. An *orientation* of a triangulated surface $\phi : |K| \xrightarrow{\cong} S$ with polyhedron $K = \{\Delta_1, \ldots, \Delta_n\}$ is a choice of cyclic ordering of the vertices of each triangle Δ_i, such that whenever $\Delta_i \cap \Delta_j$ is an edge, the ordering of the end points of this edge induced by Δ_i is opposite to the ordering induced by Δ_j. If the triangulated surface admits an orientation then it is *orientable*, otherwise it is *non-orientable*.

Like the Euler characteristic, orientability is in fact an intrinsic property of the topological manifold S and does not actually depend on the triangulation: you will also see this in Part II Algebraic Topology.

Example. $\mathbb{RP}^2$ is not orientable. In the triangulation drawn above we consider the middle horizontal strip, which is a Möbius band, and try to order the vertices in each triangle from left to right so that they fit together:



But the induced ordering on the left-hand edge points upwards (1 to 2), and the induced ordering on the right-hand edge points downwards (2 to 3), and as we identify these edges with a twist *these orderings are equal, not opposite.* Once we chose the cyclic ordering of the left-hand triangle our hands were tied, so it is not possible to orient (this triangulation of) the Möbius band, so it is not possible to orient (this triangulation of) $\mathbb{RP}^2$.

The same reasoning shows that e.g. a connected-sum $\mathbb{RP}^2 \# S$ of triangulable surfaces is non-orientable, at least when it is formed using well-chosen discs (cf. Q7 on Example Sheet 3). △

Given this preparation, we may formulate a classification theorem for surfaces.

Theorem (Classification of surfaces)**.** If two compact connected (triangulable) topological surfaces have the same orientability and Euler characteristic, then they are homeomorphic. □

We will not prove this, though we have all the ingredients. The strategy is to first show that any connected triangulable surface can be obtained by identifying pairs on edges in a polygon, and then to show that the labelled polygon description can be manipulated—using the same kinds of moves that we used to show that $\mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$ is homeomorphic to $T \# \mathbb{RP}^2$—until it is one of a list of standard forms. These standard forms describe the orientable surfaces[6]

$$S^2, T, T \# T, T \# T \# T, \ldots$$

having Euler characteristics $2, 0, -2, -4, \ldots$, and the non-orientable surfaces

$$\mathbb{RP}^2, \mathbb{RP}^2 \# \mathbb{RP}^2, \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2, \ldots$$

having Euler characteristics $1, 0, -1, -2, \ldots$, so these surfaces are all different. The proof in fact shows that these are *all* the compact connected (triangulable) topological surfaces.

[6]The orientable surface given by the connected-sum of g tori is said to have *genus* g; it has $\chi = 2 - 2g$.