# Characteristic classes and $K$-theory

Oscar Randal-Williams

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Chapter 1

# Vector bundles

Lecture 1

Throughout these notes, *map* means continuous function.

## 1.1 Vector bundles

Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$.

**Definition 1.1.1.** An *$n$-dimensional $\mathbb{F}$-vector bundle* over a space $X$ is a collection $\{E_x\}_{x\in X}$ of $\mathbb{F}$-vector spaces and a topology on the set $E := \coprod_{x\in X} E_x$ so that the map

$$\begin{aligned}\pi : E &\longrightarrow X\\ e \in E_x &\longmapsto x\end{aligned}$$

satisfies the following *local triviality* condition: each $x \in X$ has a neighbourhood $U \ni x$ and a homeomorphism

$$\varphi_U : U \times \mathbb{F}^n \longrightarrow E|_U := \pi^{-1}(U)$$

such that $\pi(\varphi_U(x, v)) = x$ and such that $\varphi_U$ gives a *linear isomorphism* $\{y\} \times \mathbb{F}^n \overset{\sim}{\to} E_y$ for all $y \in U$. We call such a $U$ a *trivialising open set*, and $\varphi_U$ a *trivialisation.*

We will usually refer to $\pi : E \to X$ as a vector bundle, leaving the vector space structure on the fibres $E_x := \pi^{-1}(x)$ implicit.

A *section* of a vector bundle $\pi : E \to X$ is a map $s : X \to E$ such that $\pi \circ s = Id_X$. In particular, letting $s_0(x)$ be the zero element in the vector space $E_x$ defines the *zero section* $s_0 : X \to E$. We write $E^\# := E \setminus s_0(X)$ for the complement of the zero section, i.e. the set of non-zero vectors.

**Example 1.1.2.** Projection to the first coordinate $\pi_1 : X \times \mathbb{F}^n \to X$ defines a vector bundle, the *trivial $n$-dimensional $\mathbb{F}$-vector bundle*, where $E_x = \{x\} \times \mathbb{F}^n$ is given the evident $\mathbb{F}$-vector space structure. To save space we will write this as $\underline{\mathbb{F}}^n$, or $\underline{\mathbb{F}}^n_X$ if we need to emphasise the base space.

**Definition 1.1.3.** If $\pi : E \to X$ is a vector bundle, then a subspace $E_0 \subset E$ is a *subbundle* if each $E_0 \cap E_x$ is a vector subspace of $E_x$, and $\pi|_{E_0} : E_0 \to X$ is locally trivial as in Definition 1.1.1.

**Example 1.1.4.** The *Grassmannian* $Gr_n(\mathbb{F}^N)$ is the set of $n$-dimensional vector subspaces of $\mathbb{F}^N$. If $Fr_n(\mathbb{F}^N) \subset (\mathbb{F}^N)^n$ is the subspace of those sequences $(v_1, \ldots, v_n)$ of vectors which are linearly independent, then there is a surjective map

$$\begin{aligned}q : Fr_n(\mathbb{F}^N) &\longrightarrow Gr_n(\mathbb{F}^N)\\ (v_1, \ldots, v_n) &\longmapsto \langle v, \ldots, v_n\rangle_{\mathbb{F}}\end{aligned}$$

given by sending a sequence of linearly independent vectors to its span, and we use this to give $Gr_n(\mathbb{F}^N)$ the quotient topology. Let

$$\gamma_{\mathbb{F}}^{n,N} := \{(P, v) \in Gr_n(\mathbb{F}^N) \times \mathbb{F}^N \mid v \in P\},$$

and $\pi : \gamma_{\mathbb{F}}^{n,N} \to Gr_n(\mathbb{F}^N)$ be given by projection to the first factor. The fibre $\pi^{-1}(P)$ is identified with $P$, so has the structure of a vector space. We claim that this is locally trivial. To see this, for $P \in Gr_n(\mathbb{F}^N)$ consider the orthogonal projection $\pi_P : \mathbb{F}^N \to P$: the set

$$U := \{Q \in Gr_n(\mathbb{F}^N) \mid \pi_P|_Q : Q \to P \text{ is an iso}\}$$

is an open neighbourhood of $P$ (this can be checked by showing that $q^{-1}(U)$ is open), and the map

$$\begin{aligned} \gamma_{\mathbb{F}}^{n,N}|_U &\longrightarrow U \times P \\ (Q, v) &\longmapsto (Q, \pi_P(v)) \end{aligned}$$

is a homeomorphism: its inverse $\varphi_U$ gives a local trivialisation. The vector bundle $\gamma_{\mathbb{F}}^{n,N}$ is called the *tautological vector bundle* on $Gr_n(\mathbb{F}^N)$.

**Definition 1.1.5.** It is conventional to call a 1-dimensional vector bundle a *line bundle*.

### 1.1.1 Morphisms of vector bundles

If $\pi : E \to X$ and $\pi' : E' \to X$ are two vector bundles over the same space $X$, then a map $f : E \to E'$ is a *vector bundle map* if it is linear on each fibre, i.e. $f_x : E_x \to E'_x$ is a linear map. Hence two vector bundles are *isomorphic* if there are mutually inverse vector bundle maps $f : E \to E'$ and $g : E' \to E$.

### 1.1.2 Pullback

If $\pi : E \to X$ is a vector bundle and $f : Y \to X$ is a continuous map, we let

$$f^*E := \{(y, e) \in Y \times E \mid f(y) = \pi(e)\},$$

and define $f^*\pi : f^*E \to Y$ by $f^*\pi(y, e) = y$. Then $(f^*E)_y \cong E_{f(y)}$ has the structure of a $\mathbb{F}$-vector space.

Suppose $\varphi_U : U \times \mathbb{F}^n \xrightarrow{\sim} E|_U$ is a local trivialisation of $\pi$. Writing $V := f^{-1}(U)$, an open set in $Y$, the map

$$\begin{aligned} V \times \mathbb{F}^n &\longrightarrow (f^*E)|_V \\ (y, e) &\longmapsto (y, \varphi_U(f(y), e)) \end{aligned}$$

is a homeomorphism. Thus $f^*\pi : f^*E \to Y$ is a $\mathbb{F}$-vector bundle.

### 1.1.3 Operations on vector bundles

Roughly speaking, any natural operation that one can perform on vector spaces, one can also perform on vector bundles. We shall not try to make this precise, but settle for the following examples.

### Sum of bundles

If $\pi : E \to X$ is a $n$-dimensional $\mathbb{F}$-vector bundle and $\pi' : E' \to X$ is a $n'$-dimensional $\mathbb{F}$-vector bundle, there is a $(n+n')$-dimensional vector bundle $\pi \oplus \pi' : E \oplus E' \to X$ with

$$E \oplus E' := \{(e, e') \in E \times E' \,|\, \pi(e) = \pi'(e')\}$$

and $(\pi \oplus \pi')(e, e') := \pi(e)$. Then $(E \oplus E')_x = E_x \times E'_x$ has a natural $\mathbb{F}$-vector space structure, and one can easily find local trivialisations over open sets given by intersecting a trivialising open set for $E$ with one for $E'$. This is often called *Whitney sum*.

### Tensor product of bundles

If $\pi : E \to X$ is a $n$-dimensional $\mathbb{F}$-vector bundle and $\pi' : E' \to X$ is a $n'$-dimensional $\mathbb{F}$-vector bundle, there is a $(n \cdot n')$-dimensional vector bundle $\pi \otimes \pi' : E \otimes E' \to X$. As a set

$$E \otimes E' = \coprod_{x \in X} E_x \otimes_{\mathbb{F}} E'_x,$$

but we must give this a topology making it into a vector bundle.

To do so, let $\{U_\alpha\}_{\alpha \in I}$ be a cover of $X$ over which both vector bundles are trivial, and let $\varphi_{U_\alpha} : U_\alpha \times \mathbb{F}^n \to E|_{U_\alpha}$ and $\varphi'_{U_\alpha} : U_\alpha \times \mathbb{F}^{n'} \to E'|_{U_\alpha}$ be trivialisations. We can form

$$\begin{aligned} U_\alpha \times (\mathbb{F}^n \otimes \mathbb{F}^{n'}) &\longrightarrow (E \otimes E')|_{U_\alpha} = \coprod_{x \in U_\alpha} E_x \otimes_{\mathbb{F}} E'_x \\ (x, v) &\longmapsto (\varphi_{U_\alpha}(x, -) \otimes \varphi'_{U_\alpha}(x, -))(v) \end{aligned}$$

and topologise $E \otimes E'$ by declaring these functions to be homeomorphisms onto open subsets.

### Homomorphisms of bundles

If $\pi : E \to X$ is a $n$-dimensional $\mathbb{F}$-vector bundle and $\pi' : E' \to X$ is a $n'$-dimensional $\mathbb{F}$-vector bundle, there is a $(n \cdot n')$-dimensional vector bundle $Hom(\pi, \pi') : Hom(E, E') \to X$. As a set it is

$$Hom(E, E') = \coprod_{x \in X} Hom_{\mathbb{F}}(E_x, E'_x),$$

and this is given a topology similarly to the case of tensor products.

A function $s : X \to Hom(E, E')$ such that $Hom(\pi, \pi') \circ s = \mathrm{Id}_X$ gives for each $x \in X$ an element $s(x) \in Hom(E_x, E'_x)$, and these assemble to a function

$$\hat{s} : E \longrightarrow E'.$$

If $s$ is continuous then $\hat{s}$ is a bundle map.

In particular, we have the *dual* vector bundle $E^\vee = Hom(E, \underline{\mathbb{F}}_X)$.

### Realification and complexification

If $\pi : E \to X$ is a $\mathbb{C}$-vector bundle, then by neglect of structure we can consider it as a $\mathbb{R}$-vector bundle; we write $E_{\mathbb{R}}$ to emphasise this. If $\pi : E \to X$ is a $\mathbb{R}$-vector bundle then there is a $\mathbb{C}$-vector bundle $\pi \otimes_{\mathbb{R}} \mathbb{C} : E \otimes_{\mathbb{R}} \mathbb{C} \to X$ with total space given by the tensor product $E \otimes (X \times \mathbb{C})_{\mathbb{R}}$ of the vector bundle $E$ and the trivial 1-dimensional complex vector bundle, considered as a real vector bundle. Complex multiplication on the second factor makes this into a complex vector bundle.

### Complex conjugate bundles

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If $\pi : E \to X$ is a $\mathbb{C}$-vector bundle, then there is a $\mathbb{C}$-vector bundle $\overline{\pi} : \overline{E} \to X$ where $\overline{E}$ is equal to $E$ as a topological space and as a real vector bundle, but the fibres $\overline{E}_x$ are given the opposite $\mathbb{C}$-vector space structure to $E_x$: multiplication by $\lambda \in \mathbb{C}$ on $\overline{E}_x$ is defined to be multiplication by $\overline{\lambda}$ on $E_x$.

### Exterior powers

The exterior algebra $\Lambda^* V$ on a $\mathbb{F}$-vector space $V$ is the quotient of the tensor algebra $T(V) := \bigoplus_{k=0}^{\infty} V^{\otimes k}$ by the two-sided ideal generated by all $v \otimes v$. As these elements are homogeneous (namely quadratic), the exterior algebra is graded and we write $\Lambda^k V$ for the degree $k$ part. Elements are written as $v_1 \wedge v_2 \wedge \cdots \wedge v_k$. Choosing a basis, it is easy to see that $\dim \Lambda^k V = \binom{\dim V}{k}$, and in particular that $\Lambda^k V = 0$ if $k > \dim V$. The formula

$$\begin{aligned} \Lambda^* V \otimes \Lambda^* W &\longrightarrow \Lambda^*(V \oplus W) \\ (v_1 \wedge \cdots \wedge v_k) \otimes (w_1 \wedge \cdots \wedge w_l) &\longmapsto v_1 \wedge \cdots \wedge v_k \wedge w_1 \wedge \cdots \wedge w_l \end{aligned}$$

defines a linear isomorphism.

We can also apply this construction to vector bundles: if $\pi : E \to X$ is a $\mathbb{F}$-vector bundle, we define

$$\Lambda^k E = \coprod_{x \in X} \Lambda^k E_x,$$

and give it a topology similarly to the case of tensor products.

## 1.2 Inner products

An *inner product* on a $\mathbb{R}$-vector bundle $\pi : E \to X$ is a bundle map

$$\langle -, - \rangle : E \otimes E \longrightarrow \underline{\mathbb{R}}_X$$

such that the map $\langle -, - \rangle_x : E_x \otimes E_x \to \mathbb{R}$ on fibres is an inner product. Equivalently, it is a section

$$x \mapsto \langle -, -, \rangle_x : X \to Hom(E \otimes E, \underline{\mathbb{R}}_X)$$

which has the property that each value $\langle -, -, \rangle_x \in Hom(E_x \otimes E_x, \mathbb{R})$ is an inner product. Such an inner product defines a bundle isomorphism

$$e \mapsto \langle e, - \rangle_{\pi(e)} : E \longrightarrow E^{\vee}.$$

Similarly, a *Hermitian inner product* on a $\mathbb{C}$-vector bundle $\pi : E \to X$ is a bundle map

$$\langle -, - \rangle : \overline{E} \otimes E \longrightarrow \underline{\mathbb{C}}_X$$

such that the map $\langle -, - \rangle_x : \overline{E_x} \otimes E_x \to \mathbb{C}$ on fibres is a Hermitian inner product on $E_x$. Such a Hermitian inner product defines a bundle isomorphism

$$e \mapsto \langle e, - \rangle_{\pi(e)} : \overline{E} \longrightarrow E^\vee.$$

It will be useful to know that (Hermitian) inner products exist as long as the base is sufficiently well-behaved. For any open cover $\{U_\alpha\}_{\alpha \in I}$ of a compact Hausdorff space $X$ one may find a *partition of unity*: maps $\lambda_\alpha : U_\alpha \to [0, \infty)$ such that

(i) $supp(\lambda_\alpha) := \overline{\{x \in X \mid \lambda_\alpha(x) > 0\}} \subset U_\alpha$,

(ii) each $x \in X$ lies in finitely-many $supp(\lambda_\alpha)$,

(iii) $\sum_{\alpha \in I} \lambda_\alpha(x) = 1$ for any $x \in X$.

**Lemma 1.2.1.** *If $\pi : E \to X$ is a $\mathbb{F}$-vector bundle over a compact Hausdorff space, then $E$ admits a (Hermitian) inner product.*

*Proof.* Via the local trivialisations $\varphi_{U_\alpha} : U_\alpha \times \mathbb{F}^n \xrightarrow{\sim} E|_{U_\alpha}$ and the standard (Hermitian) inner product on $\mathbb{F}^n$, we obtain a (Hermitian) inner product $\langle -, - \rangle_{U_\alpha}$ on $E|_{U_\alpha}$. We then define

$$\langle e, f \rangle = \sum_{\alpha \in I} \lambda_\alpha(\pi(e)) \cdot \langle e, f \rangle_{U_\alpha};$$

this is a locally-finite sum of bundle maps, so a bundle map. It is an inner product on each fibre as the space of inner products, inside the space of all bilinear forms, is convex. □

## 1.3 Embedding into trivial bundles

**Lemma 1.3.1.** *If $\pi : E \to X$ is a $\mathbb{F}$-vector bundle over a compact Hausdorff space, then*

*(i) $E$ is (isomorphic to) a subbundle of a trivial bundle $\underline{\mathbb{F}}_X^N$ for some $N \gg 0$, and*

*(ii) there is a $\mathbb{F}$-vector bundle $\pi' : E' \to X$ such that $E \oplus E' \cong \underline{\mathbb{F}}_X^N$.*

*Proof.* For (i), as $X$ is compact, let $U_1, \ldots, U_p \subset X$ be a finite open trivialising cover, $\lambda_1, \ldots, \lambda_p : X \to [0, 1]$ be a partition of unity associated with it, and let

$$\varphi_{U_i} : U_i \times \mathbb{F}^n \longrightarrow E|_{U_i}$$

be local trivialisations: write $v \mapsto (\pi(v), \rho_i(v))$ for their inverses. The map

$$\begin{aligned} \varphi : E &\longrightarrow X \times (\mathbb{F}^n)^{\times p} \\ v &\longmapsto (\pi(v), \lambda_1(\pi(v)) \cdot \rho_1(v), \ldots, \lambda_p(\pi(v)) \cdot \rho_p(v)) \end{aligned}$$

is well-defined (as $\lambda_i(\pi(v)) = 0$ if $\rho_i(v)$ is not defined), is a linear injection on each fibre, and is a homeomorphism onto its image. Furthermore, its image is a subbundle, as over $supp(\lambda_i)$ projection to $X$ times the $i$th copy of $\mathbb{F}^n$ gives a local trivialisation.

For (ii), using (i) we may assume that $E$ is a subbundle of $X \times \mathbb{F}^N$ and we let

$$E' := \{(x, v) \in X \times \mathbb{F}^N \mid v \in E_x^{\perp}\},$$

using the standard (Hermitian) inner product on $\mathbb{F}^N$, with $\pi' : E' \to X$ given by projection to the first factor; $E'_x = E_x^{\perp}$ certainly has a vector space structure, and it remains to see that $\pi'$ is locally trivial. For $x \in X$ let $U \ni x$ be a trivialising neighbourhood and $\varphi_U : U \times \mathbb{F}^n \to E|_U$ be a trivialisation. The coordinates of $\mathbb{F}^n$ define nowhere zero sections $s_1, \ldots, s_n : U \to E|_U \subset U \times \mathbb{F}^N$. Choose vectors $e_{n+1}, \ldots, e_N \in \mathbb{F}^N$ such that

$$s_1(x), \ldots, s_n(x), e_{n+1}, \ldots, e_N \in \mathbb{F}^N$$

are linearly independent. As being linearly independent is an open condition, there is a perhaps smaller neighbourhood $U \supset U' \ni x$ such that

$$s_1(y), \ldots, s_n(y), e_{n+1}, \ldots, e_N \in \mathbb{F}^N$$

are linearly independent for each $y \in U'$: these determine sections $s_1, \ldots, s_N : U' \to U' \times \mathbb{F}^N$ which are linearly independent at each point. Applying the Gram–Schmidt process to these (which is continuous) gives sections

$$s'_1, \ldots, s'_N : U' \to U' \times \mathbb{F}^N$$

which are orthogonal and such that $s'_1(y), \ldots, s'_n(y)$ form a basis of $E_y$. Then the remaining vectors $s'_{n+1}(y), \ldots, s'_N(y)$ form a basis of $E'_y$ for each $y$, so there is a homeomorphism

$$\begin{aligned} \varphi'_U : U' \times \mathbb{F}^{N-n} &\longrightarrow E'|_{U'} \subset U' \times \mathbb{F}^N \\ (y, t_{n+1}, \ldots, t_N) &\longmapsto \left(y, \sum_{i=n+1}^{N} t_i s_i(y)\right) \end{aligned}$$

giving a local trivialisation of $\pi' : E' \to X$. □

Lecture 3

## 1.4 Concordance

If $\pi : E \to X \times [0, 1]$ is a vector bundle, then the restrictions $\pi_i : E_i = E|_{X \times \{i\}} \to X$ for $i = 0, 1$ are called *concordant.*

**Lemma 1.4.1.** *If $X$ is compact Hausdorff then concordant vector bundles are isomorphic.*

*Proof.* Let $\pi_t : E_t \to X$ be the restriction of $\pi$ to $X \times \{t\}$, and $\pi'_t : E'_t \to X \times [0, 1]$ be the pullback of $E_t$ along the projection $X \times [0, 1] \to X$. Then $E$ and $E'_t$ are isomorphic when restricted to $X \times \{t\}$, so the vector bundle

$$Hom(\pi, \pi'_t) : Hom(E, E'_t) \to X \times [0, 1]$$

has a section $s_t$ over $X \times \{t\}$. By local triviality and using a partition of unity this may be extended to a section $s : X \times [0,1] \to Hom(E, E'_t)$, and as being a linear isomorphism is an open condition, and $X$ is compact, restricted to $X \times (t-\epsilon, t+\epsilon)$ it gives a linear isomorphism. Thus there is a vector bundle isomorphism

$$E|_{X\times(t-\epsilon,t+\epsilon)} \cong E'_t|_{X\times(t-\epsilon,t+\epsilon)},$$

and so

$$E_s \cong E_t$$

for any $s \in (t-\epsilon, t+\epsilon)$. As $[0,1]$ is connected, $E_0 \cong E_1$. □

On the other hand isomorphic vector bundles are also concordant. Given $\pi_i : E_i \to X$ which are isomorphic, we can take $E_0 \times [0,1/2] \to X \times [0,1/2]$ and $E_1 \times [1/2,1] \to X \times [1/2,1]$ and identify them along an isomorphism $E_0 \times \{1/2\} \cong E_1 \times \{1/2\}$ to get a vector bundle $\pi : E \to X \times [0,1]$ which agrees with $E_i$ over $X \times \{i\}$.

**Corollary 1.4.2.** *If $f_0, f_1 : X \to Y$ are homotopic maps, $X$ is compact Hausdorff, and $\pi : E \to Y$ is a vector bundle, then $f_0^*E \cong f_1^*E$.*

*Proof.* Let $F : X \times [0,1] \to Y$ be a homotopy from $f_0$ to $f_1$, and apply the lemma to $F^*\pi : F^*E \to X \times [0,1]$. □

**Corollary 1.4.3.** *A vector bundle over a contractible compact Hausdorff space is trivial.*

*Proof.* The identity map of such a space $X$ is homotopic to a constant map $c : X \to \{*\} \xrightarrow{i} X$. Thus if $E \to X$ is a vector bundle then $E \cong c^*i^*(E)$, and $i^*(E)$ is trivial (as all vector bundles over a point are) so $c^*i^*(E)$ is too. □

## 1.5 Classification by infinite Grassmannians

Let $\mathbb{F}^\infty = \bigcup_N \mathbb{F}^N$, topologised so that a subset is open if and only if its intersection with each $\mathbb{F}^N$ is open. This is an infinite-dimensional $\mathbb{F}$-vector space, and the standard (Hermitian) inner products on $\mathbb{F}^N$ extend to a (Hermitian) inner product on $\mathbb{F}^\infty$. Our discussion in Example 1.1.4 applies to this too, defining $Gr_n(\mathbb{F}^\infty)$ and a tautological vector bundle $\gamma_{\mathbb{F}}^{n,\infty}$ on it, which restricts to $\gamma_{\mathbb{F}}^{n,N}$ on the subspace $Gr_n(\mathbb{F}^N) \subset Gr_n(\mathbb{F}^\infty)$.

Corollary 1.4.2 shows that for $X$ compact there is a well-defined function

$$\begin{aligned}\{\text{maps } \phi : X \to Gr_n(\mathbb{F}^\infty)\}/\text{homotopy} &\longrightarrow \{n\text{-dim vector bundles over } X\}/\text{isomorphism}\\ \phi &\longmapsto (\phi^*\gamma_{\mathbb{F}}^{n,\infty} \to X).\end{aligned}$$

**Lemma 1.5.1.** *If $X$ is compact Hausdorff then this function is a bijection.*

*Proof.* If $\pi : E \to X$ is an $n$-dimensional $\mathbb{F}$-vector bundle, then by Lemma 1.3.1 (i) we may suppose that $E$ is a subbundle of $X \times \mathbb{F}^N$ for some $N \gg 0$. In this case each $E_x$ is an $n$-dimensional vector subspace of $\mathbb{F}^N$, and hence of $\mathbb{F}^\infty$, which defines a map

$$\begin{aligned}\phi_E : X &\longrightarrow Gr_n(\mathbb{F}^\infty)\\ x &\longmapsto E_x,\end{aligned}$$

which tautologically satisfies $\phi_E^*(\gamma_{\mathbb{F}}^{n,\infty}) \cong E$. This shows surjectivity.

To see injectivity, let $\phi_0, \phi_1 : X \to Gr_n(\mathbb{F}^\infty)$ be maps and suppose they classify isomorphic vector bundles. Consider the homotopy of linear maps

$$\begin{aligned} L : \mathbb{F}^\infty \times [0,1] &\longrightarrow \mathbb{F}^\infty \\ (x_1, x_2, x_3, \ldots; t) &\longmapsto (1-t)\cdot(x_1, x_2, x_3, \ldots) + t\cdot(x_1, 0, x_2, 0, x_3, 0, \ldots). \end{aligned}$$

This gives a homotopy

$$\begin{aligned} X \times [0,1] &\longrightarrow Gr_n(\mathbb{F}^\infty) \\ (x,t) &\longmapsto L(-,t)(\phi_0(x)) \end{aligned}$$

from $\phi_0$ to a map $\phi_0'$ which only uses the odd-numbered coordinates. Similarly we obtain a homotopy from $\phi_1$ to a $\phi_1'$ which only uses the even-numbered coordinates. The vector bundles $(\phi_0')^*\gamma_{\mathbb{F}}^{n,\infty}$ and $(\phi_1')^*\gamma_{\mathbb{F}}^{n,\infty}$ are isomorphic (as they are isomorphic to $\phi_0^*\gamma_{\mathbb{F}}^{n,\infty}$ and $\phi_1^*\gamma_{\mathbb{F}}^{n,\infty}$ respectively, which were isomorphic by assumption), say by an isomorphism $\psi$. Then linear interpolation

$$\begin{aligned} X \times [0,1] &\longrightarrow Gr_n(\mathbb{F}^\infty) \\ (x,t) &\longmapsto \langle (1-t)\cdot v + t\cdot \psi_x(v) \,|\, v \in \phi_0'(x) \rangle \end{aligned}$$

gives a homotopy from $\phi_0'$ to $\phi_1'$, as the linear map $(1-t)\cdot(-) + t\cdot\psi_x(-) : \phi_0'(x) \to \mathbb{F}^\infty$ is injective for all $t$ by the way we arranged the parity of the coordinates. □

## 1.6 The clutching construction

Corollary 1.4.3 can be used to give a useful description of vector bundles over spheres, and more generally over suspensions.

For a space $X$ recall that the *cone on* $X$ is

$$CX := (X \times [0,1])/(X \times \{0\}),$$

which is contractible. The *suspension of* $X$ is

$$\Sigma X := (X \times [0,1])/(x,0) \sim (x',0) \text{ and } (x,1) \sim (x',1).$$

If $X$ is compact Hausdorff then so is $CX$ and $\Sigma X$. Identifying $X$ with $X \times \{1\} \subset CX$, we see there is a homeomorphism $\Sigma X \cong CX \cup_X CX$; we write $CX_-$ and $CX_+$ for these two copies of $CX$.

If $f : X \to GL_n(\mathbb{F})$ is a continuous map, we can form

$$E_f := (CX_- \times \mathbb{F}^n \sqcup CX_+ \times \mathbb{F}^n)/\sim$$

where $((x,1),v) \in CX_- \times \mathbb{F}^n$ is identified with $((x,1), f(x)(v)) \in CX_+ \times \mathbb{F}^n$. This has a natural map $\pi_f : E_f \to CX_- \cup_X CX_+ \cong \Sigma X$, and it is easy to check that it is locally trivial. It is called *the vector bundle over* $\Sigma X$ *obtained by clutching along* $f$. If $F : X \times [0,1] \to GL_n(\mathbb{F})$ is a homotopy from $f$ to $g$, the same construction gives a vector bundle over $(\Sigma X) \times [0,1]$ which restricts to $E_f$ at one end and to $E_g$ at the other: thus if $X$ is compact Hausdorff then $E_f \cong E_g$ by Lemma 1.4.1.

On the other hand, if $\pi : E \to \Sigma X \cong CX_- \cup_X CX_+$ is a vector bundle and $X$ is compact Hausdorff then by Corollary 1.4.3 the restrictions $E|_{CX_\pm} \to CX_\pm$ are both trivial, and we can choose trivialisations

$$\varphi_\pm : E|_{CX_\pm} \longrightarrow CX_\pm \times \mathbb{F}^n.$$

From this we can form the map of vector bundles

$$X \times \mathbb{F}^n \xrightarrow{\varphi_-^{-1}} E|_X \xrightarrow{\varphi_+^{-1}} X \times \mathbb{F}^n$$

over $X$, which is necessarily of the form $(x, v) \mapsto (x, f(x)(v))$ for some $f : X \to GL_n(\mathbb{F})$. This identifies $E \cong E_f$ for the clutching map $f$. If we re-choose the trivialisations $\varphi_\pm$ then up to homotopy $f$ changes by multiplication by a fixed matrix.

Thus every vector bundle over $\Sigma X$ arises up to isomorphism by clutching. More precisely, there is a well-defined function

$$\frac{\{\text{maps } \phi : X \to GL_n(\mathbb{F})\}}{\text{homotopy}, \pi_0(GL_n(\mathbb{F}))} \longrightarrow \frac{\{n\text{-dim vector bundles over } \Sigma X\}}{\text{isomorphism}}$$
$$f \longmapsto (\pi_f : E_f \to \Sigma X)$$

and for $X$ compact Hausdorff it is a bijection.

Chapter 2

# Characteristic classes

Lecture 4

## 2.1 Recollections on Thom and Euler classes

Recall that to a $R$-oriented $d$-dimensional real vector bundle $\pi : E \to X$ there is associated a *Thom class*

$$u = u_E \in H^d(E, E^{\#}; R).$$

The composition

$$H^i(X;R) \xrightarrow{\pi^*} H^i(E;R) \xrightarrow{u_E \smile -} H^{i+d}(E, E^{\#}; R)$$

is then an isomorphism, the *Thom isomorphism*. Under the maps

$$H^d(E, E^{\#}; R) \xrightarrow{q^*} H^d(E;R) \xrightarrow{s_0^*} H^d(X;R)$$

the Thom class is sent to the *Euler class*

$$e = e(E) \in H^d(X;R).$$

**Remark 2.1.1.** For a real vector bundle $\pi : E \to B$ with inner product, you may have seen the Thom class defined as a class $u_E \in H^d(\mathbb{D}(E), \mathbb{S}(E); R)$. As the natural inclusions $\mathbb{D}(E) \to E$ and $\mathbb{S}(E) \to E^{\#}$ are homotopy equivalences, the natural map

$$H^d(E, E^{\#}; R) \longrightarrow H^d(\mathbb{D}(E), \mathbb{S}(E); R)$$

is an isomorphism and so these definitions correspond.

### 2.1.1 Naturality

A map $f : X' \to X$ induces a map $\hat{f} : E' \to E$ given by projection to the second coordinate, where

$$E' := f^*E = \{(x', e) \in X' \times E \mid f(x') = \pi(e)\}$$

is the pullback, and this projection map $\hat{f}$ sends $(E')^{\#}$ to $E^{\#}$. Thus there is a commutative diagram

$$\begin{array}{ccccc} H^d(E, E^{\#}; R) & \xrightarrow{q^*} & H^d(E;R) & \xrightarrow{s_0^*} & H^d(X;R) \\ \downarrow \hat{f}^* & & \downarrow \hat{f}^* & & \downarrow f^* \\ H^d(E', (E')^{\#}; R) & \xrightarrow{(q')^*} & H^d(E';R) & \xrightarrow{s_0^*} & H^d(X';R). \end{array}$$

If we orient $E' = f^*E$ by defining $u_{E'} := \hat{f}^*(u_E)$, which is the same as insisting that $\hat{f}$ be an orientation-preserving linear isomorphism on each fibre, then this diagram gives

$$e(f^*E) = f^*(e(E)) \in H^d(X';R).$$

### 2.1.2 Sum formula

If $\pi_0 : E_0 \to X$ and $\pi_1 : E_1 \to X$ are $R$-oriented real vector bundles of dimensions $d_0$ and $d_1$, recall that the underlying set of $E_0 \oplus E_1$ is the fibre product $E_0 \times_X E_1$, and so

$$(E_0 \oplus E_1)^{\#} = (E_0^{\#} \times_X E_1) \cup (E_0 \times_X E_1^{\#}).$$

Thus there are maps of pairs

$$\begin{aligned} p_0 &: (E_0 \times_X E_1, E_0^{\#} \times_X E_1) \longrightarrow (E_0, E_0^{\#}) \\ p_1 &: (E_0 \times_X E_1, E_0 \times_X E_1^{\#}) \longrightarrow (E_1, E_1^{\#}) \end{aligned}$$

and so a map

$$\begin{aligned} \varphi : H^{d_0}(E_0, E_0^{\#}; R) \otimes H^{d_1}(E_1, E_1^{\#}; R) &\longrightarrow H^{d_0+d_1}(E_0 \oplus E_1, (E_0 \oplus E_1)^{\#}; R) \\ x \otimes y &\longmapsto p_0^*(x) \smile p_1^*(y) \end{aligned}$$

using the relative cup product.

One can check that $\varphi(u_{E_0} \otimes u_{E_1})$ is a Thom class for $E_0 \oplus E_1$ (i.e. restricts to a generator of the cohomology of each fibre), which in particular gives an $R$-orientation of $E_0 \oplus E_1$. By pulling back this Thom class along the zero section it follows that

$$e(E_0 \oplus E_1) = e(E_0) \smile e(E_1) \in H^{d_0+d_1}(X; R).$$

### 2.1.3 Gysin sequence

If we choose an inner product on a $d$-dimensional $R$-oriented real vector bundle $\pi : E \to X$ and form the sphere bundle $p : \mathbb{S}(E) \to X$, then combining the equivalence $\mathbb{S}(E) \simeq E^{\#}$, the long exact sequence for the pair $(E, E^{\#})$, and the Thom isomorphism, we obtain the *Gysin sequence*

$$\cdots \longrightarrow H^i(X; R) \xrightarrow{e(E) \smile -} H^{i+d}(X; R) \xrightarrow{p^*} H^{i+d}(\mathbb{S}(E); R) \xrightarrow{p_!} H^{i+1}(X; R) \longrightarrow \cdots$$

### 2.1.4 Complex vector bundles

A complex vector bundle is $R$-oriented for any commutative ring $R$, so any $d$-dimensional complex vector bundle $\pi : E \to X$ has an associated Euler class $e(E) \in H^{2d}(X; R)$. In particular, if $\pi : L \to X$ is a complex line bundle i.e. a 1-dimensional complex vector bundle, then there is a class

$$e = e(L) \in H^2(X; R).$$

The tautological line bundle $\gamma_{\mathbb{C}}^{1,N+1} \to \mathbb{CP}^N$ is given by

$$\gamma_{\mathbb{C}}^{1,N+1} = \{(\ell, v) \in \mathbb{CP}^N \times \mathbb{C}^{N+1} \mid v \in \ell\},$$

and has an Euler class $x := e(\gamma_{\mathbb{C}}^{1,N+1}) \in H^2(\mathbb{CP}^N; R)$. Then

$$H^*(\mathbb{CP}^N; R) = R[x]/(x^{N+1}).$$

**Remark 2.1.2.** There is one small subtlety which we will have to bear in mind. If $[\mathbb{CP}^N] \in H_{2N}(\mathbb{CP}^N; R)$ denotes the fundamental class given by the orientation of $\mathbb{CP}^N$ determined by the complex structure, then

$$\langle x^N, [\mathbb{CP}^N] \rangle = (-1)^N.$$

We will explain why this holds in Example 2.9.3. It can also be seen by showing that the Poincaré dual of $x$ is $-[\mathbb{CP}^{N-1}]$.

**Remark 2.1.3.** If $\pi_i : E_i \to X$ for $i = 0, 1$ are complex vector bundles, then one can check that the $R$-orientation of $E_0 \oplus E_1$ coming from Section 2.1.2 agrees with the $R$-orientation coming from the fact that $E_0 \oplus E_1$ is a complex vector bundle (it is enough to check this on a single fibre).

### 2.1.5 Real vector bundles

A real vector bundle may or may not be $R$-orientable for a particular commutative ring $R$, but is always $\mathbb{F}_2$-oriented. Thus any $d$-dimensional real vector bundle $\pi : E \to X$ has an associated Euler class $e(E) \in H^d(X; \mathbb{F}_2)$. In particular, if $\pi : L \to X$ is a real line bundle i.e. a 1-dimensional complex vector bundle, then there is a class

$$e = e(L) \in H^1(X; \mathbb{F}_2).$$

The tautological line bundle $\gamma_{\mathbb{R}}^{1,N+1} \to \mathbb{RP}^N$ is given by

$$\gamma_{\mathbb{R}}^{1,N+1} = \{(\ell, v) \in \mathbb{RP}^N \times \mathbb{R}^{N+1} \mid v \in \ell\},$$

and has an Euler class $x := e(\gamma_{\mathbb{R}}^{1,N+1}) \in H^1(\mathbb{RP}^N; \mathbb{F}_2)$. Then

$$H^*(\mathbb{RP}^N; \mathbb{F}_2) = \mathbb{F}_2[x]/(x^{N+1}).$$

## 2.2 The projective bundle formula

Let $\pi : E \to X$ be a $d$-dimensional $\mathbb{F}$-vector bundle. We may form its *projectivisation*

$$\mathbb{P}(E) = E^\#/\mathbb{F}^\times,$$

that is, remove the zero section and then take the quotient by the $\mathbb{F}^\times$-action given by the action of scalars on each fibre. The map

$$\begin{aligned} p : \mathbb{P}(E) &\longrightarrow X \\ [v] &\longmapsto \pi(v) \end{aligned}$$

is well-defined. The fibre $p^{-1}(x)$ is the projectivisation $\mathbb{P}(E_x)$, which we identity with the set of 1-dimensional subspaces of $E_x$. Furthermore, $\mathbb{P}(E)$ has a canonical 1-dimensional $\mathbb{F}$-vector bundle over it, with total space given by

$$L_E := \{(\ell, v) \in \mathbb{P}(E) \times E \mid v \in \ell\}$$

and projection map $q : L_E \to \mathbb{P}(E)$ given by $q(\ell, v) = \ell$. (It is easy to check that the map $q$ is locally trivial over each $p^{-1}(U_\alpha) \subset \mathbb{P}(E)$, where $U_\alpha$ is a trivialising open set for $\pi$.) Thus there is defined a class

$$x_E := e(L_E) \in \begin{cases} H^2(\mathbb{P}(E); R) & \text{if } \mathbb{F} = \mathbb{C}, \text{ for any commutative ring } R \\ H^1(\mathbb{P}(E); \mathbb{F}_2) & \text{if } \mathbb{F} = \mathbb{R}. \end{cases}$$

To avoid repetition, let us write $R = \mathbb{F}_2$ in the case $\mathbb{F} = \mathbb{R}$.

**Theorem 2.2.1.** *If $\pi : E \to X$ is a $d$-dimensional $\mathbb{F}$-vector bundle then the $H^*(X; R)$-module map*

$$\begin{aligned} H^*(X; R)\{1, x_E, x_E^2, \ldots, x_E^{d-1}\} &\longrightarrow H^*(\mathbb{P}(E); R) \\ \sum_{i=0}^{d-1} y_i \cdot x_E^i &\longmapsto \sum_{i=0}^{d-1} p^*(y_i) \smile x_E^i \end{aligned}$$

*is an isomorphism.*

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*Proof.* We shall give the proof only when $X$ is compact: the theorem is true in general, but uses Zorn's lemma and so requires understanding the behaviour of cohomology with respect to infinite ascending unions, which has not been covered in Part III Algebraic Topology and is outside the scope of this course.

As $X$ is compact, we may find finitely-many open subsets $U_1, \ldots, U_n \subset B$ so that the vector bundle is trivial over each $U_i$. We let $V_i = U_1 \cup U_2 \cup \cdots \cup U_i$, and will prove the theorem for $\pi|_{V_i} : E|_{V_i} \to V_i$ by induction over $i$. The theorem holds for $V_0 = \emptyset$ as there is nothing to show. Assuming it holds for $V_{i-1}$, consider $V_{i-1} \subset V_i = V_{i-1} \cup U_i$ and the map of long exact sequences

$$\begin{array}{ccc}
H^{*-1}(V_{i-1}; R)\{1, x_E, x_E^2, \ldots, x_E^{d-1}\} & \longrightarrow & H^{*-1}(\mathbb{P}(E|_{V_{i-1}}); R) \\
\downarrow \partial & & \downarrow \partial \\
H^*(V_i, V_{i-1}; R)\{1, x_E, x_E^2, \ldots, x_E^{d-1}\} & \longrightarrow & H^*(\mathbb{P}(E|_{V_i}), \mathbb{P}(E|_{V_{i-1}}); R) \\
\downarrow & & \downarrow \\
H^*(V_i; R)\{1, x_E, x_E^2, \ldots, x_E^{d-1}\} & \longrightarrow & H^*(\mathbb{P}(E|_{V_i}); R) \\
\downarrow & & \downarrow \\
H^*(V_{i-1}; R)\{1, x_E, x_E^2, \ldots, x_E^{d-1}\} & \longrightarrow & H^*(\mathbb{P}(E|_{V_{i-1}}); R) \\
\downarrow \partial & & \downarrow \partial \\
H^{*+1}(V_i, V_{i-1}; R)\{1, x_E, x_E^2, \ldots, x_E^{d-1}\} & \longrightarrow & H^{*+1}(\mathbb{P}(E|_{V_i}), \mathbb{P}(E|_{V_{i-1}}); R)
\end{array}$$

This commutes: the only tricky point is the squares containing maps $\partial$, where one uses the general fact that if $i : B \hookrightarrow Y$ then the map $\partial : H^p(B; R) \to H^{p+1}(Y, B; R)$ is a map of $H^*(Y;R)$-modules in the sense that

$$\partial(y \smile i^*x) = \partial(y) \smile x.$$

By assumption the first and fourth horizontal maps are isomorphisms, so if we can show the second and fifth horizontal maps are too then the result follows from the 5-lemma. (Note that the second and fifth horizontal maps are the same, just with a degree shift.) By excision we can identify the second (and so fifth) horizontal map with

$$H^*(U_i, U_i \cap V_{i-1}; R)\{1, x_E, x_E^2, \ldots, x_E^{d-1}\} \longrightarrow H^*(\mathbb{P}(E|_{U_i}), \mathbb{P}(E|_{U_i \cap V_{i-1}}); R).$$

Choosing a trivialisation $E|_{U_i} \xrightarrow{\sim} \underline{\mathbb{F}^d}_{U_i}$ gives a homeomorphism of pairs

$$(\mathbb{P}(E|_{U_i}), \mathbb{P}(E|_{U_i \cap V_{i-1}})) \xrightarrow{\sim} (U_i, U_i \cap V_{i-1}) \times \mathbb{FP}^{d-1}$$

under which $L_E$ corresponds to $\pi_2^*(\gamma_{\mathbb{F}}^{1,d})$, and so $x_E$ corresponds to $\pi_2^*(x)$. Now the classes

$$1, x, x^2, \ldots, x^{d-1} \in H^*(\mathbb{FP}^{d-1}; R)$$

are an $R$-module basis, so the (relative) Künneth theorem applies and gives that

$$\begin{aligned} H^*(U_i, U_i \cap V_{i-1}; R)\{1, x, x^2, \ldots, x^{d-1}\} &= H^*(U_i, U_i \cap V_{i-1}; R) \otimes_R H^*(\mathbb{FP}^{d-1}; R) \\ &\longrightarrow H^*((U_i, U_i \cap V_{i-1}) \times \mathbb{FP}^{d-1}; R) \\ \sum a_i \cdot x^i &\longmapsto \sum \pi_1^*(a_i) \smile \pi_2^*(x^i) \end{aligned}$$

is an isomorphism, as required. □

**Corollary 2.2.2.** *If $\pi : E \to X$ is a $d$-dimensional $\mathbb{F}$-vector bundle then the map*

$$p^* : H^*(X; R) \longrightarrow H^*(\mathbb{P}(E); R)$$

*is injective.*

*Proof.* Under the isomorphism given by Theorem 2.2.1, this map corresponds to the inclusion of the $H^*(X;R)$-module summand associated to $1 = x_E^0$. □

## 2.3 Chern classes

We now specialise to the case $\mathbb{F} = \mathbb{C}$, and let $\pi : E \to X$ be a $d$-dimensional complex vector bundle.

**Definition 2.3.1.** The *Chern classes* $c_i(E) \in H^{2i}(X;R)$ are the unique classes satisfying $c_0(E) = 1$ and

$$\sum_{i=0}^{d} (-1)^i p^*(c_i(E)) \smile x_E^{d-i} = 0 \in H^{2d}(\mathbb{P}(E); R),$$

under the isomorphism of Theorem 2.2.1.

The following theorem describes the basic properties of Chern classes.

**Theorem 2.3.2.**

*(i) The class $c_i(E)$ only depends on $E$ up to isomorphism.*

*(ii) If $f : X' \to X$ is a map then $c_i(f^*E) = f^*(c_i(E)) \in H^{2i}(X'; R)$.*

*(iii) For complex vector bundles $\pi_i : E_i \to X$, $i = 1, 2$, we have*

$$c_k(E_1 \oplus E_2) = \sum_{a+b=k} c_a(E_1) \smile c_b(E_2) \in H^{2k}(X; R).$$

*(iv) If $\pi : E \to X$ is a $d$-dimensional vector bundle then $c_i(E) = 0$ for $i > d$.*

*Proof.* For (i), suppose $\phi : E_1 \to E_2$ is an isomorphism of vector bundles over $X$, then it induces a homeomorphism $\mathbb{P}(\phi) : \mathbb{P}(E_1) \xrightarrow{\sim} \mathbb{P}(E_2)$ over $B$, satisfying $\mathbb{P}(\phi)^*(L_{E_2}) = L_{E_1}$ and so satisfying $\mathbb{P}(\phi)^*(x_{E_2}) = x_{E_1}$. Thus we have

$$0 = \mathbb{P}(\phi)^* \left( \sum_{i=0}^{d} (-1)^i p_2^*(c_i(E_2)) \smile x_{E_2}^{d-i} \right) = \sum_{i=0}^{d} (-1)^i p_1^*(c_i(E_2)) \smile x_{E_1}^{d-i}$$

which is the defining formula for the $c_i(E_1)$.

The argument for (ii) is similar. There is a commutative square

$$\begin{array}{ccc} E' := f^*E & \xrightarrow{\hat{f}} & E \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{f} & X \end{array}$$

and projectivising gives a commutative square

$$\begin{array}{ccc} \mathbb{P}(E') & \xrightarrow{\mathbb{P}(\hat{f})} & \mathbb{P}(E) \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ X' & \xrightarrow{f} & X \end{array}$$

where $\mathbb{P}(\hat{f})^*(L_E) = L_{E'}$ and so $\mathbb{P}(\hat{f})^*(x_E) = x_{E'}$. Thus we have

$$0 = \mathbb{P}(\hat{f})^* \left( \sum_{i=0}^{d} (-1)^i p^*(c_i(E)) \smile x_E^{d-i} \right) = \sum_{i=0}^{d} (-1)^i (p')^*(f^* c_i(E)) \smile x_{E'}^{d-i}$$

which is the defining formula for the $c_i(E')$.

For (iii), we have $\mathbb{P}(E_1) \subset \mathbb{P}(E_1 \oplus E_2) \supset \mathbb{P}(E_2)$ disjoint closed subsets, with open complements $U_i = \mathbb{P}(E_1 \oplus E_2) \setminus \mathbb{P}(E_i)$. The inclusions

$$\mathbb{P}(E_1) \subset U_2$$
$$\mathbb{P}(E_2) \subset U_1$$

are easily seen to be deformation retractions. The line bundle $L_{E_1 \oplus E_2}$ restricts to $L_{E_i}$ over $\mathbb{P}(E_i)$, so $x_{E_1 \oplus E_2}$ restricts to $x_{E_i}$ over $\mathbb{P}(E_i)$. Supposing that $E_i$ has dimension $d_i$, the classes

$$\omega_i = \sum_{j=0}^{d_i} (-1)^j p^*(c_j(E_i)) \smile x_{E_1 \oplus E_2}^{d_i - j} \in H^{2d_i}(\mathbb{P}(E_1 \oplus E_2); R)$$

therefore have the property that $\omega_1$ restricts to zero on $\mathbb{P}(E_1)$ and so on $U_2$, and $\omega_2$ restricts to zero on $\mathbb{P}(E_2)$ and so on $U_1$. Hence

$$\omega_1 \smile \omega_2 = \sum_{k=0}^{d_1+d_2} (-1)^k p^* \left( \sum_{a+b=k} c_a(E_1) \smile c_b(E_2) \right) \smile x_{E_1 \oplus E_2}^{d_1+d_2-k}$$

is zero on $U_1 \cup U_2 = \mathbb{P}(E_1 \oplus E_2)$, so by the defining formula of $c_k(E_1 \oplus E_2)$ we get the claimed identity.

Part (iv) is true by definition. □

Due to item (iii) it is often convenient to consider the *total Chern class*

$$c(E) = 1 + c_1(E) + c_2(E) + \cdots \in \prod_i H^{2i}(X; R),$$

as item (iii) is then equivalent to the formula

$$c(E_1 \oplus E_2) = c(E_1) \smile c(E_2),$$

which is often easier to manipulate.

**Example 2.3.3.** Consider a complex line bundle $\pi : E \to X$. Then $p : \mathbb{P}(E) \to X$ is a homeomorphism, and it is easy to see that $L_E = p^*(E)$. Thus

$$x_E = e(L_E) = e(p^*(E)) = p^*(e(E))$$

and so

$$0 = 1 \smile x_E^1 - p^*(c_1(E)) \smile x_E^0$$

so that $c_1(E) = e(E) \in H^2(X; R)$ (as $p^*$ is injective).

In particular, for $\gamma_{\mathbb{C}}^{1,N+1} \to \mathbb{CP}^N$ we have $c_1(\gamma_{\mathbb{C}}^{1,N+1}) = x \in H^2(\mathbb{CP}^N; R)$, and so the total Chern class is

$$c(\gamma_{\mathbb{C}}^{1,N+1}) = 1 + x.$$

**Example 2.3.4.** Let $\pi_1 : \underline{\mathbb{C}}_X^n := X \times \mathbb{C}^n \to X$ be the trivial $n$-dimensional complex vector bundle. It is pulled back along the unique map $f : X \to *$ from the trivial bundle $\underline{\mathbb{C}}_*^n = * \times \mathbb{C}^n \to *$, and so

$$c_i(\underline{\mathbb{C}}_X^n) = c_i(f^* \underline{\mathbb{C}}_*^n) = f^*(c_i(\underline{\mathbb{C}}_*^n))$$

but this vanishes for $i > 0$ as $c_i(\underline{\mathbb{C}}_*^n) \in H^{2i}(*; R) = 0$. Thus the total Chern class satisfies

$$c(\underline{\mathbb{C}}_X^n) = 1.$$

This means that $c(E \oplus \underline{\mathbb{C}}_X^n) = c(E) \smile c(\underline{\mathbb{C}}_X^n) = c(E)$, and so, by expanding out, we see that $c_i(E \oplus \underline{\mathbb{C}}_X^n) = c_i(E)$ for all $i$.

Lecture 6

## 2.4 Stiefel–Whitney classes

We may repeat the entire discussion above with $R = \mathbb{F}_2$ and with real vector bundles. Then if $\pi : E \to X$ is a $d$-dimensional real vector bundle it has a real projectivisation $\mathbb{P}(E)$, a canonical $x_E \in H^1(\mathbb{P}(E); \mathbb{F}_2)$, Theorem 2.2.1 gives an isomorphism

$$H^*(X; \mathbb{F}_2)\{1, x_E, x_E^2, \ldots, x_E^{d-1}\} \xrightarrow{\sim} H^*(\mathbb{P}(E); \mathbb{F}_2),$$

and under this identification the *Stiefel–Whitney classes* $w_i(E) \in H^i(X; \mathbb{F}_2)$ are defined by $w_0(E) = 1$ and

$$\sum_{i=0}^{d} p^*(w_i(E)) \smile x_E^{d-i} = 0 \in H^d(\mathbb{P}(E); \mathbb{F}_2).$$

As in Example 2.3.3 we have $w_1(\gamma_{\mathbb{R}}^{1,N+1}) = x \in H^1(\mathbb{RP}^N; \mathbb{F}_2)$. The analogue of Theorem 2.3.2 holds, as follows.

**Theorem 2.4.1.**

*(i) The class $w_i(E)$ only depends on $E$ up to isomorphism.*

*(ii) If $f : X' \to X$ is a map then $w_i(f^*E) = f^*(w_i(E)) \in H^i(X'; \mathbb{F}_2)$.*

*(iii) For real vector bundles $\pi_i : E_i \to X$, $i = 1, 2$, we have*

$$w_k(E_1 \oplus E_2) = \sum_{a+b=k} w_a(E_1) \smile w_b(E_2) \in H^k(X; \mathbb{F}_2).$$

*(iv) If $\pi : E \to X$ is a $d$-dimensional vector bundle then $w_i(E) = 0$ for $i > d$.*

The *total Stiefel–Whitney class* is

$$w(E) = 1 + w_1(E) + w_2(E) + \cdots \in \prod_i H^i(X; \mathbb{F}_2),$$

and it satisfies $w(E \oplus F) = w(E) \cdot w(F)$.

## 2.5 Pontrjagin classes

If $\pi : E \to X$ is a $d$-dimensional real vector bundle, then we can form a $d$-dimensional complex vector bundle $E \otimes_{\mathbb{R}} \mathbb{C} \to X$ by forming the fibrewise complexification.

**Definition 2.5.1.** For a real vector bundle $\pi : E \to X$, we define the *Pontrjagin classes* $p_i(E)$ by

$$p_i(E) := (-1)^i c_{2i}(E \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4i}(X; R).$$

## 2.6 The splitting principle

The following is a very useful technique for establishing relations between characteristic classes.

**Theorem 2.6.1.** *For a complex vector bundle $\pi : E \to X$ over a compact Hausdorff space $X$ and a commutative ring $R$, there is an associated space $F(E)$ and map $f : F(E) \to X$ such that*

*(i) the vector bundle $f^*(E)$ is a direct sum of complex line bundles, and*

*(ii) the map $f^* : H^*(X; R) \to H^*(F(E); R)$ is injective.*

*The analogous statement holds for real vector bundles and $R = \mathbb{F}_2$.*

*Proof.* By induction it is enough to find a map $f' : F'(E) \to X$ so that $(f')^*(E) \cong E' \oplus L$ with $L$ a complex line bundle, and $(f')^* : H^*(X; R) \to H^*(F'(E); R)$ injective, as we can iteratively apply the same to the vector bundle $E'$.

For this we take $F'(E) := \mathbb{P}(E)$ and $f' = p : \mathbb{P}(E) \to X$. There is an injective bundle map

$$\begin{aligned} \phi : L_E &\longrightarrow p^*(E) \\ (\ell, v) &\longmapsto (\ell, v), \end{aligned}$$

and as $X$ is compact Hausdorff we may choose an (Hermitian) inner product on $E$, inducing one of $p^*(E)$, and hence take $E'$ to be the orthogonal complement of $\phi(L_E)$ in $p^*(E)$. Then $p^*(E) \cong L_E \oplus E'$ as required, and $p^*$ is injective by Corollary 2.2.2. □

## 2.7 The Euler class revisited

As a first application of the splitting principle, we prove the following.

**Theorem 2.7.1.** *If $\pi : E \to X$ is a $d$-dimensional complex vector bundle over a compact Hausdorff base, then $c_d(E) = e(E) \in H^{2d}(X; R)$. Similarly, if $\pi : E \to X$ is a $d$-dimensional real vector bundle, then $w_d(E) = e(E) \in H^d(X; \mathbb{F}_2)$.*

*Proof.* We consider the complex case. Let $f : F(E) \to X$ be the map provided by the splitting principle, so $f^*E \cong L_1 \oplus \cdots \oplus L_d$ with the $L_i$ complex line bundles. Then

$$f^*(c(E)) = c(f^*E) = c(L_1 \oplus \cdots \oplus L_d) = (1 + c_1(L_1)) \cdots (1 + c_1(L_d))$$

and so, expanding out, we have

$$f^*(c_d(E)) = c_1(L_1) \cdots c_1(L_d).$$

On the other hand we have

$$f^*e(E) = e(f^*E) = e(L_1 \oplus \cdots \oplus L_d) = e(L_1) \cdots e(L_d).$$

Now by Example 2.3.3 we have $c_1(L) = e(L)$ for a complex line bundle $L$, so it follows that $f^*(c_d(E)) = f^*e(E)$. But $f^*$ is injective, so $c_d(E) = e(E)$.

The argument in the real case is analogous. □

## 2.8 Examples

**Example 2.8.1.** Consider $k\gamma_{\mathbb{C}}^{1,N+1} = \gamma_{\mathbb{C}}^{1,N+1} \oplus \cdots \oplus \gamma_{\mathbb{C}}^{1,N+1} \to \mathbb{CP}^N$ the direct sum of $k$ copies of the tautological complex line bundle. We showed in Example 2.3.3 that

$$c(\gamma_{\mathbb{C}}^{1,N+1}) = 1 + x \in H^*(\mathbb{CP}^N; \mathbb{Z}),$$

so

$$c(k\gamma_{\mathbb{C}}^{1,N+1}) = (1+x)^k = 1 + kx + \binom{k}{2}x^2 + \cdots \in H^*(\mathbb{CP}^N; \mathbb{Z}).$$

**Example 2.8.2.** Similarly, $k\gamma_{\mathbb{R}}^{1,N+1} \to \mathbb{RP}^N$, the direct sum of $k$ copies of the tautological real line bundle, has

$$w(k\gamma_{\mathbb{R}}^{1,N+1}) = (1+x)^k = 1 + kx + \binom{k}{2}x^2 + \cdots \in H^*(\mathbb{RP}^N; \mathbb{F}_2),$$

but now the binomial coefficients are taken modulo 2.

**Example 2.8.3.** If $\pi : E \to X$ is a complex vector bundle, recall from Section 1.1.3 that the conjugate vector bundle $\overline{\pi} : \overline{E} \to X$ has the same underlying real bundle but with the opposite complex structure. That is, the identity map $f : \overline{E} \to E$ is complex antilinear, satisfying $f(\lambda \cdot v) = \overline{\lambda} \cdot f(v)$. This still induces a map

$$\mathbb{P}(f) : \mathbb{P}(\overline{E}) \longrightarrow \mathbb{P}(E)$$

over $X$, which interacts with the canonical line bundles $L_E$ and $L_{\overline{E}}$ as

$$\mathbb{P}(f)^*(L_E) = \overline{L_{\overline{E}}}.$$

As a real vector bundle $\overline{L_{\overline{E}}}$ is equal to $L_{\overline{E}}$, but it has the opposite complex structure and hence the opposite orientation. Thus its Euler class has the opposite sign, so we have

$$\mathbb{P}(f)^*(x_E) = \mathbb{P}(f)^*(e(L_E)) = e(\mathbb{P}(f)^*(L_E)) = e(\overline{L_{\overline{E}}}) = -e(L_{\overline{E}}) = -x_{\overline{E}}.$$

Applying this to the polynomial defining the $c_i(E)$ gives that

$$\sum_{i=0}^{d} (-1)^i p^*(c_i(E)) \smile (-x_{\overline{E}})^{d-i} = 0 \in H^{2d}(\mathbb{P}(\overline{E}); R),$$

and comparing this with the polynomial defining the $c_i(\overline{E})$ shows that

$$c_i(\overline{E}) = (-1)^i c_i(E).$$

**Example 2.8.4.** If $\pi : E \to X$ is a complex vector bundle then choosing a Hermitian inner product on $E$, which is possible if $X$ is compact Hausdorff, gives an isomorphism $\overline{E} \cong E^\vee$, so in this case we also have

$$c_i(E^\vee) = (-1)^i c_i(E).$$

**Example 2.8.5.** Over $\mathbb{CP}^n \times \mathbb{CP}^n$ consider the line bundle $L := \pi_1^*(\gamma_{\mathbb{C}}^{1,n+1}) \otimes \pi_2^*(\gamma_{\mathbb{C}}^{1,n+1})$, having $c_1(L) \in H^2(\mathbb{CP}^n \times \mathbb{CP}^n; \mathbb{Z})$. By the Künneth theorem, we must have $c_1(L) = Ax \otimes 1 + B1 \otimes x$ for some $A, B \in \mathbb{Z}$. Choose a point $\{P\} \in \mathbb{CP}^n$, and pull back $L$ along the inclusion

$$\mathbb{CP}^n = \mathbb{CP}^n \times \{P\} \hookrightarrow \mathbb{CP}^n \times \mathbb{CP}^n.$$

One one hand this pulls $L$ back to $\gamma_{\mathbb{C}}^{1,n+1} \otimes P \cong \gamma_{\mathbb{C}}^{1,n+1}$, having first Chern class $x$. On the other hand it pulls back $Ax \otimes 1 + B1 \otimes x$ to $Ax$, so $A = 1$; by symmetry we see that $B = 1$ too. As $x = c_1(\gamma_{\mathbb{C}}^{1,n+1})$, we can write this as

$$c_1(L) = c_1(\gamma_{\mathbb{C}}^{1,n+1}) \otimes 1 + 1 \otimes c_1(\gamma_{\mathbb{C}}^{1,n+1}).$$

Now if $X$ is compact Hausdorff and $\pi_i : L_i \to X$, $i = 1, 2$, are line bundles, then there is a $N \gg 0$ and maps $f_i : X \to \mathbb{CP}^N$ such that $f_i^*(\gamma_{\mathbb{C}}^{1,N+1}) \cong L_i$, by the discussion in Section 1.5. Thus $L_1 \otimes L_2 = (f_1 \times f_2)^*(L)$ for the line bundle $L \to \mathbb{CP}^N \times \mathbb{CP}^N$ described above. Thus

$$c_1(L_1 \otimes L_2) = (f_1 \times f_2)^*(c_1(L)) = (f_1 \times f_2)^*(c_1(\gamma_{\mathbb{C}}^{1,N+1}) \otimes 1 + 1 \otimes c_1(\gamma_{\mathbb{C}}^{1,N+1})) = c_1(L_1) + c_1(L_2).$$

## 2.9 Some tangent bundles

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**Example 2.9.1.** Realising $S^n$ as the unit sphere in $\mathbb{R}^{n+1}$, it has a 1-dimensional normal bundle, which is trivial (by taking the outwards-pointing unit normal vector). Thus there is an isomorphism

$$TS^n \oplus \underline{\mathbb{R}} \cong \underline{\mathbb{R}}^{n+1},$$

and so

$$w(TS^n) = w(TS^n \oplus \underline{\mathbb{R}}) = w(\underline{\mathbb{R}}^{n+1}) = 1$$

and so all Stiefel–Whitney classes of $TS^n$ are trivial.

**Example 2.9.2.** $\mathbb{RP}^n$ is obtained from $S^n$ as the quotient by the antipodal map. Inside $\mathbb{R}^{n+1}$ the antipodal map acts by inversion on each of the $n+1$ coordinate directions, but acts trivially on the normal bundle as it sends the outwards-pointing unit normal vector at $x$ to the outwards-pointing unit normal vector at $-x$. This gives an isomorphism of vector bundles

$$T\mathbb{RP}^n \oplus \underline{\mathbb{R}} \cong (n+1)\gamma_{\mathbb{R}}^{1,n+1},$$

and so

$$w(T\mathbb{RP}^n) = w(T\mathbb{RP}^n \oplus \underline{\mathbb{R}}) = w((n+1)\gamma_{\mathbb{R}}^{1,n+1}) = (1+x)^{n+1} \in H^*(\mathbb{RP}^n; \mathbb{F}_2).$$

**Example 2.9.3.** The tautological bundle $\gamma_{\mathbb{C}}^{1,n+1} \to \mathbb{CP}^n$ is naturally a subbundle of the trivial bundle $\underline{\mathbb{C}}_{\mathbb{CP}^n}^{n+1}$; let us write $\omega^n \to \mathbb{CP}^n$ for its ($n$-dimensional) orthogonal complement. There is a map of vector bundles

$$\phi : Hom(\gamma_{\mathbb{C}}^{1,n+1}, \omega^n) \longrightarrow T\mathbb{CP}^n$$

given as follows: on the fibre over a point $\ell \in \mathbb{CP}^n$, given a linear map $f : \ell \to \ell^\perp$ we can obtain a nearby line $\ell_f$ as the image of the linear map

$$Id \oplus f : \ell \longrightarrow \ell \oplus \ell^\perp = \mathbb{C}^{n+1}.$$

For $t \in \mathbb{R}$ we therefore get a smooth path $t \mapsto \ell_{t \cdot f}$ through $\ell$, defining a vector $\phi(f) \in T_\ell \mathbb{CP}^n$. It is easy to see that $\phi$ so defined is a linear isomorphism on each fibre.

In particular, this describes $T\mathbb{CP}^n$ as an $n$-dimensional complex vector bundle.

Adding on $\underline{\mathbb{C}}_{\mathbb{CP}^n} = Hom(\gamma_{\mathbb{C}}^{1,n+1}, \gamma_{\mathbb{C}}^{1,n+1})$ to each side and using $\gamma_{\mathbb{C}}^{1,n+1} \oplus \omega^n = \underline{\mathbb{C}}^{n+1}_{\mathbb{CP}^n}$ gives

$$T\mathbb{CP}^n \oplus \underline{\mathbb{C}}_{\mathbb{CP}^n} \cong Hom(\gamma_{\mathbb{C}}^{1,n+1}, \underline{\mathbb{C}}^{n+1}_{\mathbb{CP}^n}) \cong (n+1)\overline{\gamma_{\mathbb{C}}^{1,n+1}},$$

and so

$$c(T\mathbb{CP}^n) = (1 + c_1(\overline{\gamma_{\mathbb{C}}^{1,n+1}}))^{n+1} = (1-x)^{n+1} \in H^*(\mathbb{CP}^n; R).$$

In particular, by Theorem 2.7.1 we have

$$e(T\mathbb{CP}^n) = c_n(T\mathbb{CP}^n) = (-1)^n (n+1) x^n$$

and so

$$\langle e(T\mathbb{CP}^n), [\mathbb{CP}^n] \rangle = (n+1)(-1)^n \langle x^n, [\mathbb{CP}^n] \rangle.$$

We know that this calculates the Euler characteristic of $\mathbb{CP}^n$, which is $n+1$: thus we must have $\langle x^n, [\mathbb{CP}^n] \rangle = (-1)^n$, as claimed in Remark 2.1.2.

**Example 2.9.4.** Consider the manifold $M = \mathbb{RP}^n \# \mathbb{RP}^n$, with tangent bundle $TM \to M$. Let us write $H^*(M; \mathbb{F}_2) = \mathbb{F}_2[x,y]/(x^{n+1}, y^{n+1}, xy, x^n - y^n)$, so

$$w(TM) = 1 + \sum_{i=1}^{n-1} (a_i \cdot x^i + b_i \cdot y^i) + c \cdot x^n$$

for some scalars $a_i$, $b_i$ and $c$. As the restriction of $TM$ to each copy of $\mathbb{RP}^n \setminus D^n \subset M$ is isomorphic to the restriction of $T\mathbb{RP}^n$, we have that

$$a_i = b_i = \binom{n+1}{i} \text{ for } 1 \leq i \leq n-1.$$

We cannot determine $c$ this way, as $\mathbb{RP}^n \setminus D^n$ has no $n$th cohomology. But as $w_n(TM)$ agrees with the Euler class of $M$ mod 2, we know that $\langle w_n(TM), [M] \rangle$ is the Euler characteristic of $M$, which is 0 mod 2 so $c = 0$.

## 2.10 Nonimmersions

Suppose that there is an immersion $i : M^n \looparrowright \mathbb{R}^{n+k}$, i.e. a smooth map whose derivative is injective at each point. This gives an injective map of vector bundles

$$Di : TM \longrightarrow \underline{\mathbb{R}}^{n+k}_M$$

with orthogonal complement $\nu_i \to M$ a $k$-dimensional real vector bundle. Then $TM \oplus \nu_i \cong \underline{\mathbb{R}}^{n+k}_{\mathbb{RP}^n}$, and so

$$w(TM) \smile w(\nu_i) = w(TM \oplus \nu_i) = w(\underline{\mathbb{R}}^{n+k}_M) = 1$$

and so

$$w(\nu_i) = \frac{1}{w(M)} \in H^*(M; \mathbb{F}_2).$$

As $\nu_i$ has dimension $k$, we must have $w_j(\nu_i) = 0$ for $j > k$.

**Example 2.10.1.** If there is an immersion $i : \mathbb{RP}^n \looparrowright \mathbb{R}^{n+k}$, then

$$w(\nu_i) = \frac{1}{(1+x)^{n+1}} \in H^*(\mathbb{RP}^n; \mathbb{F}_2)$$

is a polynomial of degree at most $k$. We may consider the following table,

| $n$ | $\frac{1}{(1+x)^{n+1}} \in H^*(\mathbb{RP}^n; \mathbb{F}_2) = \mathbb{F}_2[x]/(x^{n+1})$ | does not immerse in $\mathbb{R}^N$ |
|---|---|---|
| 2 | $1+x$ | 2 |
| 3 | $1$ | |
| 4 | $1+x+x^2+x^3$ | 6 |
| 5 | $1+x^2$ | 6 |
| 6 | $1+x$ | 6 |
| 7 | $1$ | |
| 8 | $1+x+x^2+x^3+x^4+x^5+x^6+x^7$ | 14 |
| 9 | $1+x^2+x^4+x^6$ | 14 |
| 10 | $1+x+x^4+x^5$ | 14 |
| 11 | $1+x^4$ | 14 |
| 12 | $1+x+x^2+x^3$ | 14 |
| 13 | $1+x^2$ | 14 |
| 14 | $1+x$ | 14 |
| 15 | $1$ | |
| 16 | $1+x+x^2+\cdots+x^{15}$ | 30 |
| 17 | $1+x^2+x^4+x^6+x^8+x^{10}+x^{12}+x^{14}$ | 30 |
| 18 | $1+x+x^4+x^5+x^8+x^9+x^{12}+x^{13}$ | 30 |
| 19 | $1+x^4+x^8+x^{12}$ | 30 |
| 20 | $1+x+x^2+x^3+x^8+x^9+x^{10}+x^{11}$ | 30 |

The smallest dimensional real projective plane for which the smallest Euclidean space it immerses into is not known is $\mathbb{RP}^{24}$: it is known to immerse in $\mathbb{RP}^{39}$, and known to not immerse in $\mathbb{R}^{37}$.

## 2.11 Cohomology of infinite Grassmannians

In this section we will see that the Chern, Stiefel–Whitney, and Pontrjagin classes fully account for the cohomology of the infinite Grassmannians. In view of the fact that the infinite Grassmannians classify vector bundles (Section 1.5), this shows that these classes are “the only” characteristic classes of vector bundles.

**Lemma 2.11.1.** *There is a homotopy equivalence* $\mathbb{S}(\gamma_{\mathbb{F}}^{n,\infty}) \simeq Gr_{n-1}(\mathbb{F}^\infty)$.

*Proof.* The space $\mathbb{S}(\gamma_{\mathbb{F}}^{n,\infty})$ consists of pairs $(V, v)$ of an $n$-dimensional subspace $V \leq \mathbb{F}^\infty$ and a unit vector $v \in V$. To this we can associate the $(n-1)$-dimensional vector space $\langle v \rangle^\perp := \{x \in V \mid x \perp v\} \leq \mathbb{F}^\infty$, which defines a map

$$\Phi : \mathbb{S}(\gamma_{\mathbb{F}}^{n,\infty}) \longrightarrow Gr_{n-1}(\mathbb{F}^\infty).$$

On the other hand assigning to an $(n-1)$-dimensional subspace $W \leq \mathbb{F}^\infty$ the pair $(\mathbb{F} \oplus W \leq \mathbb{F} \oplus \mathbb{F}^\infty, e_1)$, where we identify $\mathbb{F} \oplus \mathbb{F}^\infty \cong \mathbb{F}^\infty$ by shifting coordinates, defines a map

$$\Psi : Gr_{n-1}(\mathbb{F}^\infty) \longrightarrow \mathbb{S}(\gamma_{\mathbb{F}}^{n,\infty}).$$

The composition $\Phi \circ \Psi$ is almost the identity: it sends $W \leq \mathbb{F}^\infty$ to the subspace $0 \oplus W \leq \mathbb{F} \oplus \mathbb{F}^\infty \cong \mathbb{F}^\infty$, i.e. its image under the linear map $(x_1, x_2, x_3, \ldots) \mapsto (0, x_1, x_2, \ldots)$. The straight line homotopy $(x_1, x_2, \ldots; t) \mapsto (1-t)\cdot(x_1, x_2, \ldots) + t\cdot(0, x_1, x_2, \ldots)$ through linear maps shows that this is homotopic to the identity.

The composition $\Psi \circ \Phi$ sends $(V, v)$ to $(\mathbb{F} \oplus \langle v \rangle^\perp \leq \mathbb{F} \oplus \mathbb{F}^\infty \cong \mathbb{F}^\infty, e_1)$. A sequence of straight-line homotopies similar to the proof of Lemma 1.5.1 shows that this is also homotopic to the identity. □

**Theorem 2.11.2.** *The map*

$$\begin{aligned} R[c_1, c_2, \ldots, c_n] &\longrightarrow H^*(Gr_n(\mathbb{C}^\infty); R) \\ c_i &\longmapsto c_i(\gamma_{\mathbb{C}}^{n,\infty}) \end{aligned}$$

*is an isomorphism. Similarly, the map*

$$\begin{aligned} \mathbb{F}_2[w_1, w_2, \ldots, w_n] &\longrightarrow H^*(Gr_n(\mathbb{R}^\infty); \mathbb{F}_2) \\ w_i &\longmapsto w_i(\gamma_{\mathbb{R}}^{n,\infty}) \end{aligned}$$

*is an isomorphism.*

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*Proof.* We do the complex case, and neglect the ring $R$ from the notation; the real case is completely analogous. We proceed by induction on $n$; the case $n = 0$ is immediate as $Gr_0(\mathbb{C}^\infty) = \{*\}$. For $n > 0$ we consider the Gysin sequence for the sphere bundle $p : \mathbb{S}(\gamma_{\mathbb{C}}^{n,\infty}) \to Gr_n(\mathbb{C}^\infty)$. By Lemma 2.11.1, a portion of this sequence takes the form

$$H^i(Gr_n(\mathbb{C}^\infty)) \xrightarrow{e(\gamma_{\mathbb{F}}^{n,\infty})\smile -} H^{i+2n}(Gr_n(\mathbb{C}^\infty)) \xrightarrow{inc^*} H^{i+2n}(Gr_{n-1}(\mathbb{C}^\infty)) \longrightarrow H^{i+1}(Gr_n(\mathbb{C}^\infty)).$$

By induction $H^*(Gr_{n-1}(\mathbb{C}^\infty))$ is generated by the classes $c_i(\gamma_{\mathbb{C}}^{n-1,\infty}) = inc^* c_i(\gamma_{\mathbb{C}}^{n,\infty})$, so the map $inc^*$ is surjective. By Theorem 2.7.1 we have $e(\gamma_{\mathbb{C}}^{n,\infty}) = c_n(\gamma_{\mathbb{C}}^{n,\infty})$. Thus we find that cup product with $c_n(\gamma_{\mathbb{C}}^{n,\infty})$ is injective on $H^*(Gr_n(\mathbb{C}^\infty))$, and that there are isomorphisms

$$inc^* : H^*(Gr_n(\mathbb{C}^\infty))/(c_n(\gamma_{\mathbb{C}}^{n,\infty})) \cong H^*(Gr_{n-1}(\mathbb{C}^\infty)) = R[c_1(\gamma_{\mathbb{C}}^{n-1,\infty}), \ldots, c_{n-1}(\gamma_{\mathbb{C}}^{n-1,\infty})].$$

This implies the claim, by comparing with the exact sequence

$$0 \longrightarrow R[c_1, \ldots, c_n] \xrightarrow{c_n \cdot -} R[c_1, \ldots, c_n] \longrightarrow R[c_1, \ldots, c_{n-1}] \longrightarrow 0$$

and using the 5-lemma. □

There is not such a pleasant formula for the integral cohomology of $Gr_n(\mathbb{R}^\infty)$, but we can understand it away from the prime 2 by introducing the *oriented Grassmannian*

$$\widetilde{Gr}_n(\mathbb{R}^N) = \{(V,\omega) \mid V \in Gr_n(\mathbb{R}^N), \omega \text{ is an orientation of } V\}.$$

Forgetting the orientation gives a 2-to-1 covering map $p : \widetilde{Gr}_n(\mathbb{R}^N) \to Gr_n(\mathbb{R}^N)$. We may think of an orientation of $V$ as a choice of unit vector in the 1-dimensional inner product space $\Lambda^n V$, which identifies $\widetilde{Gr}_n(\mathbb{R}^N)$ with $\mathbb{S}(\Lambda^n \gamma_{\mathbb{R}}^{n,N})$.

Writing $\tilde{\gamma}_{\mathbb{R}}^{n,N} := p^*(\gamma_{\mathbb{R}}^{n,N})$ for the pullback of the tautological bundle, the advantage is that this vector bundle is canonically oriented. Thus we have a Gysin sequence for $\widetilde{Gr}_{n-1}(\mathbb{R}^\infty) \simeq \mathbb{S}(\tilde{\gamma}_{\mathbb{R}}^{n,\infty}) \xrightarrow{p} \widetilde{Gr}_n(\mathbb{R}^\infty)$ and so can try to proceed as in the argument above.

**Theorem 2.11.3.** *If 2 is invertible in $R$ and $k > 0$ then the maps*

$$R[e, p_1, \ldots, p_{k-1}] \longrightarrow H^*(\widetilde{Gr}_{2k}(\mathbb{R}^\infty); R)$$
$$R[p_1, \ldots, p_{k-1}] \longrightarrow H^*(\widetilde{Gr}_{2k-1}(\mathbb{R}^\infty); R),$$

*sending $e$ and $p_i$ to the Euler and Pontrjagin classes of $\tilde{\gamma}_{\mathbb{R}}^{n,\infty}$, and the map*

$$\mathbb{F}_2[w_2, \ldots, w_n] \longrightarrow H^*(\widetilde{Gr}_n(\mathbb{R}^\infty); \mathbb{F}_2)$$

*sending $w_i$ to the Stiefel–Whitney class of $\tilde{\gamma}_{\mathbb{R}}^{n,\infty}$, are isomorphisms.*

In the first case the class $p_k(\tilde{\gamma}_{\mathbb{R}}^{2k,\infty})$ is also defined, but on Example Sheet 1 Q5 (ii) you will show that it is equal to $e(\tilde{\gamma}_{\mathbb{R}}^{2k,\infty})^2$.

*Proof.* For the third case we proceed exactly as in Theorem 2.11.2, but starting the induction with $\widetilde{Gr}_1(\mathbb{R}^\infty) = S^\infty \simeq *$.

For the first two cases we use the Gysin sequences to proceed by induction on $n$ (i.e. $2k$ or $2k-1$); the case $n = 1$ is immediate as above. If $n = 2k$ then, as $p^*$ is surjective on $R$-cohomology by induction, the Gysin sequence for $p$ gives short exact sequences

$$0 \longrightarrow H^i(\widetilde{Gr}_{2k}(\mathbb{R}^\infty); R) \xrightarrow{e(\tilde{\gamma}_{\mathbb{R}}^{2k,\infty}) \smile -} H^{i+2k}(\widetilde{Gr}_{2k}(\mathbb{R}^\infty); R) \xrightarrow{p^*} H^{i+2k}(\widetilde{Gr}_{2k-1}(\mathbb{R}^\infty); R) \longrightarrow 0$$

and we finish the argument just like in Theorem 2.11.2. If $n = 2k-1$ then by Example Sheet 1 Q2 we have $2e(\tilde{\gamma}_{\mathbb{R}}^{2k-1,\infty}) = 0$ and so $e(\tilde{\gamma}_{\mathbb{R}}^{2k-1,\infty}) = 0$ by our assumption that 2 is invertible in $R$. In this case the Gysin sequence for $p$ gives short exact sequences

$$0 \longrightarrow H^i(\widetilde{Gr}_{2k-1}(\mathbb{R}^\infty); R) \xrightarrow{p^*} H^i(\widetilde{Gr}_{2k-2}(\mathbb{R}^\infty); R) \xrightarrow{p_!} H^{i-(2k-2)}(\widetilde{Gr}_{2k-1}(\mathbb{R}^\infty); R) \longrightarrow 0.$$

By our observation about the top Pontrjagin class being the square of the Euler class, and induction, the composition

$$R[p_1, \ldots, p_{k-1}] \longrightarrow H^*(\widetilde{Gr}_{2k-1}(\mathbb{R}^\infty); R) \xrightarrow{p^*} H^*(\widetilde{Gr}_{2k-2}(\mathbb{R}^\infty); R)$$

is split injective (with image the subring $R[p_1, \ldots, p_{k-2}, e^2] \subset R[e, p_1, \ldots, p_{k-2}]$). The claim then follows from the Gysin sequence above by counting ranks. $\square$

To make a final conclusion about the unoriented real Grassmannians $Gr_n(\mathbb{R}^\infty)$ we will need to use a general tool.

**Lemma 2.11.4.** *If $\pi : \widetilde{X} \to X$ is a covering space with fibres of finite cardinality $m$, and $R$ is a ring in which $m$ is invertible, then $\pi^* : H^*(X; R) \to H^*(\widetilde{X}; R)$ is split injective.*

*Proof.* The construction

$$\begin{aligned} \tau : C_*(X;\mathbb{Z}) &\longrightarrow C_*(\widetilde{X};\mathbb{Z}) \\ \sigma : \Delta^p \to X &\longmapsto \sum_{\text{lifts } \tilde{\sigma} \text{ of } \sigma} \tilde{\sigma} : \Delta^p \to \widetilde{X} \end{aligned}$$

defines a chain map, and $\pi_\# \circ \tau$ is multiplication by $m$ as $p \circ \tilde{\sigma} = \sigma$ and there are $m$ lifts. Applying $Hom_{\mathbb{Z}}(-, R)$ gives a chain map $\tau^\vee : C^*(\widetilde{X}; R) \to C^*(X; R)$ satisfying $\tau^\vee \pi^*([x]) = m \cdot [x]$ on cohomology, and as $m \in R$ is assumed to be invertible the claim follows. ☐

**Corollary 2.11.5.** *If $2$ is invertible in $R$ and $k > 0$ then the maps*

$$\begin{aligned} R[p_1, \ldots, p_k] &\longrightarrow H^*(Gr_{2k}(\mathbb{R}^\infty); R) \\ R[p_1, \ldots, p_{k-1}] &\longrightarrow H^*(Gr_{2k-1}(\mathbb{R}^\infty); R), \end{aligned}$$

*sending $p_i$ to the Pontrjagin classes of $\gamma_{\mathbb{R}}^{n,\infty}$, are isomorphisms.*

*Proof.* We apply the lemma to the 2-to-1 covering space $p : \widetilde{Gr}_n(\mathbb{R}^\infty) \to Gr_n(\mathbb{R}^\infty)$, so as $2 \in R$ is invertible we find that the map

$$p^* : H^*(Gr_n(\mathbb{R}^\infty); R) \longrightarrow H^*(\widetilde{Gr}_n(\mathbb{R}^\infty); R) = \begin{cases} R[e, p_1, \ldots, p_{k-1}] & n = 2k \\ R[p_1, \ldots, p_{k-1}] & n = 2k - 1 \end{cases}$$

is (split) injective.

Reversing the orientation gives an involution $t : \widetilde{Gr}_n(\mathbb{R}^N) \to \widetilde{Gr}_n(\mathbb{R}^N)$ which commutes with $p$, and pulls back $\tilde{\gamma}_{\mathbb{R}}^{n,\infty}$ to the same vector bundle but with the opposite orientation. Thus $t^* p_i = p_i$ and $t^* e = -e$. As $t$ commutes with $p$, the map $p^*$ lands in the $t$-invariant subring. This is the whole ring in the second case, and the subring $R[e^2, p_1, \ldots, p_{k-1}] = R[p_1, \ldots, p_{k-1}, p_k]$ in the first case. As the Pontrjagin classes certainly are all pulled back along $p$, it follows that $p^*$ is an isomorphism onto this subring. ☐

# Chapter 3

# $K$-theory

Lecture 9

## 3.1 The functor $K$

For a compact Hausdorff space $X$, we let $Vect(X)$ denote the set of isomorphism classes of finite-dimensional complex vector bundles $\pi : E \to X$.

*In distinction with Definition 1.1.1 we only ask for the dimension of a vector bundle to be locally constant, not globally constant: thus a vector bundle may have different dimensions over different connected components of $X$.*

Whitney sum $\oplus$ and the zero-dimensional vector bundle $\underline{\mathbb{C}^0}_X$ make $(Vect(X), \oplus, \underline{\mathbb{C}^0}_X)$ into an *abelian monoid*, i.e.

(i) $\oplus$ is associative and commutative, and

(ii) $\underline{\mathbb{C}^0}_X$ is a unit element for $\oplus$.

**Definition 3.1.1.** We let $K^0(X)$ be the *Grothendieck completion* of the abelian monoid $(Vect(X), \oplus, \underline{\mathbb{C}^0}_X)$.

That is

$$K^0(X) = Vect(X) \times Vect(X)/\sim$$

where $([E_0], [F_0]) \sim ([E_1], [F_1])$ if and only if there exists a $[C] \in Vect(X)$ such that $[E_0 \oplus F_1 \oplus C] = [E_1 \oplus F_0 \oplus C]$. This has a sum operation defined by

$$[([E_0], [F_0])] + [([E_1], [F_1])] := [([E_0 \oplus E_1], [F_0 \oplus F_1])],$$

which is well-defined with respect to $\sim$, and $[([\underline{\mathbb{C}^0}_X], [\underline{\mathbb{C}^0}_X])]$ is a unit for this sum operation. This is an abelian group, with $[([F_0], [E_0])]$ inverse to $[([E_0], [F_0])]$.

For ease of notation, we usually write

$$\begin{aligned} E - F &:= [([E], [F])] \\ 0 &:= [([\underline{\mathbb{C}^0}_X], [\underline{\mathbb{C}^0}_X])]. \end{aligned}$$

**Example 3.1.2.** A vector bundle over a point $*$ is determined up to isomorphism by its dimension, so $(Vect(*), \oplus, \underline{\mathbb{C}^0}_*) \cong (\mathbb{N}, +, 0)$ as an abelian monoid. Thus $K^0(*) \cong \mathbb{Z}$ as an abelian group, as the Grothendieck completion is precisely the usual construction of the integers from the natural numbers.

Explicitly, the isomorphism is given by $E - F \in K^0(*) \mapsto \dim(E) - \dim(F) \in \mathbb{Z}$.

By construction $[E] \mapsto ([E], [\underline{\mathbb{C}^0}_X]) : Vect(X) \to K^0(X)$ is a homomorphism of abelian monoids. It has the following universal property among homomorphisms to abelian groups.

**Lemma 3.1.3.** *If $(A, +, 0)$ is an abelian group, then any homomorphism*

$$\phi : (Vect(X), \oplus, \underline{\mathbb{C}^0}_X) \longrightarrow (A, +, 0)$$

*of abelian monoids extends uniquely to a homomorphism*

$$\hat{\phi} : (K^0(X), +, 0) \longrightarrow (A, +, 0)$$

*of abelian groups.*

*Proof.* If $[E_0 \oplus F_1 \oplus C] = [E_1 \oplus F_0 \oplus C]$ then

$$\phi([E_0]) + \phi([F_1]) + \phi([C]) = \phi([E_1]) + \phi([F_0]) + \phi([C]) \in A$$

and we can cancel the $\phi([C])$'s *as $A$ is a group.* Hence

$$\phi([E_0]) - \phi([F_0]) = \phi([E_1]) - \phi([F_1]) \in A$$

and so $\hat{\phi}([([E_0], [F_0])]) := \phi([E_0]) - \phi([F_0])$ is well defined. This is a homomorphism of groups. □

If $(X, x_0)$ is a based space, then there is a homomorphism of abelian monoids

$$\begin{aligned} rk_{x_0} : Vect(X) &\longrightarrow \mathbb{Z} \\ [E] &\longmapsto \dim(E_{x_0}) \end{aligned}$$

which, as the target is an abelian group, extends to a homomorphism

$$rk_{x_0} : K^0(X) \longrightarrow \mathbb{Z},$$

which is split via the homomorphism $n \mapsto \mathrm{sign}(n)\underline{\mathbb{C}^{|n|}}_X$.

**Definition 3.1.4.** The *reduced $K$-theory* of $(X, x_0)$ is

$$\tilde{K}^0(X) := \mathrm{Ker}(rk_{x_0} : K^0(X) \to \mathbb{Z}),$$

so the splitting gives a canonical isomorphism $K^0(X) \cong \mathbb{Z} \oplus \tilde{K}^0(X)$.

For example, by Example 3.1.2 we have $\tilde{K}^0(*) = 0$.

**Lemma 3.1.5.** *If $X$ is compact Hausdorff, then every element of $K^0(X)$ may be written as $E - \underline{\mathbb{C}^N}_X$ for some vector bundle $E \to X$ and some $N$.*

*Proof.* An element of $K^0(X)$ is of the form $[([E], [F])]$. By Lemma 1.3.1, as $X$ is compact Hausdorff there is a vector bundle $F'$ so that $F \oplus F' \cong \underline{\mathbb{C}^N}_X$ for some $N \gg 0$. Then

$$([\underline{\mathbb{C}}^0_X], [F]) \sim ([F'], [\underline{\mathbb{C}^N}_X]),$$

so adding $([E], [\underline{\mathbb{C}}^0_X])$ to both sides shows that

$$([E], [F]) \sim ([E \oplus F'], [\underline{\mathbb{C}^N}_X]),$$

as required. □

It follows from this lemma that we may give another description of $\tilde{K}^0(X)$ when $X$ is compact Hausdorff, namely

$$\tilde{K}^0(X) \cong Vect(X)/\approx$$

where $[E_0] \approx [E_1]$ if and only if $[E_0 \oplus \underline{\mathbb{C}^N}_X] = [E_1 \oplus \underline{\mathbb{C}^M}_X]$ for some $N, M \in \mathbb{N}$. The isomorphism is given by the function

$$[E] \in Vect(X)/\approx \longmapsto E - \underline{\mathbb{C}^{\dim(E_{x_0})}}_X \in \tilde{K}^0(X)$$

This is easily seen to be well-defined, is surjective by the previous lemma, and if

$$E - \underline{\mathbb{C}^{\dim(E_{x_0})}}_X = E' - \underline{\mathbb{C}^{\dim(E'_{x_0})}}_X \in \tilde{K}^0(X)$$

then

$$[E \oplus \underline{\mathbb{C}^{\dim(E'_{x_0})}}_X \oplus C] = [E' \oplus \underline{\mathbb{C}^{\dim(E_{x_0})}}_X \oplus C] \in Vect(X)$$

for some $C$, and further adding on a $C'$ such that $C \oplus C' \cong \underline{\mathbb{C}^L}_X$, which uses again that $X$ is compact Hausdorff, gives that $[E] \approx [E']$.

So for compact Hausdorff spaces we can also think of reduced $K$-theory as being given by vector bundles up to such *stable isomorphism*.

**Corollary 3.1.6.** $\tilde{K}^0(S^1) = 0$

*Proof.* As $S^1 = \Sigma S^0$, by the clutching construction there is a bijection between isomorphism classes of $n$-dimensional complex vector bundles over $S^1$ and homotopy classes of maps $S^0 \to GL_n(\mathbb{C})$. As $GL_n(\mathbb{C})$ is path-connected, all such maps are homotopic, so all $n$-dimensional complex vector bundles over $S^1$ are isomorphic: that is, they are all trivial. The claim now follows by the above description of reduced $K$-theory. □

### 3.1.1 Functoriality

If $f : X \to Y$ is a continuous map, there is an induced map of abelian monoids $f^* : Vect(Y) \to Vect(X)$ given by $f^*([E]) = [f^*E]$. This extends to the Grothendieck completion, to give a homomorphism of abelian groups

$$\begin{aligned} f^* : K^0(Y) &\longrightarrow K^0(X) \\ E - F &\longmapsto f^*(E) - f^*(F). \end{aligned}$$

If $f : (X, x_0) \to (Y, y_0)$ is a based map it induces a map on reduced $K$-theory by the same formula.

**Lemma 3.1.7.** *If $X$ is compact Hausdorff and $f, g : X \to Y$ are homotopic maps, then*

$$f^* = g^* : K^0(Y) \longrightarrow K^0(X).$$

*Proof.* It is enough to show that $f^*, g^* : Vect(Y) \to Vect(X)$ are equal. This is immediate from Corollary 1.4.2. □

Lecture 10

### 3.1.2 Ring structure

The tensor product of vector bundles induces a multiplication on $K$-theory. More precisely, tensor product gives a function

$$\begin{aligned} - \otimes - : Vect(X) \times Vect(X) &\longrightarrow Vect(X) \\ ([E],[F]) &\longmapsto [E \otimes F] \end{aligned}$$

which is additive in each variable, and the Grothendieck completion promotes this to a bilinear map

$$\begin{aligned} - \otimes - : K^0(X) \times K^0(X) &\longrightarrow K^0(X) \\ (E_0 - F_0, E_1 - F_1) &\longmapsto E_0 \otimes E_1 - E_0 \otimes F_1 - E_1 \otimes F_0 + F_0 \otimes F_1. \end{aligned}$$

This is associative, as tensor product of vector bundles is, and has unit $\underline{\mathbb{C}^1}_X$. It makes $K^0(X)$ into a unital commutative ring, and if $f : X \to Y$ is a map then

$$f^* : K^0(Y) \to K^0(X)$$

is a homomorphism of unital rings. To ease notation we write

$$1 := \underline{\mathbb{C}^1}_X \in K^0(X)$$

and so for $n \in \mathbb{Z}$ we write $n := \mathrm{sign}(n)\underline{\mathbb{C}^{|n|}}_X$.

**Remark 3.1.8.** If $(X, x_0)$ is a based space with $i : \{*\} \to X$ the map $i(*) = x_0$, then the rank homomorphism $rk_{x_0} : K^0(X) \to \mathbb{Z}$ can be identified with

$$i^* : K^0(X) \longrightarrow K^0(*)$$

under the isomorphism $K^0(*) \cong \mathbb{Z}$ of Example 3.1.2. Thus the reduced $K$-theory

$$\tilde{K}^0(X) = \mathrm{Ker}(i^* : K^0(X) \to K^0(*))$$

is an *ideal* of $K^0(X)$. In particular, it still has a multiplication, but no longer a unit.

As usual there is also an external product given by

$$\begin{aligned} - \boxtimes - : K^0(X) \times K^0(Y) &\longrightarrow K^0(X \times Y) \\ A \otimes B &\longmapsto \pi_X^*(A) \otimes \pi_Y^*(B). \end{aligned}$$

## 3.2 The fundamental product theorem

Let us write $H = [\overline{\gamma_{\mathbb{C}}^{1,2}}] \in K^0(\mathbb{CP}^1)$ for the $K$-theory class of the complex conjugate (eqivalently, the dual) of the tautological line bundle.[1]

[1] One might think it more natural to just take $[\gamma_{\mathbb{C}}^{1,2}]$, as I once did, but $[\overline{\gamma_{\mathbb{C}}^{1,2}}]$ is more natural from the point of view of Algebraic Geometry, and will make sure that our formulas match everybody else's.

**Lemma 3.2.1.** *We have* $H + H = H^2 + 1 \in K^0(\mathbb{CP}^1)$*, or in other words* $(H-1)^2 = 0$.

*Proof.* We identify $\mathbb{CP}^1 = S^2$, and will use the clutching construction, in particular the bijection

$$\{\text{maps } \phi : S^1 \to GL_n(\mathbb{C})\}/\text{homotopy} \longrightarrow \{n\text{-dim vector bundles over } S^2\}/\text{isomorphism}$$
$$f \longmapsto (\pi_f : E_f \to S^2)$$

It is easy to see from the clutching construction that: Whitney sum of vector bundles (on the right) corresponds to block-sum of matrices (on the left); tensor product of vector bundles (on the right) corresponds to Kronecker product of matrices (on the left).

Considering the clutching construction for $\gamma_{\mathbb{C}}^{1,2} \to \mathbb{CP}^1$, we express $\mathbb{CP}^1$ as the suspension of $S^1$ via the sets

$$C_0 := \{[1 : z] \mid |z| \leq 1\}$$
$$C_1 := \{[w : 1] \mid |w| \leq 1\}$$

whose intersection is identified via $z \mapsto [1 : z] = [\frac{1}{z} : 1]$ with the unit complex numbers $S^1$, and which are both discs and so cones on this circle. The maps

$$\varphi_{C_0} : C_0 \times \mathbb{C} \longrightarrow \gamma_{\mathbb{C}}^{1,2}|_{C_0}$$
$$([1 : z], \lambda) \longmapsto ([1 : z], \lambda \cdot (1, z))$$

$$\varphi_{C_1} : C_1 \times \mathbb{C} \longrightarrow \gamma_{\mathbb{C}}^{1,2}|_{C_1}$$
$$([w : 1], \lambda) \longmapsto ([w : 1], \lambda \cdot (w, 1))$$

are local trivialisations, and they determine the corresponding map

$$f : S^1 \cong C_0 \cap C_1 \longrightarrow GL_1(\mathbb{C}) = \mathbb{C}^\times$$
$$z \mapsto [1 : z] \longmapsto z.$$

Thus the bundle $\gamma_{\mathbb{C}}^{1,2} \oplus \gamma_{\mathbb{C}}^{1,2}$ corresponds to the map

$$f' : S^1 \longrightarrow GL_2(\mathbb{C})$$
$$z \longmapsto \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix},$$

and the bundle $(\gamma_{\mathbb{C}}^{1,2} \otimes \gamma_{\mathbb{C}}^{1,2}) \oplus \underline{\mathbb{C}}_{\mathbb{CP}^1}$ corresponds to the map

$$f'' : S^1 \longrightarrow GL_2(\mathbb{C})$$
$$z \longmapsto \begin{bmatrix} z^2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now

$$\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} = \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} z^2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and as $GL_2(\mathbb{C})$ is path connected there is a path from $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to the identity matrix. This gives a homotopy from $f'$ to $f''$, so $\gamma_{\mathbb{C}}^{1,2} \oplus \gamma_{\mathbb{C}}^{1,2}$ and $(\gamma_{\mathbb{C}}^{1,2} \otimes \gamma_{\mathbb{C}}^{1,2}) \oplus \underline{\mathbb{C}}_{\mathbb{CP}^1}$ are isomorphic: thus their complex conjugates are too. □

This lemma defines a ring homomorphism

$$\phi : \mathbb{Z}[H]/((H-1)^2) \longrightarrow K^0(\mathbb{CP}^1).$$

**Theorem 3.2.2** (Fundamental Product Theorem)**.** *If $X$ is compact Hausdorff then the ring homomorphism*

$$\begin{aligned} \mu : K^0(X) \otimes \mathbb{Z}[H]/((H-1)^2) &\longrightarrow K^0(X \times \mathbb{CP}^1) \\ x \otimes y &\longmapsto \pi_1^*(x) \otimes \pi_2^*(\phi(y)) \end{aligned}$$

*is an isomorphism.*

**Corollary 3.2.3.** *Taking $X$ to be a point it follows that*

$$\phi : \mathbb{Z}[H]/((H-1)^2) \longrightarrow K^0(\mathbb{CP}^1)$$

*is an isomorphism.*

We defer the proof of the Fundamental Product Theorem to Chapter 4, as it is extensive and will take several lectures.

## 3.3 Bott periodicity and the cohomological structure of $K$-theory

It turns out that it is conceptually simpler to develop the cohomological structure for reduced $K$-theory and pointed spaces. We will do so, and deduce the consequences for unreduced $K$-theory and unpointed spaces at the end.

### 3.3.1 Beginning the long exact sequence of a pair

**Lemma 3.3.1.** *Let $E \to X$ be a vector bundle over a compact Hausdorff space, and $A \subset X$ be a closed set such that $E|_A \to A$ is trivial. Then there is an open neighbourhood $U \supset A$ such that $E|_U \to U$ is trivial.*

*Proof.* Choose a trivialisation of $E|_A \to A$, which is equivalent to choosing sections $s_1, \ldots, s_n : A \to E|_A$ such that the $s_i(a)$ are a basis for $E_a$ for each $a \in A$. Let $U_1, \ldots, U_r$ be open subsets in $X$ over which the bundle $E$ is trivial, and which cover $X$. Choosing a trivialisation of $E|_{U_j}$, one gets maps

$$\begin{aligned} s_i|_{U_j \cap A} : U_j \cap A &\longrightarrow E|_{U_j \cap A} \cong (U_j \cap A) \times \mathbb{C}^n \\ a &\longmapsto (a, \Sigma_{i,j}(a)) \end{aligned}$$

and by the Tietze extension theorem[2] there are extensions of of the $\Sigma_{i,j}$ to maps $\Sigma'_{i,j}$ : $U_j \to \mathbb{C}^n$. Under the trivialisations $E|_{U_j} \cong U_j \times \mathbb{C}^n$ these define sections

$$s_{i,j} : U_j \longrightarrow E|_{U_j}.$$

Using a partition of unity $\{\varphi_j\}$ subordinate to the cover $\{U_j\}$, we can form sections

$$s'_i = \sum \varphi_j \cdot s_{i,j} : X \longrightarrow E$$

which agree with the $s_i$ over $A$. As the sections $s'_i$ are linearly independent over each point of $A$, they are linearly independent over some open neighbourhood $U$ of $A$, where they define a trivialisation of $E|_U \to U$. □

**Proposition 3.3.2.** *If $X$ is a compact Hausdorff space, $A \subset X$ is a closed subspace, and $* \in A$ is a basepoint, then the based maps*

$$(A, *) \xrightarrow{i} (X, *) \xrightarrow{q} (X/A, A/A)$$

*induce homomorphisms*

$$\tilde{K}^0(A) \xleftarrow{i^*} \tilde{K}^0(X) \xleftarrow{q^*} \tilde{K}^0(X/A)$$

*which are exact at $\tilde{K}^0(X)$.*

Note that $A$, as a closed subspace of a compact Hausdorff space, is again compact
Lecture 11 Hausdorff; $X/A$ is too, by an elementary argument.

*Proof.* The composition $i^*q^*$ is $(q \circ i)^*$, and $q \circ i$ is the constant map to the basepoint. Since $\tilde{K}(*) = 0$, it follows that $i^*q^* = 0$.

For the converse, we use the description $\tilde{K}^0(X) \cong Vect(X)/\approx$, vector bundles modulo stable isomorphism. Let $E \to X$ be a vector bundle so that $i^*(E) = E|_A \to A$ is stably trivial. After adding on a trivial bundle to $E$, we may therefore suppose that $E|_A \to A$ is trivial, and choose a trivialisation $h : E|_A \xrightarrow{\sim} A \times \mathbb{C}^n$. Let

$$E/h = E/h^{-1}(a, v) \sim h^{-1}(a', v),$$

which has an induced projection $E/h \to X/A$.

We claim this is locally trivial. Over the open set $(X/A) \setminus (A/A) \cong X \setminus A$ it is identified with $E|_{X\setminus A} \to X \setminus A$ so is locally trivial. By the previous lemma it follows that there is an open neighbourhood $U \supset A$ such that $E|_U \to U$ is trivial. Restricted to $U/A \subset X/A$ we can identify, via a trivialisation extending $h$,

$$(E/h)|_{U/A} \cong (U/A) \times \mathbb{C}^n$$

[2] A continuous real-valued function defined on a closed subset of a normal space extends to the entire space. Compact Hausdorff spaces are normal.

so it is trivial, and so in particular locally trivial. Thus $E/h \to X/A$ is a vector bundle. The square

$$\begin{array}{ccc} E & \xrightarrow{\text{quotient}} & E/h \\ \downarrow & & \downarrow \\ X & \xrightarrow{q} & X/A \end{array}$$

is a pullback, which identifies $E = q^*(E/h)$. Hence $\mathrm{Ker}(i^*) \subset \mathrm{Im}(q^*)$ as required. □

We want to extend this to a long exact sequence, analogous to the long exact sequence of a pair for cohomology.

**Definition 3.3.3.** Let $f : X \to Y$ be a map. The *mapping cylinder* of $f$ is

$$M_f := ((X \times [0,1]) \sqcup Y)/(x,1) \sim f(x).$$

There are inclusions

$$\begin{aligned} i : X &\longrightarrow M_f \\ x &\longmapsto [(x,0)] \end{aligned}$$

$$\begin{aligned} j : Y &\longrightarrow M_f \\ y &\longmapsto [y] \end{aligned}$$

and a (deformation) retraction

$$\begin{aligned} r : M_f &\longrightarrow Y \\ [x,t] &\longmapsto f(x) \\ [y] &\longmapsto y. \end{aligned}$$

This gives a factorisation

$$f : X \xrightarrow{i} M_f \underset{\sim}{\xrightarrow{r}} Y,$$

where $r$ is a homotopy equivalence.

The *mapping cone* of $f$ is $C_f = M_f/i(X)$. We consider $C_f$ as a based space with basepoint $[i(X)]$. There is a map

$$\begin{aligned} c : C_f &\longrightarrow Y/f(X) \\ [x,t] &\longmapsto f(X)/f(X) \\ [y] &\longmapsto [y]. \end{aligned}$$

The point $f(X)/f(X)$ serves as a basepoint.

If $X$ and $Y$ are compact Hausdorff, so are $M_f$ and $C_f$.

**Lemma 3.3.4.** *Let $X$ be a compact Hausdorff space, $A \subset X$ be a contractible closed subspace, and $* \in A$ a basepoint. Then the collapse map $c : X \to X/A$ induces an isomorphism on $\tilde{K}^0$.*

*Proof.* If $E \to X$ is a vector bundle then as $A$ is contractible, compact, and Hausdorff by Corollary 1.4.3 the restriction $E|_A \to A$ must be trivial. Choosing a trivialisation $h : E|_A \to A \times \mathbb{C}^n$, there is therefore a vector bundle

$$E/h \to X/A$$

by the construction in the previous proposition, which satisfies $c^*(E/h) = E$. This shows that $\tilde{K}^0(c)$ is surjective. To show it is injective we must show that $E/h$ is independent of $h$ up to isomorphism.

If $h_0$ and $h_1$ are trivialisations, they differ by postcomposition with

$$\begin{aligned} h_0^{-1}h_1 : A \times \mathbb{C}^n &\longrightarrow A \times \mathbb{C}^n \\ (a, v) &\longmapsto (a, g(a)(v)) \end{aligned}$$

for a map $g : A \to GL_n(\mathbb{C})$. As $A$ is contractible, this map is homotopic to the constant map to the identity matrix. From such a homotopy we produce a homotopy $H$ of trivialisations over $A$, and hence a trivialisation of $E|_A \times [0,1] \to A \times [0,1]$ which is $h_0$ at one end and $h_1$ at the other. In the same way that we constructed $E/h$, we construct

$$(E \times [0,1])/H \to (X/A) \times [0,1]$$

which is $E/h_0$ at one end and $E/h_1$ at the other. By Lemma 1.4.1 these bundles are isomorphic. □

**Corollary 3.3.5.** *If $f : X \to Y$ is the inclusion of a closed subspace into a compact Hausdorff space and $* \in X$ is a basepoint, then the map $c : C_f \to Y/X$ induces an isomorphism on $\tilde{K}^0$.*

*Proof.* As $f$ is the inclusion of a closed subspace, the cone on $X$

$$C(X) := (X \times [0,1])/X \times \{0\}$$

is a closed subspace of $C_f$, and the map $c$ is given by collapsing $C(X)$ to a point. Furthermore, $C(X)$ contracts to its cone-point. Apply the previous lemma. □

Now given a compact Hausdorff space $X$ and a closed (and hence compact Hausdorff) subspace $A \subset X$ containing a basepoint $* \in A$, with $f : A \to X$ the inclusion map, we can form the following diagram:

$$\begin{array}{ccccccc} A & \xhookrightarrow{f} & X & \xhookrightarrow{j} & C_f & \xhookrightarrow{k} & C_j \\ & & & \searrow^{q} & \downarrow{\scriptstyle c} & \searrow^{q'} & \downarrow{\scriptstyle c'} \\ & & & & X/A & & C_f/X \end{array}$$

and notice that $C_f/X \cong \Sigma A$. Using this we can join the exact sequence of Proposition 3.3.2 with the exact sequence

$$\tilde{K}^0(X) \xleftarrow{q^*} \tilde{K}^0(X/A) \cong \tilde{K}^0(C_f) \xleftarrow{(q')^*} \tilde{K}^0(\Sigma A)$$

to get a sequence

$$\tilde{K}^0(A) \xleftarrow{i^*} \tilde{K}^0(X) \xleftarrow{q^*} \tilde{K}^0(X/A) \xleftarrow{\partial} \tilde{K}^0(\Sigma A)$$

which is exact at the two middle positions. Here we write

$$\partial : \tilde{K}^0(\Sigma A) \cong \tilde{K}^0(C_f/X) \xrightarrow{(q')^*} \tilde{K}^0(C_f) \underset{\sim}{\xleftarrow{c^*}} \tilde{K}^0(X/A).$$

Continuing in this way, we have $C_k \cong \Sigma X$ and so on, giving a sequence

$$\tilde{K}^0(A) \xleftarrow{i^*} \tilde{K}^0(X) \xleftarrow{q^*} \tilde{K}^0(X/A) \xleftarrow{\partial} \tilde{K}^0(\Sigma A) \xleftarrow{(\Sigma i)^*} \tilde{K}^0(\Sigma X) \xleftarrow{(\Sigma q)^*} \tilde{K}^0(\Sigma X/A) \xleftarrow{\Sigma\partial} \cdots$$

which is exact at every position except perhaps the leftmost. Recall that cohomology satisfies $\tilde{H}^{i+1}(\Sigma X) \cong \tilde{H}^i(X)$, so by analogy we define

$$\tilde{K}^{-i}(X) := \tilde{K}^0(\Sigma^i X) \quad \text{for } i \geq 0.$$

Then this sequence may be written as

$$\tilde{K}^0(A) \xleftarrow{i^*} \tilde{K}^0(X) \xleftarrow{q^*} \tilde{K}^0(X/A) \xleftarrow{\partial} \tilde{K}^{-1}(A) \xleftarrow{i^*} \tilde{K}^{-1}(X) \xleftarrow{q^*} \tilde{K}^{-1}(X/A) \xleftarrow{\partial} \cdots.$$

In order to deal with the failure of exactness at the left-hand end, we must define $\tilde{K}^i(X)$ for $i > 0$, and extend this sequence to the left.

### 3.3.2 The external product on reduced $K$-theory

We have defined $\tilde{K}^0(X)$ as the subgroup of $K^0(X)$ of those virtual vector bundles whose virtual dimension at $x_0$ is zero. We may therefore restrict the external product on $K^0$ to reduced $K$-theory, giving

$$\tilde{K}^0(X) \otimes \tilde{K}^0(X) \longrightarrow \tilde{K}^0(X \times Y).$$

However, for pointed spaces the cartesian product $\times$ is less appropriate than the *smash product*

$$X \wedge Y = (X \times Y)/(X \vee Y).$$

We want to explain how our discussion so far can be used to improve the external product above to a product

$$\tilde{K}^0(X) \otimes \tilde{K}^0(X) \longrightarrow \tilde{K}^0(X \wedge Y).$$

To do so, we look at the sequence Lecture 12

$$\tilde{K}^0(X \vee Y) \xleftarrow{i^*} \tilde{K}^0(X \times Y) \xleftarrow{q^*} \tilde{K}^0(X \wedge Y) \xleftarrow{\partial} \tilde{K}^{-1}(X \vee Y) \xleftarrow{i^*} \tilde{K}^{-1}(X \times Y) \longleftarrow \cdots$$

which is exact except perhaps at the left.

**Lemma 3.3.6.** *The inclusions $i_X : X \to X \vee Y$ and $i_Y : Y \to X \vee Y$ induce an isomorphism*

$$i_X^* \oplus i_Y^* : \tilde{K}^{-i}(X \vee Y) \longrightarrow \tilde{K}^{-i}(X) \oplus \tilde{K}^{-i}(Y)$$

*for all $i \geq 0$.*

*Proof.* As $(X \vee Y)/X = Y$, we have a sequence

$$\tilde{K}^0(X) \xleftarrow{i_X^*} \tilde{K}^0(X \vee Y) \xleftarrow{r_Y^*} \tilde{K}^0(Y) \xleftarrow{\partial} K^{-1}(X) \xleftarrow{i_X^*} \tilde{K}^{-1}(X \vee Y) \longleftarrow \cdots$$

exact except perhaps at the left. Here $r_Y : X \vee Y \to Y$ collapses $X$ to a point, and $r_X : X \vee Y \to X$ collapses $Y$. Now $r_X \circ i_X = Id_X$ shows that $i_X^*$ is surjective, so this sequence is actually exact and is in fact a collection of short exact sequences

$$0 \longleftarrow \tilde{K}^{-i}(X) \xleftarrow{i_X^*} \tilde{K}^{-i}(X \vee Y) \xleftarrow{r_Y^*} \tilde{K}^{-i}(Y) \longleftarrow 0,$$

which are split by $i_Y^*$. □

Returning to the exact sequence above, we see that the map

$$i^* : \tilde{K}^{-i}(X \times Y) \longrightarrow \tilde{K}^{-i}(X \vee Y) \cong \tilde{K}^{-i}(X) \oplus \tilde{K}^{-i}(Y)$$

is also surjective, and is split by $\pi_X^* \oplus \pi_Y^*$, the maps induced by projection to the factors. This splitting in particular gives a decomposition

$$\tilde{K}^0(X \times Y) \cong \tilde{K}^0(X \wedge Y) \oplus \tilde{K}^0(X) \oplus \tilde{K}^0(Y).$$

If $x \in \tilde{K}^0(X)$ and $y \in \tilde{K}^0(Y)$, then we have

$$\pi_X^*(x) \otimes \pi_Y^*(y) \in \tilde{K}^0(X \times Y).$$

This vanishes when restricted to $\{x_0\} \times Y$ or $X \times \{y_0\}$, i.e. vanishes under $i^*$, as then it is the tensor product of a $K$-theory class with the zero $K$-theory class. Thus $\pi_X^*(x) \otimes \pi_Y^*(y)$ lies in the canonical summand $\tilde{K}^0(X \wedge Y)$ of $\tilde{K}^0(X \times Y)$. This defines the external product

$$- \boxtimes - : \tilde{K}^0(X) \otimes \tilde{K}^0(Y) \longrightarrow \tilde{K}^0(X \wedge Y).$$

As usual, the internal product is obtained from the external product by pulling back along the diagonal map $\Delta : X \to X \wedge X$.

**Example 3.3.7.** If $A, B \subset X$ then there is a commutative square

$$\begin{array}{ccc} X & \xrightarrow{x \mapsto [x,x]} & X \wedge X \\ \downarrow & & \downarrow \\ X/(A \cup B) & \xrightarrow{[x] \mapsto [[x],[x]]} & (X/A) \wedge (X/B) \end{array}$$

where the vertical maps are the evident quotient maps. Thus if $A$ and $B$ are closed subsets which cover $X$ and are contractible, then we find that the internal product

$$- \otimes - : \tilde{K}^0(X) \otimes \tilde{K}^0(X) \longrightarrow \tilde{K}^0(X)$$

is the zero map (as $\tilde{K}^0(X/A) \to \tilde{K}^0(X)$ is an isomorphism, and similarly for $B$). (As an application, this gives another proof of the identity $(H-1)^2 = 0 \in K^0(\mathbb{CP}^1)$ from Lemma 3.2.1, because $H - 1$ lies in reduced $K$-theory.)

More generally, this argument shows that if $X$ can be covered by $n$ contractible closed sets, then all $n$-fold products in the ring $\tilde{K}^0(X)$ are trivial.

### 3.3.3 Bott periodicity

There is a map

$$\begin{aligned} c : \Sigma X &\longrightarrow S^1 \wedge X \\ [t, x] &\longmapsto [e^{2\pi i t}, x] \end{aligned}$$

which collapses $[0, 1] \times \{x_0\} \subset \Sigma X$. This is a contractible closed subspace, so by Lemma 3.3.4 the map

$$c^* : \tilde{K}^0(S^1 \wedge X) \longrightarrow \tilde{K}^0(\Sigma X)$$

is an isomorphism. We will therefore freely identify these groups. As $S^{n+1} = S^1 \wedge S^n$, and $\Sigma^{n+1}(-) = \Sigma(\Sigma^n(-))$, we can iterate.

As a consequence of the Fundamental Product Theorem we calculated

$$K^0(S^2) = \mathbb{Z}[H]/((H-1)^2) = \mathbb{Z}\{1\} \oplus \mathbb{Z}\{H-1\}$$

and so $\tilde{K}^0(S^2) = \mathbb{Z}\{H-1\}$. We may therefore form the map

$$\begin{aligned} \beta : \tilde{K}^0(X) &\longrightarrow \tilde{K}^0(S^2 \wedge X) \cong \tilde{K}^0(\Sigma^2 X) \\ x &\longmapsto (H-1) \boxtimes x, \end{aligned}$$

called the *Bott map.*

**Theorem 3.3.8.** *The Bott map $\beta : \tilde{K}^0(X) \to \tilde{K}^0(\Sigma^2 X)$ is an isomorphism for all compact Hausdorff spaces $X$.*

*Proof.* The Fundamental Product Theorem implies that the external product map

$$K^0(S^2) \otimes K^0(X) \longrightarrow K^0(S^2 \times X)$$

is an isomorphism. Writing $K^0(X) = \mathbb{Z} \oplus \tilde{K}^0(X)$, and similarly for $S^2$, we obtain an isomorphism

$$\tilde{K}^0(S^2) \oplus \tilde{K}^0(X) \oplus \Big(\tilde{K}^0(S^2) \otimes \tilde{K}^0(X)\Big) \longrightarrow \tilde{K}^0(S^2 \times X),$$

and comparing it with the decomposition of $\tilde{K}^0(S^2 \times X)$ produced above it shows that the external product map

$$\tilde{K}^0(S^2) \otimes \tilde{K}^0(X) \longrightarrow \tilde{K}^0(S^2 \wedge X)$$

is an isomorphism. With the identification $\tilde{K}^0(S^2) = \mathbb{Z}\{H-1\}$, this is the Bott map. □

### 3.3.4 Finishing the long exact sequence of a pair

The above gives, in the notation we have introduced, an isomorphism

$$\beta : \tilde{K}^0(X) \longrightarrow \tilde{K}^{-2}(X)$$

and so, replacing $X$ by $\Sigma^i X$, an isomorphism

$$\beta : \tilde{K}^{-i}(X) \longrightarrow \tilde{K}^{-i-2}(X)$$

for all $i \geq 0$. Thus we have actually only defined two distinct $K$-groups, $\tilde{K}^0$ and $\tilde{K}^{-1}$, and so we re-define

$$\tilde{K}^i(X) := \begin{cases} \tilde{K}^0(X) & \text{if } i \in \mathbb{Z} \text{ is even} \\ \tilde{K}^0(\Sigma X) & \text{if } i \in \mathbb{Z} \text{ is odd.} \end{cases}$$

Using Bott Periodicity we can immediately extend the half-exact sequence we developed in Section 3.3.1 to a long exact sequence, but by the periodicity we may roll this up into the following six-term exact sequence:

$$\begin{array}{ccccc} \tilde{K}^0(A) & \xleftarrow{i^*} & \tilde{K}^0(X) & \xleftarrow{q^*} & \tilde{K}^0(X/A) \\ \downarrow \partial & & & & \uparrow \partial \\ \tilde{K}^{-1}(X/A) & \xrightarrow{q^*} & \tilde{K}^{-1}(X) & \xrightarrow{i^*} & \tilde{K}^{-1}(A). \end{array}$$

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### 3.3.5 $K$-theory of spheres

Let us use Bott periodicity to calculate the $K$-theory of spheres, and to say something about maps between spheres.

**Corollary 3.3.9.** *We have*

$$\tilde{K}^i(S^{2n}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = -1 \end{cases}$$

*and*

$$\tilde{K}^i(S^{2n+1}) = \begin{cases} 0 & i = 0 \\ \mathbb{Z} & i = -1. \end{cases}$$

*Proof.* By Bott Periodicity (Theorem 3.3.8) it is enough to do this calculation for $S^0$ and $S^1$, and as $\tilde{K}^i(S^1) \cong K^{i-1}(S^0)$ it is enough to do it for $S^0$. We have $K^0(*) = \mathbb{Z}$ given by the dimension, so $K^0(S^0) = \mathbb{Z} \oplus \mathbb{Z}$, and hence $\tilde{K}^0(S^0) = \mathbb{Z}$. On the other hand $K^{-1}(S^0) = \tilde{K}^0(S^1) = 0$ by Corollary 3.1.6. □

**Example 3.3.10.** Let $f : S^1 \to S^1$ have degree $d$, and let us show that the map $f^* : \tilde{K}^{-1}(S^1) \to \tilde{K}^{-1}(S^1)$ is given by multiplication by $d$.

By definition, this is the map $(\Sigma f)^* : \tilde{K}^0(S^2) \to \tilde{K}^0(S^2)$, and $g := \Sigma f$ has degree $d$ too. We have $\tilde{K}^0(S^2) = \mathbb{Z}\{H - 1\}$, so suppose that $g^*(H-1) = k \cdot (H-1)$; we must show that $k = d$. This identity in $K$-theory, after taking complex conjugates, means that we have isomorphisms

$$g^*(\gamma_{\mathbb{C}}^{1,2}) \oplus \underline{\mathbb{C}}^N \cong (\gamma_{\mathbb{C}}^{1,2})^{\oplus k} \oplus \underline{\mathbb{C}}^{N+1-k}$$

for some $N \gg 0$. Taking total Chern classes we get

$$g^*(1+x) = (1+x)^k = 1 + kx \in H^*(S^2; \mathbb{Z})$$

so $g^*(x) = kx$ and as $x \neq 0$ and $g$ has degree $d$, it follows that $k = d$ as required.

It is a fact from Algebraic Topology that maps $h : S^p \to S^p$ are determined up to homotopy by their degree: so if $h$ has degree $d$ then it is homotopic to $\Sigma^{p-1} f$ for a degree $d$ map $f : S^1 \to S^1$ as above, and so $h^* : \tilde{K}^*(S^p) \to \tilde{K}^*(S^p)$ is give by multiplication by $d$ too. In Section 5.2 we will see how to deduce this without using the fact from Algebraic Topology.

### 3.3.6 The graded multiplication

If $X$ is a pointed space then the diagonal map $d : X \to X \wedge X$ induces for $i, j \geq 0$ maps

$$S^{i+j} \wedge d : S^{i+j} \wedge X \longrightarrow S^{i+j} \wedge X \wedge X \cong (S^i \wedge X) \wedge (S^j \wedge X)$$

and so maps

$$\tilde{K}^0(S^i \wedge X) \otimes \tilde{K}^0(S^j \wedge X) \xrightarrow{-\boxtimes-} \tilde{K}^0(S^i \wedge X \wedge S^j \wedge X) \xrightarrow{d^*} \tilde{K}^0(S^{i+j} \wedge X)$$

hence, using that $\Sigma X \to S^1 \wedge X$ is an isomorphism on $K$-theory, a map $- \otimes' - : \tilde{K}^{-i}(X) \otimes \tilde{K}^{-j}(X) \to \tilde{K}^{-i-j}(X)$. For future ease[3] we define the graded multiplication $- \otimes -$ to be $(-1)^{ij}$ times $- \otimes' -$. Using Bott Periodicity this extends to a multiplication defined for all $i, j \in \mathbb{Z}$. I did not say this in lectures!

**Remark 3.3.11.** One may show that this multiplication is graded-commutative (like the cup product), but we shall not do so.

### 3.3.7 Unpointed spaces and unreduced $K$-theory

If $X$ is a space we can make a based space $X_+ = X \sqcup \{*\}$. This satisfies $\tilde{K}^0(X_+) = K^0(X)$, and so we can redefine unreduced $K$-theory in terms of reduced $K$-theory as

$$K^i(X) := \tilde{K}^i(X_+).$$

With this definition $K^{-1}(*) = \tilde{K}^{-1}(*_+) = \tilde{K}^0(S^1) = 0$ by Corollary 3.3.9. With this definition we also obtain the six-term exact sequence

$$\begin{array}{ccccc} K^0(A) & \xleftarrow{i^*} & K^0(X) & \xleftarrow{q^*} & \tilde{K}^0(X/A) \\ \downarrow{\scriptstyle\partial} & & & & \uparrow{\scriptstyle\partial} \\ \tilde{K}^{-1}(X/A) & \xrightarrow{q^*} & K^{-1}(X) & \xrightarrow{i^*} & K^{-1}(A), \end{array}$$

and a graded multiplication $- \otimes - : K^i(X) \otimes K^j(X) \to K^{i+j}(X)$.

[3] See https://mathoverflow.net/questions/441484/the-graded-multiplication-on-topological-k-theory

## 3.4 The Mayer–Vietoris sequence

If $X$ be a compact Hausdorff space which is the union of closed subspaces $A$ and $B$, then $X/A \cong B/(A \cap B)$, so there is a map of long exact sequences as follows.

$$\begin{array}{ccccccccc}
\tilde{K}^{-1}(X/A) & \xleftarrow{\partial} & K^0(A) & \xleftarrow{i_A^*} & K^0(X) & \xleftarrow{q_A^*} & \tilde{K}^0(X/A) & \xleftarrow{\partial} & K^{-1}(A) \\
\downarrow \cong & & j_A^* \downarrow & & \downarrow i_B^* & & \downarrow \cong & & \downarrow j_A^* \\
\tilde{K}^{-1}(B/A \cap B) & \xleftarrow{\partial} & K^0(A \cap B) & \xleftarrow{j_B^*} & K^0(B) & \xleftarrow{q_{A\cap B}^*} & \tilde{K}^0(B/A \cap B) & \xleftarrow{\partial} & K^{-1}(A \cap B)
\end{array}$$

It is an exercise in homological algebra to see that

$$\begin{array}{ccccc}
K^0(A \cap B) & \xleftarrow{j_A^* - j_B^*} & K^0(A) \oplus K^0(B) & \xleftarrow{i_A^* \oplus i_B^*} & K^0(X) \\
\partial' \downarrow & & & & \uparrow \partial' \\
K^{-1}(X) & \xrightarrow{i_A^* \oplus i_B^*} & K^{-1}(A) \oplus K^{-1}(B) & \xrightarrow{j_A^* - j_B^*} & K^{-1}(A \cap B)
\end{array}$$

is then exact, where $\partial'$ is defined as

$$\partial' : K^i(A \cap B) \xrightarrow{\partial} \tilde{K}^{i+1}(B/A \cap B) \xleftarrow{\sim} \tilde{K}^{i+1}(X/A) \xrightarrow{q_A^*} K^{i+1}(X).$$

## 3.5 The Fundamental Product Theorem for $K^{-1}$

There is a useful technique that will let us upgrade many statements about $K^0$ to $K^*$. Firstly, the long exact sequence for the pair $(S^1 \vee X, X)$ takes the form

$$\begin{array}{ccccccc}
K^0(X) & \xleftarrow{j^*} & K^0(S^1 \vee X) & \xleftarrow{p^*} & \tilde{K}^0(S^1) & = & 0 \\
\downarrow \partial & & & & \partial \uparrow & & \\
\tilde{K}^{-1}(S^1) & \xrightarrow{p^*} & K^{-1}(S^1 \vee X) & \xrightarrow{j^*} & K^{-1}(X), & &
\end{array}$$

and the two maps $j^*$ are split surjective (via the collapse map $S^1 \vee X \to X$) so the map $j^* : K^0(S^1 \vee X) \to K^0(X)$ is an isomorphism. Now the long exact sequence of $K$-theory for the pair $(S^1 \times X, S^1 \vee X)$ takes the form

$$\begin{array}{ccccccc}
K^0(S^1 \vee X) & \xleftarrow{i^*} & K^0(S^1 \times X) & \xleftarrow{q^*} & \tilde{K}^0(S^1 \wedge X) & = & K^{-1}(X) \\
\downarrow \partial & & & & \partial \uparrow & & \\
\tilde{K}^{-1}(S^1 \wedge X) & \xrightarrow{q^*} & K^{-1}(S^1 \times X) & \xrightarrow{i^*} & K^{-1}(S^1 \vee X). & &
\end{array}$$

and the two maps $i^*$ are split surjective (via the projections $S^1 \leftarrow S^1 \times X \to X$ and the description of $K^*(S^1 \vee X)$ above). Thus the map

$$\pi_X^* \oplus q^* : K^0(X) \oplus K^{-1}(X) \longrightarrow K^0(S^1 \times X)$$

is an isomorphism. This gives a natural isomorphism

$$K^{-1}(X) \cong \operatorname{Coker}(\pi_X^* : K^0(X) \to K^0(S^1 \times X)),$$

which can be used to reduce certain questions about $K^{-1}$ to questions about $K^0$. An example of this type of argument is the following.

**Corollary 3.5.1.** *If $X$ is compact Hausdorff then the left $K^0(X)$-module homomorphism*

$$\begin{aligned} K^{-1}(X) \otimes \mathbb{Z}[H]/((H-1)^2) &\longrightarrow K^{-1}(X \times \mathbb{CP}^1) \\ x \otimes y &\longmapsto \pi_1^*(x) \otimes \pi_2^*(\phi(y)) \end{aligned}$$

*is an isomorphism.*

*Proof.* The exact sequence

$$0 \longrightarrow K^0(X) \xrightarrow{\pi_X^*} K^0(S^1 \times X) \longrightarrow K^{-1}(X) \longrightarrow 0$$

is split, so stays exact after applying $- \otimes \mathbb{Z}[H]/((H-1)^2)$. Thus we have a commutative diagram

$$\begin{array}{ccc} K^0(X) \otimes \mathbb{Z}[H]/((H-1)^2) & \xrightarrow{-\boxtimes-} & K^0(X \times \mathbb{CP}^1) \\ \downarrow & & \downarrow \\ K^0(S^1 \times X) \otimes \mathbb{Z}[H]/((H-1)^2) & \xrightarrow{-\boxtimes-} & K^0(S^1 \times X \times \mathbb{CP}^1) \\ \downarrow & & \downarrow \\ K^{-1}(X) \otimes \mathbb{Z}[H]/((H-1)^2) & \xrightarrow{-\boxtimes-} & K^{-1}(X \times \mathbb{CP}^1) \end{array}$$

where the columns are short exact sequences, and the top two horizontal maps are isomorphisms by the Fundamental Product Theorem (Theorem 3.2.2). Hence the bottom horizontal map is an isomorphism too. □

Chapter 4

# Proof of the Fundamental Product Theorem

Lecture 14

The argument we give in the chapter essentially follows Hatcher[1], which in turn follows Husemoller[2] and ultimately Atiyah and Bott[3]. We will give a fairly complete argument, but skip lightly over some analytic details. This material is not examinable.

## 4.1 A more general clutching construction

The Fundamental Product Theorem concerns vector bundles over $X \times \mathbb{CP}^1 = X \times S^2$, and we will make use of a clutching construction adapted to this situation.

Given a vector bundle $\pi : E \to X$ and an automorphism

$$\begin{array}{ccc} E \times S^1 & \xrightarrow[\sim]{f} & E \times S^1 \\ & {\scriptstyle \pi\times S^1} \searrow \quad \swarrow {\scriptstyle \pi\times S^1} & \\ & X \times S^1 & \end{array}$$

of this bundle pulled back to $X \times S^1$, called a *clutching map*, we can form a vector bundle over $X \times S^2 = (X \times D^2_-) \cup_{X \times S^1} (X \times D^2_+)$ with total space

$$((E \times D^2_-) \sqcup (E \times D^2_+))/(e, z) \in E \times S^1 \subset E \times D^2_- \sim f(e, z) \in E \times S^1 \subset E \times D^2_+.$$

Call this bundle $[E, f]$. As with the usual clutching construction, changing the map $f$ by a homotopy of vector bundle automorphisms changes the result only up to isomorphism. It is evident from the construction that $[E_1, f_1] \oplus [E_2, f_2] \cong [E_1 \oplus E_2, f_1 \oplus f_2]$. As an example, $[E, Id] = \pi_1^*(E)$. We may as well assume that $f$ is *normalised* to be the identity over $X \times \{1\}$: if not, acting on $E \times D^2_+$ by the inverse of $f|_{X\times\{1\}} \times D^2_+$ gives an isomorphism from $[E, f]$ to $[E, f']$ with $f' := (f|_{X\times\{1\}} \times D^2_+)^{-1} \circ f$ which is normalised.

Given any vector bundle $\pi : F \to X \times S^2$, there are restrictions $\pi_\pm : F_\pm \to X \times D^2_\pm$. As the projection maps $X \times D^2_\pm \to X$ are homotopy equivalences, with homotopy inverses $X = X \times \{1\} \subset X \times S^1 \subset X \times D^2_\pm$, there is a vector bundle $E := F|_{X\times\{1\}} \to X$ and bundle isomorphisms

$$F_- \cong_{\varphi_-} E \times D^2_- \text{ and } F_+ \cong_{\varphi_+} E \times D^2_+$$

which are the identity over $X \times \{1\} \subset X \times D^2_\pm$. Combining this with the identity $F_-|_{X\times S^1} = F_+|_{X\times S^1}$ gives a normalised clutching map

$$f : E \times S^1 \cong_{\varphi_-} F_-|_{X\times S^1} = F_+|_{X\times S^1} \cong_{\varphi_+} E \times S^1$$

[1] *Vector bundles and K-theory*, available at `https://www.math.cornell.edu/~hatcher/VBKT/VB.pdf`
[2] Chapter 11 of *Fibre bundles*, Third edition, Graduate Texts in Mathematics, 20. (1994).
[3] *On the periodicity theorem for complex vector bundles*, Acta Math. 112 (1964), 229–247.

such that $F \cong [E, f]$. Thus any vector bundle on $X \times S^2$ arises by clutching with a normalised clutching map. Similar reasoning shows that $f$ is unique up to homotopy of (normalised) clutching maps.

We will think of such an $f : E \times S^1 \xrightarrow{\sim} E \times S^1$ as a continuous family of automorphisms of the vector bundle $E \to X$ parameterised by $z \in S^1$. For $x \in X$ and $z \in S^1$ we write $f(x, z) : E_x \to E_x$ for the restriction of $f$ to the fibre over $(x, z)$.

## 4.2 Reduction to Laurent polynomial clutching maps

A clutching map $f : E \times S^1 \to E \times S^1$ of the form

$$f(x, z) = \sum_{|k| \leq n} a_k(x) z^k$$

for bundle maps $a_k : E \to E$ (which need not be isomorphisms) is called a *Laurent polynomial clutching map*. We will argue that there is no loss of generality in only considering such clutching maps, using some Fourier theory.

**Proposition 4.2.1.** *Any vector bundle $[E, f]$ is isomorphic to $[E, f']$ for a Laurent polynomial clutching map $f'$.*

*Proof.* The vector space $End(E \times S^1)$ of all bundle maps $a : E \times S^1 \to E \times S^1$ can be given the operator norm $||a|| = \sup_{x \in X, v \in E_x - \{0\}} |a(v)|/|v|$, after choosing a Hermitian inner product on $E$. As $X \times S^1$ is compact, the invertible bundle maps form an open subset. Thus any bundle map sufficiently close to $f$ is invertible, and so is a clutching map, and furthermore is homotopic through clutching maps to $f$. It therefore suffices to show that there is a sequence of Laurent polynomial bundle maps converging uniformly to $f$.

Define a bundle map $a_k : E \to E$ by the formula[4]

$$a_k(x) := \frac{1}{2\pi i} \int_{S^1} z^{-k} f(x, z) \frac{dz}{z},$$

and let $s_n(x, z) := \sum_{|k| \leq n} a_k(x) z^k$ and $\sigma_m(x, z) := \frac{1}{m+1} \sum_{n=0}^{m} s_n(x, z)$. The latter are the Cesàro sums of the Fourier series for the continuous map $f$, so $\sigma_m(x, z) \to f(x, z)$ uniformly by (an elaboration of) Fejér's theorem. On the other hand all $\sigma_m(x, z)$'s are Laurent polynomial bundle maps. □

## 4.3 Reduction to polynomial clutching maps

We have seen in the proof of Lemma 3.2.1 that the clutching map for the tautological bundle $\gamma_{\mathbb{C}}^{1,2} \to \mathbb{CP}^1 = S^2$ is $z \mapsto z : S^1 \to GL_1(\mathbb{C}) = \mathbb{C}^\times$, and so that for $\overline{\gamma_{\mathbb{C}}^{1,2}}$ is $z \mapsto z^{-1}$. Thus the bundle $\pi_2^* H \to X \times S^2$ is presented as $[\underline{\mathbb{C}}^1_X, z^{-1}]$. More generally we have the following.

[4] Here $f(x, z)$ are elements in the finite-dimensional vector space $Hom(E_x, E_x)$, and the integral is taken in this vector space.

**Proposition 4.3.1.** *For any $E \to X$ and clutching map $f$,*

$$[E, f] \otimes \pi_2^*(H) \cong [E, fz^{-1}].$$

*Proof.* Apply the procedure of Section 4.1 to the bundle $[E, f] \otimes \pi_2^*(H)$. □

If $f$ is a Laurent polynomial clutching map then it has the form $p \cdot z^{-n}$ for a clutching map $p$ which is *polynomial*, i.e. $p(x, z) = \sum_{k=0}^{2n} b_k(x)z^k$, and so $[E, f] \cong [E, p] \otimes \pi_2^*(H^n)$.

## 4.4 Reduction to linear clutching maps

The Fundamental Product Theorem is about $K^0(-)$, not about $Vect_{\mathbb{C}}(-)$, and so at some point we must take advantage of the fact that we can differences as well as sums of vector bundles. This is that point.

Say that a clutching map is *linear* if it has the form $f(x, z) = a_0(x) + a_1(x)z$.

**Proposition 4.4.1.** *If $p$ is a polynomial clutching map of degree $\leq n$ for the vector bundle $E \to X$, then there is a linear clutching map $L^n p$ for $E^{\oplus n+1} \to X$ such that $[E, p] \oplus [E^{\oplus n}, Id] \cong [E^{\oplus n+1}, L^n p]$.*

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In particular, we have $[E, p] = [E^{\oplus n+1}, L^n p] - [E^{\oplus n}, Id] \in K^0(X \times S^2)$.

*Proof.* Write $p(x, z) = \sum_{k=0}^{n} a_k(x)z^k$. Define $L^n p$ to be the bundle endomorphism of $E^{\oplus n+1}$ given by the matrix

$$L^n p = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} & a_n \\ -z & Id & 0 & \cdots & 0 & 0 \\ 0 & -z & Id & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & Id & 0 \\ 0 & 0 & 0 & \cdots & -z & Id \end{pmatrix},$$

which may be written as the product

$$\begin{pmatrix} Id & a_1^*(z) & \cdots & a_{n-1}^*(z) & a_n^*(z) \\ 0 & Id & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & Id & 0 \\ 0 & 0 & \cdots & 0 & Id \end{pmatrix} \begin{pmatrix} p(z) & 0 & \cdots & 0 & 0 \\ 0 & Id & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & Id & 0 \\ 0 & 0 & \cdots & 0 & Id \end{pmatrix} \begin{pmatrix} Id & 0 & \cdots & 0 & 0 \\ -z & Id & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & Id & 0 \\ 0 & 0 & \cdots & -z & Id \end{pmatrix},$$

where $a_i^*(z) := \sum_{k=i}^{n} a_k(x)z^{k-i}$. The middle matrix is invertible, as $p$ is a clutching map. The outer two matrices are of the form $Id + N$ for $N$ nilpotent, so are also invertible, and hence $L^n p$ is too and so is indeed a clutching map.

Furthermore the outer two matrices are homotopic to the identity through invertible matrices by a homotopy of the form $Id + t \cdot N$. Thus $[E^{\oplus n+1}, L^n p] \cong [E^{\oplus n+1}, p \oplus Id] \cong [E, p] \oplus [E^{\oplus n}, Id]$. □

## 4.5 Analysis of linear clutching maps

For a linear clutching map $a_0(x) + a_1(x)z$ for the vector bundle $E \to X$, the goal of this section is to show that there is a decomposition $E = E^+ \oplus E^-$, depending of course on the clutching map, such that

$$[E, a_0(x) + a_1(x)z] \cong [E^+, Id] \oplus [E^-, z].$$

We first show that we can neglect the coefficient $a_1(x)$ of $z$.

**Lemma 4.5.1.** *We have $[E, a_0(x) + a_1(x)z] \cong [E, a_0'(x) + z]$ for some bundle endomorphism $a_0' : E \to E$.*

*Proof.* Consider the expression

$$(1 + tz)(a_1(x)\tfrac{z+t}{1+tz} + a_0(x)) = (a_1(x) + ta_0(x))z + ta_1(x) + a_0(x).$$

For $t < 1$ the left-hand side is well-defined and gives an invertible endomorphism of $E_x$, since it is obtained from $z \mapsto a_0(x) + a_1(x)z$ by a reparametrisation of $S^1$. The right-hand side gives a well-defined bundle map for all $t$, and $a_1(x) + ta_0(x)$ is invertible for $t$ near 1 (it is invertible at $t = 1$ as this is $a_0(x) + a_1(x)z$ at $z = 1$) so we can write it as $(a_1(x) + a_0(x)t_0)\left(z + \frac{t_0a_1(x)+a_0(x)}{a_1(x)+a_0(x)t_0}\right)$ for $t_0 < 1$ near 1. Thus

$$[E, a_0(x) + a_1(x)z] \cong [E, (a_1(x) + a_0(x)t_0)\left(z + \tfrac{t_0a_1(x)+a_0(x)}{a_1(x)+a_0(x)t_0}\right)] \cong [E, z + \tfrac{t_0a_1(x)+a_0(x)}{a_1(x)+a_0(x)t_0}],$$

where the first isomorphism is given by the homotopy of clutching maps with $t \in [0, t_0]$, and the second isomorphism is because multiplying a clutching map by an automorphism of $E$ gives an isomorphic bundle. □

For a clutching map of the form $a_0(x) + z$, each linear map $a_0(x) : E_x \to E_x$ must have all eigenvalues off the unit circle (if $\lambda$ was an eigenvalue on the unit circle then $a_0(x) + z$ would not be invertible at $z = -\lambda \in S^1$). Let us write $E_x^+ \leq E_x$ for the sum of the generalised eigenspaces of $a_0(x)$ with eigenvalues of absolute value $> 1$, and $E_x^- \leq E_x$ for the sum of the generalised eigenspaces of $a_0(x)$ with eigenvalues of absolute value $< 1$. Then $E_x = E_x^+ \oplus E_x^-$, and this decomposition is preserved by $a_0(x)$ and hence by $a_0(x) + z$ too.

**Lemma 4.5.2.** *The subspaces $E^\pm := \bigcup_{x\in X} E_x^\pm \leq E$ are subbundles.*

*Proof.* The problem is local in $X$, so we may work inside a local trivialisation $U \times \mathbb{C}^n$. Under this trivialisation the bundle maps $a_0(x)$ assemble to a continuous map $a_0 : U \to M_{n\times n}(\mathbb{C})$, landing in the subspace $S \subset M_{n\times n}(\mathbb{C})$ of those matrices having no eigenvalues on the unit circle. If we define

$$T^+ = \left\{(A, v) \in S \times \mathbb{C}^n \,\middle|\, \begin{matrix} v \text{ lies in the sum of the generalised eigenspaces} \\ \text{of } A \text{ for eigenvalues of absolute value} > 1 \end{matrix}\right\}$$

and $T^-$ similarly, then $E^\pm|_U = a_0^*(T^\pm)$, so it suffices to show that $T^\pm$ are subbundles of the trivial bundle $S \times \mathbb{C}^n$.

If $\chi_A(t)$ is the characteristic polynomial of $A \in S$, then it factors $\chi_A(t) = \chi_A^+(t) \cdot \chi_A^-(t)$ uniquely as a product of monic polynomials having roots of absolute value greater or less than 1. As the polynomials $\chi_A(t)$ vary continuously with $A$, on the space $S$ where no roots have absolute value 1 the monic polynomials $\chi_A^\pm(t)$ vary continuously with $A$ too. In particular the dimension of the fibres of $T^\pm$, which are the degrees of these polynomials, are locally constant on $S$. The bundle maps

$$(A, v) \mapsto \chi_A^\pm(A) \cdot v : S \times \mathbb{C}^n \longrightarrow S \times \mathbb{C}^n$$

have kernels $T^\pm$, so by the Constant Rank Theorem (Example Sheet 1 Q10) the $T^\pm$ are subbundles. $\square$

We therefore have $[E, a_0(x) + z] = [E^+, a_0(x)|_{E^+} + z] \oplus [E^-, a_0(x)|_{E^-} + z]$. As all eigenvalues of $a_0(x)|_{E^+}$ have absolute value $> 1$, the formula $t \mapsto a_0(x)|_{E^+} + t \cdot z$ for $t \in [0, 1]$ is a homotopy of clutching maps, so

$$[E^+, a_0(x)|_{E^+} + z] \cong [E^+, a_0(x)|_{E^+}] \cong [E^+, Id]$$

using that $a_0(x)|_{E^+}$ is invertible. As all eigenvalues of $a_0(x)|_{E^-}$ have absolute value $< 1$, the formula $t \mapsto t \cdot a_0(x)|_{E^-} + z$ for $t \in [0, 1]$ is a homotopy of clutching maps, so

$$[E^-, a_0(x)|_{E^-} + z] \cong [E^-, z].$$

Observe that extracting $E^\pm$ from $[E, a_0(x) + a_1(x)z]$ is additive: if the $a_i(x)$ preserve
Lecture 16 a decomposition of $E$, then we obtain a corresponding decomposition of $E^\pm$.

## 4.6 The inverse

The discussion so far shows that the map $\mu : K^0(X) \otimes \mathbb{Z}[H]/((H - 1)^2) \to K^0(X \times S^2)$ from the Fundamental Product Theorem is surjective. To see this, first represent a vector bundle over $X \times S^2$ as $[E, f]$ for some clutching map $f$, and then approximate $f$ by a Laurent clutching map $pz^{-n}$ with $p$ a polynomial clutching map of degree $2n$. Then in $K$-theory $[E, f] = [E, pz^{-n}] = [E, p] \otimes \pi_2^*(H^n)$. Now we showed that for $p$ a polynomial clutching map of degree $2n$ one has

$$[E, p] = [E^{\oplus 2n+1}, L^{2n}p] - [E^{\oplus 2n}, Id]$$

with $L^{2n}p$ a linear clutching map, and by the previous section we have

$$\begin{aligned}[E^{\oplus 2n+1}, L^{2n}p] &= [(E^{\oplus 2n+1})^+, Id] + [(E^{\oplus 2n+1})^-, z] \\ &= [(E^{\oplus 2n+1})^+, Id] + [(E^{\oplus 2n+1})^-, Id] \otimes \pi_2^*(H^{-1}).\end{aligned}$$

Putting it all together we have

$$[E, pz^{-n}] = ([(E^{\oplus 2n+1})^+, Id] + [(E^{\oplus 2n+1})^-, Id] \otimes \pi_2^*(H^{-1}) - [E^{\oplus 2n}, Id]) \otimes \pi_2^*(H^n)$$

and each term is of the form $\{\text{vector bundle on } X\} \otimes \pi_2^*(H^r)$ so is in the image of $\mu$. This shows that all vector bundles are hit my $\mu$, so all differences of vector bundles are too.

To establish the injectivity of $\mu$ we must understand the ambiguity in the above construction. The ambiguity enters via $n$, the degree of the Laurent clutching map approximating $f$. In the above setting define

$$\nu_n(E,f) := (E^{\oplus 2n+1})^- \otimes (1-H) + E \otimes H^n \in K^0(X) \otimes \mathbb{Z}[H]/((H-1)^2).$$

Now if we write $pz^{-n} = (pz) \cdot z^{-(n+1)}$ then we also have

$$\nu_{n+1}(E,f) = (E^{\oplus 2(n+1)+1})^- \otimes (1-H) + E \otimes H^{n+1}$$

and would like to know that it does not depend on the degree which we consider the Laurent polynomial as having.

**Lemma 4.6.1.** *We have* $\nu_{n+1}(E,f) = \nu_n(E,f) \in K^0(X) \otimes \mathbb{Z}[H]/((H-1)^2)$.

*Proof.* The eigenspace $(E^{\oplus 2(n+1)+1})^-$ is determined using the linear clutching map $L^{2n+2}(pz)$. We have $p(x,z) = \sum_{k=0}^{2n} a_k(x) z^k$, so on $E^{\oplus 2n+3}$ we have the linear clutching map

$$L^{2n+2}(pz) = \begin{pmatrix} 0 & a_0 & a_1 & \cdots & a_{2n} & 0 \\ -z & Id & 0 & \cdots & 0 & 0 \\ 0 & -z & Id & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & Id & 0 \\ 0 & 0 & 0 & \cdots & -z & Id \end{pmatrix}.$$

Doing column operations, which are homotopic to the identity as they are given by right multiplication by a matrix of the form $Id$+nilpotent, gives a homotopy of linear clutching maps from this to

$$\begin{pmatrix} 0 & a_0 & a_1 & \cdots & a_{2n-1} & a_{2n} & 0 \\ -z & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -z & Id & \cdots & 0 & 0 & 0 \\ 0 & 0 & -z & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -z & Id & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & Id \end{pmatrix}.$$

and then left multiplying by a rotation matrix in the first two coordinates gives a homotopy of linear clutching maps from this to

$$\begin{pmatrix} -z & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_0 & a_1 & \cdots & a_{2n-1} & a_{2n} & 0 \\ 0 & -z & Id & \cdots & 0 & 0 & 0 \\ 0 & 0 & -z & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -z & Id & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & Id \end{pmatrix}.$$

Thus $[E^{\oplus 2n+3}, L^{2n+2}(pz)] \cong [E, -z] \oplus [E^{\oplus 2n+1}, L^{2n}p] \oplus [E, Id]$.

It is easy to see that $E^- = E$ for the linear clutching map $-z$, and $E^- = 0$ for the linear clutching map $Id$. It follows that $(E^{\oplus 2n+3})^- = E \oplus (E^{\oplus 2n+1})^- \oplus 0$. Thus

$$\begin{aligned}\nu_{n+1}(E,f) &= (E^{\oplus 2(n+1)+1})^- \otimes (1-H) + E \otimes H^{n+1} \\ &= (E \oplus (E^{\oplus 2n+1})^-) \otimes (1-H) + E \otimes H^{n+1} \\ &= (E^{\oplus 2n+1})^- \otimes (1-H) + E \otimes (1 - H + H^{n+1})\end{aligned}$$

but as $(H-1)^2 = 0 \in \mathbb{Z}[H]/((H-1)^2)$ we have $H(1-H) = (1-H)$ and so $H^n(1-H) = (1-H)$, hence $1 - H + H^{n+1} = H^n$. Thus the above is $\nu_n(E,f)$. □

We can therefore write $\nu(E,f)$ for the class obtained from any Laurent polynomial approximation of $f$. This is easily seen to be additive, and so extends to a homomorphism

$$\nu : K^0(X \times S^2) \longrightarrow K^0(X) \otimes \mathbb{Z}[H]/((H-1)^2).$$

To check it is a left inverse to $\mu$, we use the basis $\{1, H\}$ of $\mathbb{Z}[H]/((H-1)^2)$.

Firstly $\nu\mu(E \otimes 1) = \nu([E, Id]) = \nu_0([E, Id]) = E^- \otimes (1-H) + E \otimes 1 = E \otimes 1$ as $E^- = 0$ for the linear clutching map $Id$.

Secondly $\nu\mu(E \otimes H) = \nu([E, z^{-1}]) = (E^{\oplus 3})^- \otimes (1-H) + E \otimes H$ and $p = Id$ so

$$L^2 p = \begin{pmatrix} Id & 0 & 0 \\ -z & Id & 0 \\ 0 & -z & Id \end{pmatrix} \simeq \begin{pmatrix} Id & 0 & 0 \\ 0 & Id & 0 \\ 0 & 0 & Id \end{pmatrix}$$

so $[E^{\oplus 3}, L^2 p] \cong [E, Id]^{\oplus 3}$ and hence $(E^{\oplus 3})^- = 0$.

Chapter 5

# Further structure of $K$-theory

Lecture 17

## 5.1 The yoga of symmetric polynomials

A multivariable polynomial $p(x_1, x_2, \dots, x_n) \in \mathbb{Z}[x_1, \dots, x_n]$ is called *symmetric* if

$$p(x_1, x_2, \dots, x_n) = p(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$$

for any permutation $\sigma \in \Sigma_n$. In other words, it is a fixed point of the action of $\Sigma_n$ on the ring $\mathbb{Z}[x_1, \dots, x_n]$ by $x_i \mapsto x_{\sigma(i)}$; one might write this as $\mathbb{Z}[x_1, \dots, x_n]^{\Sigma_n} \subset \mathbb{Z}[x_1, \dots, x_n]$, and it is a subring.

The *elementary symmetric polynomials* $e_i(x_1, \dots, x_n)$ are defined intrinsically by

$$\prod_{i=1}^{n}(t + x_i) = \sum_{i=0}^{n} e_i(x_1, \dots, x_n) \cdot t^{n-i}.$$

These are symmetric polynomials, as the left-hand side is clearly invariant under reordering the $x_i$. For example, we have

$$\begin{aligned}
e_0(x_1, x_2, x_3) &= 1\\
e_1(x_1, x_2, x_3) &= x_1 + x_2 + x_3\\
e_2(x_1, x_2, x_3) &= x_1x_2 + x_1x_3 + x_2x_3\\
e_3(x_1, x_2, x_3) &= x_1x_2x_3.
\end{aligned}$$

**Theorem 5.1.1** (Fundamental Theorem of Symmetric Polynomials)**.** *The ring homomorphism*

$$\begin{aligned}
\mathbb{Z}[e_1, e_2, \dots, e_n] &\longrightarrow \mathbb{Z}[x_1, \dots, x_n]^{\Sigma_n}\\
e_i &\longmapsto e_i(x_1, \dots, x_n)
\end{aligned}$$

*is an isomorphism.*

In other words, any symmetric polynomial $p(x_1, x_2, \dots, x_n)$ has a unique representation as $\bar{p}(e_1(x_1, \dots, x_n), \dots, e_n(x_1, \dots, x_n))$.

*Proof.* For surjectivity we proceed by simultaneous induction on the degree and the number of variables. Let $p(x_1, \dots, x_n)$ be a symmetric polynomial and consider the ring homomorphism

$$q : \mathbb{Z}[x_1, \dots, x_n] \longrightarrow \mathbb{Z}[x_1, \dots, x_n]/(x_n) \cong \mathbb{Z}[x_1, \dots, x_{n-1}].$$

Now $q(p(x_1,\ldots,x_n))$ is $\Sigma_{n-1}$-invariant, so as it has fewer variables we may write

$$q(p(x_1,\ldots,x_n)) = \bar{q}(e_1(x_1,\ldots,x_{n-1}),\ldots,e_{n-1}(x_1,\ldots,x_{n-1})).$$

Thus

$$p(x_1,\ldots,x_n) - \bar{q}(e_1(x_1,\ldots,x_n),\ldots,e_{n-1}(x_1,\ldots,x_n))$$

is a symmetric polynomial, and lies in $\mathrm{Ker}(q)$ so is divisible by $x_n$. As it is symmetric it is therefore divisible by each $x_i$, and hence is divisible by $x_1\cdots x_n$ (as $\mathbb{Z}[x_1,\ldots,x_n]$ is a UFD). Then we can write

$$p(x_1,\ldots,x_n) = \bar{q}(e_1(x_1,\ldots,x_n),\ldots,e_{n-1}(x_1,\ldots,x_n)) + e_n(x_1,\ldots,x_n)\cdot f(x_1,\ldots,x_n)$$

for a symmetric polynomial $f(x_1,\ldots,x_n)$; this has strictly lower degree, so is a polynomial in the $e_i(x_1,\ldots,x_n)$, as required.

For injectivity, suppose that $p(e_1,\ldots,e_n)$ is a non-zero polynomial lying in the kernel. As it is non-zero, there exist $E_1,\ldots,E_n\in\mathbb{C}$ such that $p(E_1,\ldots,E_n)\neq 0$. Let $X_1,\ldots,X_i\in\mathbb{C}$ be such that $\prod_{i=1}^n(t+X_i) = \sum_{i=0}^n E_i\cdot t^{n-i}$ (i.e. the $X_i$ are minus the roots of the polynomial on the right-hand side). Let $\phi:\mathbb{Z}[x_1,\ldots,x_i]\to\mathbb{C}$ be defined to send $x_i$ to $X_i$. Then

$$\begin{aligned}\phi(p(e_1(x_1,\ldots,x_n),\ldots,e_n(x_1,\ldots,x_n))) &= p(e_1(X_1,\ldots,X_n),\ldots,e_n(X_1,\ldots,X_n))\\ &= p(E_1,\ldots,E_n)\neq 0\end{aligned}$$

and so $p(e_1(x_1,\ldots,x_n),\ldots,e_n(x_1,\ldots,x_n))\neq 0\in\mathbb{Z}[x_1,\ldots,x_n]$. □

An important type of symmetric polynomial we will meet are the *power sum polynomials*

$$p_k(x_1,\ldots,x_n) = \sum_{i=1}^n x_i^k.$$

By the Fundamental Theorem of Symmetric Polynomials these may be expressed in terms of the $e_i$; the first few are

$$\begin{aligned}p_1 &= e_1\\ p_2 &= e_1^2 - 2e_2\\ p_3 &= e_1^3 - 3e_1e_2 + 3e_3.\end{aligned}$$

**Lemma 5.1.2.** *We have the identity*

$$p_n - e_1p_{n-1} + e_2p_{n-2} - \cdots \mp e_{n-1}p_1 \pm ne_n = 0.$$

*Proof.* Substitute $t=-x_i$ into

$$\prod_{i=1}^n(t+x_i) = \sum_{i=0}^n e_i(x_1,\ldots,x_n)\cdot t^{n-i}$$

and then sum over $i$. □

The coefficient of $e_n$ in $p_n(e_1, \ldots, e_n)$ is $\pm n \neq 0$, so *over the rational numbers* one may also express the $e_i$ in terms of the $p_n$; the first few are

$$\begin{aligned} e_1 &= p_1 \\ e_2 &= (p_1^2 - p_2)/2 \\ e_3 &= (p_1^3 - 3p_1p_2 + 2p_3)/6. \end{aligned}$$

### 5.1.1 Relation to Chern classes

If $E = L_1 \oplus L_2 \oplus \cdots \oplus L_n \to X$ is a sum of $n$ complex line bundles, then we have

$$\begin{aligned} c(E) &= c(L_1)c(L_2)\cdots c(L_n) \\ &= (1 + c_1(L_1))(1 + c_1(L_2))\cdots(1 + c_1(L_n)) \\ &= \sum_{i=0}^{n} e_i(c_1(L_1), \ldots, c_1(L_n)) \end{aligned}$$

so $c_i(E) = e_i(c_1(L_1), \ldots, c_1(L_n))$ is the $i$th elementary symmetric polynomial in the first Chern classes of the $L_i$.

If $E \to X$ is not a sum of line bundles, we nonetheless know that there is a map $f : F(E) \to X$ such that $f^*(E) \cong L_1 \oplus L_2 \oplus \cdots \oplus L_n$ is a sum of line bundles, and $f^* : H^*(X) \to H^*(F(E))$ is injective: thus although $c_i(E)$ is not an elementary symmetric polynomial in some degee 2 cohomology classes, $H^*(X)$ injects into a ring where it is.

## 5.2 The Chern character

We wish to construct a natural ring homomorphism

$$ch : K^0(X) \longrightarrow H^{2*}(X; \mathbb{Q})$$

so that if $E \to X$ is a complex vector bundle then the degree $2i$ component of $ch(E)$ is given by a polynomial $ch_i(c_1(E), \ldots, c_i(E)) \in \mathbb{Q}[c_1(E), \ldots, c_i(E)]$.

Supposing such a natural ring homomorphism exists, then by evaluating it on the tautological line bundles $\gamma_{\mathbb{C}}^{1,N+1} \to \mathbb{CP}^N$ for each $N$ we find that there is a formal power series $f(t) \in \mathbb{Q}[[t]]$ such that

$$ch(\gamma_{\mathbb{C}}^{1,N+1}) = f(x) \in H^*(\mathbb{CP}^N; \mathbb{Q}) = \mathbb{Q}[x]/(x^{N+1})$$

for every $N$. As any line bundle $L \to X$ over a compact Hausdorff space is classified by a map $g : X \to \mathbb{CP}^N$ for some $n \gg 0$, and $x = c_1(\gamma_{\mathbb{C}}^{1,N+1})$, it follows that

$$ch(L) = f(c_1(L)) \in H^*(X; \mathbb{Q})$$

for each such line bundle. On the other hand, as $ch$ is supposed to be a *ring* homomorphism, and $c_1(l_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$ by Example 2.8.5, we must also have

$$f(c_1(L_1) + c_1(L_2)) = ch(L_1 \otimes L_1) = ch(L_1)ch(L_2) = f(c_1(L_1))f(c_1(L_2)),$$

and applying this to the external tensor product of the two natural line bundles on $\mathbb{CP}^N \times \mathbb{CP}^N$ shows that we must have the identity

$$f(s+t) = f(s) \cdot f(t) \in \mathbb{Q}[[s,t]].$$

By good old-fashioned calculus, the formal power series must then be $f(t) = \exp(a \cdot t)$ for some $a \in \mathbb{Q}$. *As a normalisation we choose* $a = 1$.

Thus for each sum of line bundles $L_1 \oplus \cdots \oplus L_n \to X$ over a compact Hausdorff space we have

$$\begin{aligned} ch(L_1 \oplus \cdots \oplus L_n) &= \exp(c_1(L_1)) + \cdots + \exp(c_1(L_n)) \in H^*(X;\mathbb{Q}) \\ &= \sum_{k=0}^{\infty} \frac{c_1(L_1)^k + \cdots + c_1(L_n)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{\bar{p}_k(c_1(L_1 \oplus \cdots \oplus L_n), \ldots, c_n(L_1 \oplus \cdots \oplus L_n))}{k!} \end{aligned}$$

where the power sum polynomials are written in terms of elementary symmetric polynomials by $p_k(x_1, \ldots, x_n) = \bar{p}_k(e_1(x_1, \ldots, x_n), \ldots, e_k(x_1, \ldots, x_n))$. It then follows from the splitting principle that for any complex vector bundle $E \to X$ we must have

$$ch(E) = \sum_{k=0}^{\infty} \frac{\bar{p}_k(c_1(E), \ldots, c_n(E))}{k!}. \tag{5.2.1}$$

Via $ch(E - F) = ch(E) - ch(F)$, this describes the homomorphism $ch$ completely.

We arrived at this description by positing the existence of a homomorphism $ch$, but by reversing the logic above it follows that the formula (5.2.1) *defines* a monoid homomorphism

$$ch : (Vect(X), \oplus, 0) \longrightarrow H^{ev}(X;\mathbb{Q})$$

which therefore extends to a unique group homomorphism $ch : K^0(X) \to H^{ev}(X;\mathbb{Q})$ by definition of the Grothendieck completion, and that this is actually a ring homomorphism. We write

$$ch_k(E) := \frac{\bar{p}_k(c_1(E), \ldots, c_n(E))}{k!} \in H^{2k}(X;\mathbb{Q})$$

for the degree $2k$ component. The first few are

$$\begin{aligned} ch_0(E) &= \dim(E) \\ ch_1(E) &= c_1(E) \\ ch_2(E) &= (c_1(E)^2 - 2c_2(E))/2 \\ ch_3(E) &= (c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E))/6 \end{aligned}$$

In particular, if $E - F \in \tilde{K}^0(X)$ then $ch_0(E - F) = 0$ so $ch(E - F) \in \tilde{H}^*(X;\mathbb{Q})$. We can similarly define the Chern character on $K^{-1}$ by

$$ch : K^{-1}(X) = \tilde{K}^0(\Sigma X) \xrightarrow{ch} \tilde{H}^{ev}(\Sigma X;\mathbb{Q}) = H^{odd}(X;\mathbb{Q})$$

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landing in odd-degree cohomology.

**Lemma 5.2.1.** *The homomorphisms*

$$
\begin{aligned}
ch : \tilde{K}^0(S^{2n}) &\longrightarrow \tilde{H}^{ev}(S^{2n};\mathbb{Q}) \\
ch : \tilde{K}^{-1}(S^{2n+1}) &\longrightarrow \tilde{H}^{odd}(S^{2n+1};\mathbb{Q})
\end{aligned}
$$

*are isomorphisms onto* $\mathbb{Z} = \tilde{H}^d(S^d;\mathbb{Z}) \subset \tilde{H}^d(S^d;\mathbb{Q})$.

*Proof.* On $S^2 = \mathbb{CP}^1$ we have $\tilde{K}^0(\mathbb{CP}^1) = \mathbb{Z}\{H-1\}$ so $ch(H-1) = \exp(c_1(H)) - \exp(0) = c_1(H)$ because $c_1(H)^2 = 0$. Now $c_1(H) = c_1(\overline{\gamma_{\mathbb{C}}^{1,2}}) = -c_1(\gamma_{\mathbb{C}}^{1,2}) = -x$, which generates $H^2(\mathbb{CP}^1;\mathbb{Z}) \subset H^2(\mathbb{CP}^1;\mathbb{Q})$ as claimed. For an even-dimensional sphere, by Bott Periodicity the external product

$$\tilde{K}^0(S^2) \otimes \tilde{K}^0(S^{2n-2}) \longrightarrow \tilde{K}^0(S^{2n})$$

is an isomorphism so the target is $\mathbb{Z}$ generated by the external product $(H-1)^{\boxtimes n}$. Thus $ch(\tilde{K}^0(S^{2n}))$ is generated by $c_1(H)^{\boxtimes n}$, so is equal to $\tilde{H}^{2n}(S^{2n};\mathbb{Z}) \subset \tilde{H}^{2n}(S^{2n};\mathbb{Q})$.

It then follows for all odd-dimensional spheres by our definition of $ch$ for $K^{-1}$. □

**Theorem 5.2.2.** *The total Chern character*

$$ch : K^*(X) \longrightarrow H^*(X;\mathbb{Q})$$

*is a homomorphism of* $\mathbb{Z}/2$*-graded rings, and if* $X$ *is a finite CW-complex then it extends to an isomorphism*

$$ch : K^*(X) \otimes \mathbb{Q} \longrightarrow H^*(X;\mathbb{Q}).$$

*Proof.* Recall from Section 3.3.6 that for $i, j \in \{0,-1\}$ the graded multiplication on $K^*(X)$ is given by the (suspended) diagonal map

$$S^{-i-j} \wedge d : S^{-i-j} \wedge X_+ \longrightarrow S^{-i-j} \wedge X_+ \wedge X_+ \cong S^{-i} \wedge X_+ \wedge S^{-j} \wedge X_+,$$

the external product on reduced $K$-theory, and, if $i = j = -1$, a sign and the Bott Periodicity isomorphism. As the Chern character on $K^0$ is multiplicative and is natural with respect to maps of spaces, only the case $i = j = -1$ needs to be checked, which is the claim that the outer part of the following diagram commutes

$$
\begin{array}{ccccc}
K^{-1}(X) \otimes K^{-1}(X) & & & & \\
\Vert & & & & \\
\tilde{K}^0(S^1 \wedge X_+) \otimes \tilde{K}^0(S^1 \wedge X_+) & \xrightarrow{(\Sigma^2 d)^*} & \tilde{K}^0(S^2 \wedge X_+) & \xleftarrow[\sim]{-\beta} & \tilde{K}^0(X_+) \\
\downarrow{\scriptstyle ch \otimes ch} & & \downarrow{\scriptstyle ch} & & \\
\tilde{H}^{ev}(S^1 \wedge X_+) \otimes \tilde{H}^{ev}(S^1 \wedge X_+) & \xrightarrow{(S^2 \wedge d)^*} & \tilde{H}^{ev}(S^2 \wedge X_+) & & \downarrow{\scriptstyle ch} \\
\uparrow{\scriptstyle (t\boxtimes -)\otimes(t\boxtimes -)} & & & \nwarrow{\scriptstyle x \boxtimes -}\ {\scriptstyle \sim} & \\
\tilde{H}^{odd}(X_+) \otimes \tilde{H}^{odd}(X_+) & & \xrightarrow{\quad - \smile - \quad} & & \tilde{H}^{ev}(X_+).
\end{array}
$$

The middle square commutes as $ch$ is natural with respect to maps of spaces, and as $\beta = (H-1) \boxtimes -$, $ch(H-1) = -x$, and $ch$ is multiplicative, the right-hand trapezium commutes. This leaves just the bottom trapezium, which only concerns cohomology. The Künneth theorem on reduced (rational) cohomology for pointed spaces $Y$ and $Z$ is

$$- \boxtimes - : \tilde{H}^*(Y;\mathbb{Q}) \otimes \tilde{H}^*(Y;\mathbb{Q}) \xrightarrow{\sim} \tilde{H}^*(Y \wedge Z;\mathbb{Q}),$$

so using $\Sigma X_+ \simeq S^1 \wedge X_+$ we see that the suspension isomorphism is given by multiplication by the standard generator $t \in \tilde{H}^1(S^1;\mathbb{Q})$. The clockwise composition then first sends $a \otimes b$ to $(t \boxtimes a) \boxtimes (t \boxtimes b)$, the applies the swap homeomorphism which gives $(-1)(t\boxtimes t)\boxtimes(a\boxtimes b)$, then pulls back along the diagonal to give $(-1)(t\boxtimes t)\boxtimes(a \smile b)$. This agrees with the anticlockwise composition $a \otimes b \mapsto x \boxtimes (a \smile b)$, because $\langle x, [S^2]\rangle = -1$ by Example 2.9.3, whereas $\langle t \boxtimes t, [S^2]\rangle = \langle t \boxtimes t, [S^1 \wedge S^1]\rangle = +1$, so $t \boxtimes t = -x$.

For the second part, note that it follows from the previous lemma that

$$\begin{aligned} ch : \tilde{K}^0(S^{2n}) \otimes \mathbb{Q} &\longrightarrow \tilde{H}^{ev}(S^{2n};\mathbb{Q}) \\ ch : \tilde{K}^{-1}(S^{2n+1}) \otimes \mathbb{Q} &\longrightarrow \tilde{H}^{odd}(S^{2n+1};\mathbb{Q}) \end{aligned}$$

are isomorphisms. Proceed by induction on the number of cells of $X$: if $X = Y \cup_f D^d$ then we have a map of long exact sequences

$$\begin{array}{ccccccccc} K^{-1}(Y)\otimes\mathbb{Q} & \longrightarrow & \tilde{K}^0(S^d)\otimes\mathbb{Q} & \longrightarrow & K^0(X)\otimes\mathbb{Q} & \longrightarrow & K^0(Y)\otimes\mathbb{Q} & \longrightarrow & \tilde{K}^{-1}(S^d)\otimes\mathbb{Q} \\ \downarrow{\scriptstyle ch} & & \downarrow{\scriptstyle ch} & & \downarrow{\scriptstyle ch} & & \downarrow{\scriptstyle ch} & & \downarrow{\scriptstyle ch} \\ H^{odd}(Y;\mathbb{Q}) & \longrightarrow & \tilde{H}^{ev}(S^d;\mathbb{Q}) & \longrightarrow & H^{ev}(X;\mathbb{Q}) & \longrightarrow & H^{ev}(Y;\mathbb{Q}) & \longrightarrow & \tilde{H}^{odd}(S^d;\mathbb{Q}). \end{array}$$

By assumption $ch : K^*(Y) \otimes \mathbb{Q} \to H^*(Y;\mathbb{Q})$ is an isomorphism, and by the previous lemma $ch : \tilde{K}^*(S^d) \otimes \mathbb{Q} \to \tilde{H}^*(S^d;\mathbb{Q})$ is an isomorphism, so it follows by the 5-lemma that $ch : K^*(X) \otimes \mathbb{Q} \to H^*(X;\mathbb{Q})$ is an isomorphism too. ☐

## 5.3 $K$-theory of $\mathbb{CP}^n$ and the projective bundle formula

Recall that we write $H = [\overline{\gamma_{\mathbb{C}}^{1,n+1}}] \in K^0(\mathbb{CP}^n)$, so that $H - 1 \in \tilde{K}^0(\mathbb{CP}^n)$. As $\mathbb{CP}^n$ can be covered by $(n+1)$ contractible spaces, it follows from Example 3.3.7 that all $(n+1)$-fold products of elements of $\tilde{K}^0(\mathbb{CP}^n)$ are trivial. In particular we have $(H-1)^{n+1} = 0 \in \tilde{K}^0(\mathbb{CP}^n)$.

**Theorem 5.3.1.** *We have* $K^0(\mathbb{CP}^n) = \mathbb{Z}[H]/((H-1)^{n+1})$ *and* $K^{-1}(\mathbb{CP}^n) = 0$.

*Proof.* For $n = 1$ the $K^0$ part follows from the Fundamental Product Theorem, as Corollary 3.2.3, and the $K^{-1}$ part from Corollary 3.3.9. Supposing it holds for $\mathbb{CP}^{n-1}$, consider the exact sequence

$$\begin{array}{ccccccccc} \mathbb{Z}[H]/((H-1)^n) & = & K^0(\mathbb{CP}^{n-1}) & \xleftarrow{i^*} & K^0(\mathbb{CP}^n) & \xleftarrow{q^*} & \tilde{K}^0(S^{2n}) & = & \mathbb{Z} \\ & & \downarrow{\scriptstyle \partial} & & & & \uparrow{\scriptstyle \partial} & & \\ 0 & = & \tilde{K}^{-1}(S^{2n}) & \xrightarrow{q^*} & K^{-1}(\mathbb{CP}^n) & \xrightarrow{i^*} & K^{-1}(\mathbb{CP}^{n-1}) & = & 0. \end{array}$$

We immediately see that $K^{-1}(\mathbb{CP}^{n-1}) = 0$, and that $(H-1)^n$ lies in

$$\mathbb{Z} \cong Ker(i^* : K^0(\mathbb{CP}^n) \to K^0(\mathbb{CP}^{n-1}))$$

so is of the form $q^*(Y)$ for some $Y \in \tilde{K}^0(S^{2n})$. Thus

$$q^*(ch(Y)) = ch((H-1)^n) = (\exp(-x) - 1)^n = (-x)^n = q^*(ch((H-1)^{\boxtimes n})),$$

and as $q^* : H^*(S^{2n}; \mathbb{Q}) \to H^*(\mathbb{CP}^n; \mathbb{Q})$ is injective we have by Lemma 5.2.1 that $Y$ generates $\tilde{K}^0(S^{2n})$. Thus $(H-1)^n$ generates $Ker(i^* : K^0(\mathbb{CP}^n) \to K^0(\mathbb{CP}^{n-1})) \cong \mathbb{Z}$, and the result follows. □

Recall that if $\pi : E \to X$ is a complex vector bundle then the projectivisation $p : \mathbb{P}(E) \to X$ carries a tautological complex line bundle $L_E \to \mathbb{P}(E)$. If $X$ is compact Hausdorff then so is $\mathbb{P}(E)$, and the conjugate of this tautological line bundle represents a class $H_E := [\overline{L_E}] \in K^0(\mathbb{P}(E))$. This is chosen so that when $X = *$ it agrees with the class $H$.

**Theorem 5.3.2.** *If $\pi : E \to X$ is a $d$-dimensional complex vector bundle over a compact Hausdorff space $X$, then the $K^0(X)$-module map*

$$\begin{aligned} K^j(X)\{1, H_E, H_E^2, \ldots, H_E^{d-1}\} &\longrightarrow K^j(\mathbb{P}(E)) \\ \sum_{i=0}^{d-1} Y_i \cdot H_E^i &\longmapsto \sum_{i=0}^{d-1} p^*(Y_i) \otimes H_E^i \end{aligned}$$

*is an isomorphism for $j = 0$ and $j = -1$.*

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*Proof.* Once we show that this holds for all trivial bundles, then it holds for any complex vector bundle $E \to X$ by following the proof of Theorem 2.2.1 (and using a finite cover of $X$ by closed sets over which the bundle is trivial, which exist using e.g. a partition of unity).

For the trivial bundle we are exactly asking for the external product map

$$- \boxtimes - : K^j(X) \otimes K^0(\mathbb{CP}^{d-1}) \longrightarrow K^j(X \times \mathbb{CP}^{d-1})$$

to be an isomorphism. When $d = 1$ there is nothing to show, and when $d = 2$ this is precisely the Fundamental Product Theorem. Looking at the map of long exact sequences of the pairs $(\mathbb{CP}^{d-1}, \mathbb{CP}^{d-2})$ and $(X \times \mathbb{CP}^{d-1}, X \times \mathbb{CP}^{d-2})$ (and using that the first is split, so stays being exact after applying $K^j(X) \otimes -$) we reduce to showing that the external product map

$$- \boxtimes - : K^j(X) \otimes \tilde{K}^0(\mathbb{CP}^{d-1}/\mathbb{CP}^{d-2}) \longrightarrow \tilde{K}^j(X \times \mathbb{CP}^{d-1}/X \times \mathbb{CP}^{d-2})$$

is an isomorphism. This is the external product map

$$- \boxtimes - : \tilde{K}^j(X_+) \otimes \tilde{K}^0(S^{2d-2}) \longrightarrow \tilde{K}^j(X_+ \wedge S^{2d-2})$$

which is simply $(d-1)$ iterations of the Bott periodicity isomorphism, so is an isomorphism. □

Just as for the projective bundle formula in cohomology, we deduce that $p : \mathbb{P}(E) \to X$ is injective in $K$-theory.

**Corollary 5.3.3.** *If $\pi : E \to X$ is a $d$-dimensional complex vector bundle over a compact Hausdorff space $X$ then the map $p^* : K^*(X) \to K^*(\mathbb{P}(E))$ is injective.*

Similarly, we obtain a splitting principle in $K$-theory.

**Corollary 5.3.4.** *For a complex vector bundle $\pi : E \to X$ over a compact Hausdorff space $X$, there is an associated space $F(E)$ and map $f : F(E) \to X$ such that*

*(i) the vector bundle $f^*(E)$ is a direct sum of complex line bundles, and*

*(ii) the map $f^* : K^*(X) \to K^*(F(E))$ is injective.*

*Furthermore, the map $f : F(E) \to X$ is the same as that of Theorem 2.6.1.*

By the projective bundle formula we must be able to express $H_E^d$ as a $K^0(X)$-linear combination of the classes $1, H_E, H_E^2, \ldots, H_E^{d-1}$. In the analogous expression in the case of cohomology, expressing $x_E^d$ in terms of $1, x_E, x_E^2, \ldots, x_E^{d-1}$, the coefficients were tautologically given by the Chern classes (see Definition 2.3.1). In this case we will show beelow the coefficients have a simple expression in terms of exterior powers (see Section 1.1.3).

To manipulate exterior powers it is convenient to package them all together as the formal power series

$$\Lambda_t(E) = \sum_{i=0}^{\infty} \Lambda^k(E) \cdot t^k \in K^0(X)[[t]].$$

This satisfies $\Lambda_t(E \oplus F) = \Lambda_t(E) \cdot \Lambda_t(F)$ by the exponential property of the exterior algebra. As the coefficient of $1 = t^0$ in $\Lambda_t(E)$ is 1, a unit, this formal power series has a multiplicative inverse. If we let

$$\Lambda_t(E - F) = \frac{\Lambda_t(E)}{\Lambda_t(F)} \in K^0(X)[[t]]$$

then this defines a natural function $\Lambda_t : K^0(X) \to K^0(X)[[t]]$. It lands in the units of the ring $K^0(X)[[t]]$, and defines a homomorphism

$$\Lambda_t : (K^0(X), +, 0) \longrightarrow (K^0(X)[[t]]^\times, \times, 1).$$

**Theorem 5.3.5.** *In the setting of Theorem 5.3.2 the identity*

$$\sum_{i=0}^{d} (-1)^i p^*(\Lambda^i(\overline{E})) \cdot H_E^{d-i} = 0 \in K^0(\mathbb{P}(E))$$

*holds.*

*Proof.* We have an inclusion of complex vector bundles $L_E \to p^*(E)$ over $\mathbb{P}(E)$, and so choosing a Hermitian inner product on $p^*(E)$ we can write $p^*(E) = L_E \oplus W$ for some $(d-1)$-dimensional vector bundle $W$. Thus we have

$$p^*(\Lambda_t(E)) = (1 + L_E t) \cdot \Lambda_t(W)$$

and so rearranging gives $\Lambda_t(W) = p^*\left(\sum_{i=0}^{\infty} \Lambda^k(E) \cdot t^k\right) \cdot (1 - L_E t + L_E^2 t^2 - \cdots)$. Extracting the coefficient of $t^d$ gives

$$\Lambda^d W = \sum_{i=0}^{d} p^*(\Lambda^i(E)) \cdot (-L_E)^{d-i}.$$

But the vector bundle $W$ is $(d-1)$-dimensional so $\Lambda^d(W) = 0$. Taking complex conjugates gives the identity. □

## 5.4 The $K$-theory Thom isomorphism, Euler class, and Gysin sequence

If $\pi : E \to X$ is a complex vector bundle then $E/E^\#$ is not Hausdorff. As we have defined relative $K$-theory of a space $X$ and a subspace $A$ to be $\tilde{K}^0(X/A)$, and $X/A$ needs to be compact Hausdorff, this means that we cannot make sense of "the relative $K$-theory of the pair $E \supset E^\#$". Instead, if $\pi : E \to X$ is a complex vector bundle over a compact Hausdorff base $X$ then we may choose a (Hermitian) inner product on $E$ and define the unit disc and sphere bundles as

$$\mathbb{D}(E) = \{v \in E \mid \langle v, v\rangle \leq 1\}$$
$$\mathbb{S}(E) = \{v \in E \mid \langle v, v\rangle = 1\}.$$

We then define the *Thom space* $Th(E)$ of $E$ to be the quotient space $\mathbb{D}(E)/\mathbb{S}(E)$. This is again compact and Hausdorff. We will discuss the Thom isomorphism in terms of this space. We already mentioned in Remark 2.1.1 that $H^i(E, E^\#; R) \xrightarrow{\sim} H^i(\mathbb{D}(E), \mathbb{S}(D); R)$, and by excision the latter is $\tilde{H}^i(Th(E); R)$.

The following theorem gives the existence of a theory of Thom classes for complex vector bundles.

**Theorem 5.4.1.** *To each complex vector bundle $\pi : E \to X$ over a compact Hausdorff base there is associated a class $\lambda_E \in \tilde{K}^0(Th(E))$ such that:*

*(i) The map $\Phi : K^0(X) \xrightarrow{\sim} K^0(\mathbb{D}(E)) \xrightarrow{\lambda_E \cdot -} \tilde{K}^0(Th(E))$ is an isomorphism.*

*(ii) If $f : X' \to X$ is a map and $E' = f^*(E)$, with induced map $Th(f) : Th(E') \to Th(E)$, then*

$$Th(f)^*(\lambda_E) = \lambda_{E'} \in \tilde{K}^0(Th(E')).$$

*(iii) If $X = *$ then $\lambda \in \tilde{K}^0(Th(\mathbb{C}^n)) = \tilde{K}^0(S^{2n})$ is the generator $(H-1)^{\boxtimes n}$.*

*Proof.* Consider the inclusion $\mathbb{P}(E) \to \mathbb{P}(E \oplus \underline{\mathbb{C}}_X)$. The map

$$\begin{aligned} E &\longrightarrow \mathbb{P}(E \oplus \underline{\mathbb{C}}_X) \\ v &\longmapsto [v, 1] \end{aligned}$$

is a homeomorphism onto the complement of $\mathbb{P}(E)$, which gives an identification of $\mathbb{P}(E \oplus \underline{\mathbb{C}}_X)/\mathbb{P}(E)$ with the 1-point compactification $E^+$. Choosing a homeomorphism $[0, 1) \cong [0, \infty)$, we get a radial homeomorphism $Th(E) \cong E^+$. Thus there is an exact sequence

$$\begin{array}{ccccc} K^0(\mathbb{P}(E)) & \xleftarrow{\;i^*\;} & K^0(\mathbb{P}(E \oplus \underline{\mathbb{C}}_X)) & \xleftarrow{\;q^*\;} & \tilde{K}^0(Th(E)) \\ \Big\downarrow \partial & & & & \Big\uparrow \partial \\ \tilde{K}^{-1}(Th(E)) & \xrightarrow{\;q^*\;} & K^{-1}(\mathbb{P}(E \oplus \underline{\mathbb{C}}_X)) & \xrightarrow{\;i^*\;} & K^{-1}(\mathbb{P}(E)). \end{array}$$

By the projective bundle formula the two maps labelled $i^*$ are surjective, so writing $H$ for both $H_{E\oplus\mathbb{C}}$ and $H_E = i^*(H_{E\oplus\mathbb{C}})$, $L$ for both $L_{E\oplus\mathbb{C}}$ and $L_E = i^*(L_{E\oplus\mathbb{C}})$, and $n$ for the dimension of $E$, we have

$$\tilde{K}^0(Th(E)) = Ker\left( \frac{K^0(X)[H]}{(\sum_{i=0}^{n+1}(-1)^i \Lambda^i(\overline{E} \oplus \mathbb{C}) \cdot H^{n+1-i})} \to \frac{K^0(X)[H]}{(\sum_{i=0}^{n}(-1)^i \Lambda^i(\overline{E}) \cdot H^{n-i})} \right).$$

Using that $H^{-1} = L$, the generator for the ideal on the left is

$$H^{n+1} \cdot \sum_{i=0}^{n+1} (-1)^i \Lambda^i(\overline{E} \oplus \mathbb{C}) \otimes H^{-i} = H^{n+1} \cdot \Lambda_{-L}(\overline{E} \oplus \mathbb{C}) = H^{n+1} \cdot \Lambda_{-L}(\overline{E}) \cdot (1 - L)$$

and that for the ideal on the right is $H^n \cdot \Lambda_{-L}(\overline{E})$. As $H$ is a unit, the class $\Lambda_{-L}(\overline{E})$ lies in the kernel of $i^*$, and we define $\lambda_E \in \tilde{K}^0(Th(E))$ to be the unique class mapping to $\Lambda_{-L}(\overline{E})$ under $q^*$.

Under these identifications the proposed Thom isomorphism map is

$$\Phi : K^0(X) \xrightarrow{\sim} \frac{K^0(X)[L]}{(1 - L)} \underset{\sim}{\xrightarrow{\lambda_E \cdot -}} \frac{(\lambda_E)}{(\lambda_E \cdot (1 - L))} = Ker(i^*) = \tilde{K}^0(Th(E))$$

so is an isomorphism as required. Property (ii) holds as the construction is natural in $E$. For property (iii), when the bundle is $E = \mathbb{C}^n \to *$ we have $\mathbb{P}(E \oplus \mathbb{C}) = \mathbb{CP}^n$ and the degree 1 map $q : \mathbb{CP}^n \to Th(\mathbb{C}^n) = S^{2n}$ comes from collapsing $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$. Then $\lambda \in \tilde{K}^0(S^{2n})$ is characterised by satisfying $q^*\lambda = \Lambda_{-L}(\mathbb{C}^n) = (\Lambda_{-1}(\mathbb{C}))^n = (1 - L)^n \in K^0(\mathbb{CP}^n)$. Taking Chern characters gives $q^*ch(\lambda) = (1 - e^x)^n = (-x)^n$, and as $\langle(-x)^n, [\mathbb{CP}^n]\rangle = 1$ it follows that $\langle ch(\lambda), [S^{2n}]\rangle = 1$ too. By the proof of Lemma 5.2.1 it follows that $\lambda = (H - 1)^{\boxtimes n}$. ☐

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The inclusion $s : X \to E$ as the zero section extends to a based map $s : X_+ \to Th(E)$ from $X$ with a disjoint basepoint added, giving a map

$$s^* : \tilde{K}^0(Th(E)) \longrightarrow \tilde{K}^0(X_+) \cong K^0(X),$$

and, by analogy with characteristic classes in cohomology, the *$K$-theory Euler class* is

$$e^K(E) := s^*(\lambda_E) \in K^0(X).$$

**Lemma 5.4.2.** *We have* $e^K(E) = \Lambda_{-1}(\overline{E}) = \sum_{i=0}^n (-1)^i \Lambda^i(\overline{E}) \in K^0(X)$.

*Proof.* Under the identifications in the proof of Theorem 5.4.1, the zero section factors through the section

$$s' : X \longrightarrow \mathbb{P}(E \oplus \mathbb{C})$$

where $s'(x)$ is the line $\{0\} \oplus \mathbb{C} \subset E_x \oplus \mathbb{C}$. This satisfies $(s')^*(L_{E\oplus\mathbb{C}}) = \underline{\mathbb{C}}_X$, whence the claim follows from the formula for $\lambda_E$. □

The long exact sequence for the pair $(\mathbb{D}(E), \mathbb{S}(E))$, combined with $\mathbb{D}(E) \simeq X$ and the Thom isomorphism, gives the *$K$-theory Gysin sequence*

$$\begin{array}{ccccc}
K^0(\mathbb{S}(E)) & \xleftarrow{p^*} & K^0(X) & \xleftarrow{e^K(E)\cdot -} & K^0(X) \\
\big\downarrow{\scriptstyle p_!} & & & & \big\uparrow{\scriptstyle p_!} \\
K^{-1}(X) & \xrightarrow{e^K(E)\cdot -} & K^{-1}(X) & \xrightarrow{p^*} & K^{-1}(\mathbb{S}(E)).
\end{array}$$

where $p : \mathbb{S}(E) \to X$, and $p_!$ is simply a name for the connecting homomorphisms

$$K^i(\mathbb{S}(E)) \xrightarrow{\partial} \tilde{K}^{i+1}(\mathbb{D}(E)/\mathbb{S}(E)) \cong \tilde{K}^{i+1}(Th(E)) \xrightarrow{\Phi^{-1}} K^{i+1}(X).$$

## 5.5 $K$-theory of $\mathbb{RP}^n$

We will now compute the $K$-theory of $\mathbb{RP}^n$, albeit in a somewhat indirect way. Let us write $\nu := [\gamma_{\mathbb{R}}^{1,n+1} \otimes_{\mathbb{R}} \mathbb{C}] - 1 \in K^0(\mathbb{RP}^n)$.

**Theorem 5.5.1.** *We have*

$$K^0(\mathbb{RP}^{2n+1}) = \mathbb{Z}\{1\} \oplus \mathbb{Z}/2^n\{\nu\} \qquad K^{-1}(\mathbb{RP}^{2n+1}) = \mathbb{Z}$$

*with* $\nu^2 = -2\nu$.

*We have*

$$K^0(\mathbb{RP}^{2n}) = \mathbb{Z}\{1\} \oplus \mathbb{Z}/2^n\{\nu\} \qquad K^{-1}(\mathbb{RP}^{2n}) = 0$$

*with* $\nu^2 = -2\nu$.

Let us begin the proof of Theorem 5.5.1 for odd-dimensional projective spaces.

**Lemma 5.5.2.** *There is a homeomorphism* $\mathbb{RP}^{2n+1} \cong \mathbb{S}(\gamma_{\mathbb{C}}^{1,n+1} \otimes \gamma_{\mathbb{C}}^{1,n+1})$*, under which the projection map* $p : \mathbb{RP}^{2n+1} \to \mathbb{CP}^n$ *pulls back* $\gamma_{\mathbb{C}}^{1,n+1}$ *to* $\gamma_{\mathbb{R}}^{1,2n+2} \otimes_{\mathbb{R}} \mathbb{C}$.

*Proof.* Consider the map $\psi : S^{2n+1} \to \mathbb{S}(\gamma_{\mathbb{C}}^{1,n+1} \otimes \gamma_{\mathbb{C}}^{1,n+1})$ which sends $x \in S^{2n+1} \subset \mathbb{C}^{n+1}$ to

$$x \otimes x \in \langle x \rangle_{\mathbb{C}} \otimes_{\mathbb{C}} \langle x \rangle_{\mathbb{C}}.$$

This satisfies $\psi(-x) = \psi(x)$ and so induces a continuous map $\bar{\psi} : \mathbb{RP}^{2n+1} \to \mathbb{S}(\gamma_{\mathbb{C}}^{1,n+1} \otimes \gamma_{\mathbb{C}}^{1,n+1})$. Given a line $\ell \in \mathbb{CP}^n$ and a vector $z \in \ell$, the 1-dimensional space $\ell \otimes_{\mathbb{C}} \ell$ is spanned by $z \otimes z$, so $\bar{\psi}$ is onto. If $\psi(x) = \psi(y)$ then $\langle x \rangle_{\mathbb{C}} = \langle y \rangle_{\mathbb{C}}$ so $y = \lambda z$ for some

$\lambda \in \mathbb{C}^\times$, but also $x \otimes x = y \otimes y = \lambda^2 x \otimes x$ and so $\lambda^2 = 1$. Thus $\lambda = \pm 1$, and so $y = \pm x$. This $\bar{\psi}$ is injective too. Thus it is a continuous bijection, and so a homeomorphism.

The map $p$ sends $\langle x \rangle_\mathbb{R}$ to $\langle x \rangle_\mathbb{C}$. Thus the fibre of $p^*(\gamma_\mathbb{C}^{1,n+1})$ at $\ell \in \mathbb{RP}^{2n+1}$ is given by the (unique) complex line in $\mathbb{C}^{n+1}$ containing $\ell$. This complex line is $\ell \oplus i\ell$, the complexification of $\ell$, so $p^*(\gamma_\mathbb{C}^{1,n+1})$ can be identified with $\gamma_\mathbb{R}^{1,2n+2} \otimes_\mathbb{R} \mathbb{C}$. □

We have $\Lambda_{-1}(L^2) = 1 - \overline{L}^2 = 1 - H^2 \in K^0(\mathbb{CP}^n)$, so applying the $K$-theory Gysin sequence to $\gamma_\mathbb{C}^{1,n+1} \otimes \gamma_\mathbb{C}^{1,n+1} \to \mathbb{CP}^n$ gives

$$\begin{array}{ccccc} K^0(\mathbb{RP}^{2n+1}) & \xleftarrow{p^*} & K^0(\mathbb{CP}^n) & \xleftarrow{(1-H^2)\cdot -} & K^0(\mathbb{CP}^n) \\ \downarrow{\scriptstyle p_!} & & & & \uparrow{\scriptstyle p_!} \\ K^{-1}(\mathbb{CP}^n) & \xrightarrow{(1-H^2)\cdot -} & K^{-1}(\mathbb{CP}^n) & \xrightarrow{p^*} & K^{-1}(\mathbb{RP}^{2n+1}). \end{array}$$

Using that $K^{-1}(\mathbb{CP}^n) = 0$ this gives an exact sequence

$$0 \to K^{-1}(\mathbb{RP}^{2n+1}) \overset{p_!}{\to} \mathbb{Z}[H]/((H-1)^{n+1}) \overset{(1-H^2)\cdot -}{\longrightarrow} \mathbb{Z}[H]/((H-1)^{n+1}) \overset{p^*}{\to} K^0(\mathbb{RP}^{2n+1}) \to 0$$

The kernel of the middle map consists of those polynomials $p(H)$ such that $(H-1)^{n+1} \mid (H-1)(H+1)p(H)$, i.e. $(H-1)^n \mid (H+1)p(H)$, i.e. $(H-1)^n \mid p(H)$, so

$$K^{-1}(\mathbb{RP}^{2n+1}) \cong \mathbb{Z}.$$

On the other hand as $p^*(H) = \nu + 1$, we have

$$K^0(\mathbb{RP}^{2n+1}) \cong \mathbb{Z}[\nu]/(\nu^{n+1}, (1-(\nu+1)^2)).$$

The second relation gives $\nu^2 = -2\nu$, using which we can re-write the first relation as $2^n \nu = 0$. Thus

$$K^0(\mathbb{RP}^{2n+1}) \cong \mathbb{Z}\{1\} \oplus \mathbb{Z}/2^n\{\nu\}$$

where the ring structure is determined by $\nu^2 = -2\nu$.

For even-dimensional projective spaces we can first consider the inclusion $\mathbb{RP}^{2n-1} \to \mathbb{RP}^{2n}$ with $\mathbb{RP}^{2n}/\mathbb{RP}^{2n-1} \cong S^{2n}$, giving an exact sequence

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & \tilde{K}^0(\mathbb{RP}^{2n}) & \longrightarrow & \mathbb{Z}/2^{n-1} \\ \uparrow{\scriptstyle \partial} & & & & \downarrow \\ \mathbb{Z} & \longleftarrow & \tilde{K}^{-1}(\mathbb{RP}^{2n}) & \longleftarrow & 0. \end{array}$$

We can then consider the inclusion $\mathbb{RP}^{2n} \to \mathbb{RP}^{2n+1}$ with $\mathbb{RP}^{2n+1}/\mathbb{RP}^{2n} \cong S^{2n+1}$, giving an exact sequence

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathbb{Z}/2^n & \longrightarrow & \tilde{K}^0(\mathbb{RP}^{2n}) \\ \uparrow{\scriptstyle \partial} & & & & \downarrow \\ \tilde{K}^{-1}(\mathbb{RP}^{2n}) & \longleftarrow & \mathbb{Z} & \longleftarrow & \mathbb{Z}. \end{array}$$

By the first diagram $\tilde{K}^{-1}(\mathbb{RP}^{2n})$ is torsion-free, and by the second it is cyclic, so it is 0 or $\mathbb{Z}$. In the latter case we have a short exact sequence

$$0 \longrightarrow \mathbb{Z}/2^n \longrightarrow \tilde{K}^0(\mathbb{RP}^{2n}) \longrightarrow \mathbb{Z} \longrightarrow 0$$

which must be split, and so $2^{n-1}\tilde{K}^0(\mathbb{RP}^{2n}) \cong \mathbb{Z} \oplus \mathbb{Z}/2$. But by the first sequence $2^{n-1}\tilde{K}^0(\mathbb{RP}^{2n})$ is a cyclic group, a contradiction. Thus $\tilde{K}^{-1}(\mathbb{RP}^{2n}) = 0$, so by the second diagram we have $\tilde{K}^0(\mathbb{RP}^{2n}) = \mathbb{Z}/2^n$. Furthermore, we see that this is generated by the pullback of $\nu \in \tilde{K}^0(\mathbb{RP}^{2n+1})$, which is the class also called $\nu$ in $\tilde{K}^0(\mathbb{RP}^{2n})$, as required. Lecture 21

## 5.6 Adams operations

We have seen that the Chern classes in $K$-theory are given by the exterior powers $\Lambda^k(E)$. These define functions $\Lambda^k : Vect(X) \to K^0(X)$, but they are not additive so are difficult to work with algebraically. One solution, which we have already used, is to consider the total exterior power $\Lambda_t : K^0(X) \to K^0(X)[[t]]^\times$, which sends addition to multiplication (of units). Another is to proceed similarly to the Chern character.

**Theorem 5.6.1.** *There are natural ring homomorphisms* $\psi^k : K^0(X) \to K^0(X)$ *for* $k \in \mathbb{N}$ *satisfying*

*(i)* $\psi^k(L) = L^k$ *if* $L \to X$ *is a line bundle,*

*(ii)* $\psi^k \circ \psi^l = \psi^{kl}$,

*(iii) for* $p$ *any prime number,* $\psi^p(x) = x^p \mod p$.

*Proof.* If $E \to X$ is a complex vector bundle which is a sum $L_1 \oplus \cdots \oplus L_n$ of complex line bundles, then we are obliged to have $\psi^k(E) = L_1^k + \cdots + L_n^k$. By the Fundamental Theorem of Symmetric Polynomials we can write $p_k(x_1, \ldots, x_n) = \sum x_i^k$ as $\bar{p}_k(e_1(x_1, \ldots, x_n), \ldots, e_k(x_1, \ldots, x_n))$ in terms of the elementary symmetric polynomials. Thus

$$\psi^k(E) = \bar{p}_k(e_1(L_1, \ldots, L_n), \ldots, e_k(L_1, \ldots, L_n)).$$

Now

$$\begin{aligned}\Lambda_t(L_1 \oplus \cdots \oplus L_n) &= \Lambda_t(L_1) \cdots \Lambda_t(L_n) = (1 + L_1 t) \cdots (1 + L_n t) \\ &= \sum_{i=0}^{n} e_i(L_1, \ldots, L_n) t^i\end{aligned}$$

so $e_i(L_1, \ldots, L_n) = \Lambda^i(L_1 \oplus \cdots \oplus L_n)$, and hence

$$\psi^k(E) = \bar{p}_k(\Lambda^1(E), \ldots, \Lambda^k(E)). \tag{5.6.1}$$

By the splitting principle in $K$-theory this must hold for all vector bundles.

This proves the uniqueness of such operations, and we can use the formula (5.6.1) to attempt to define them (as $\psi^k : Vect(X) \to K^0(X)$, extended to the Grothendieck completion). By the splitting principle, the operations so defined are additive, satisfy

$\psi^k(L) = L^k$ if $L$ is a line bundle, and satisfy $\psi^k \circ \psi^l = \psi^{kl}$. To check that they respect the multiplication, again by the splitting principle it is enough to check on $L_1 \otimes L_2$ the tensor product of two line bundles. But $\psi^k$ sends this to $(L_1 \otimes L_2)^k \cong L_1^k \otimes L_2^k = \psi^k(L_1)\psi^k(L_2)$ as required.

For the final property, on a complex vector bundle $E$ the claim is equivalent to saying that

$$\bar{p}_p(e_1, \ldots, e_p) - e_1^p \in p\mathbb{Z}[e_1, \ldots, e_n],$$

or in other words that $x_1^p + \cdots + x_n^p - (x_1 + \cdots + x_n)^p \in p\mathbb{Z}[x_1, \ldots, x_n]$. This holds as when $(x_1 + \cdots + x_n)^p$ is expanded out all terms apart from $x_1^p + \cdots + x_n^p$ have a multinomial coefficient $\binom{p}{k_1, k_2, \ldots, k_n}$ with all $k_i < p$, which is divisible by $p$. Now on a general $K$-theory class $E - F$ on $X$ we have

$$\psi^p(E - F) - (E - F)^p = (\psi^p(E) - E^p) - (\psi^p(F) - F^p) - \sum_{i=1}^{p-1} \binom{p}{i} E^i(-F)^{p-i}$$

and each term on the right-hand side is divisible by $p$. □

As these operations are natural, if $X$ is a space with a basepoint then they also induce operations $\psi^k : \tilde{K}^0(X) \to \tilde{K}^0(X)$ on reduced zeroth $K$-theory. Using these and $\tilde{K}^{-1}(X) = \tilde{K}^0(\Sigma X)$ we get operations on reduced $(-1)$st $K$-theory. Using $K^{-1}(X) = \tilde{K}^{-1}(X_+)$ we get operations on unreduced $(-1)$st $K$-theory.

**Lemma 5.6.2.** *The action of $\psi^k$ on $\tilde{K}^0(S^{2n}) \cong \mathbb{Z}$ is by multiplication by $k^n$. The action of $\psi^k$ on $\tilde{K}^{-1}(S^{2n+1}) \cong \mathbb{Z}$ is by multiplication by $k^{n+1}$.*

*Proof.* When $n = 1$ we identify $S^2 = \mathbb{CP}^1$, then this group is generated by $x = H - 1$, which satisfies $x^2 = 0$. Now

$$\psi^k(x) = H^k - 1 = (x + 1)^k - 1 = kx$$

so $\psi^k$ acts by multiplication by $k = k^1$.

We identified the generator of $\tilde{K}^0(S^{2n})$ as the $n$-fold external tensor power of $x$, and as $\psi^k$ is a ring homomorphism it commutes with external tensor powers too, so acts by $k^n$.

For $K^{-1}$ of odd-dimensional spheres, we have defined the action of $\psi^k$ via its natural action on $\tilde{K}^{-1}(S^{2n+1}) = \tilde{K}^0(\Sigma S^{2n+1}) = \tilde{K}^0(S^{2n+2})$, so the action is by multiplication by $k^{n+1}$. □

In particular, the Bott isomorphism does *not* commute with the $\psi^k$.

**Example 5.6.3.** In $\tilde{K}^0(\mathbb{RP}^n) = \mathbb{Z}/2^{\lfloor n/2 \rfloor}\{\nu\}$ with $\nu = [\gamma_{\mathbb{R}}^{1,n+1} \otimes_{\mathbb{R}} \mathbb{C}] - 1$ we have

$$\begin{aligned} \psi^k(\nu) &= \psi^k([\gamma_{\mathbb{R}}^{1,n+1} \otimes_{\mathbb{R}} \mathbb{C}] - 1) = [\gamma_{\mathbb{R}}^{1,n+1} \otimes_{\mathbb{R}} \mathbb{C}]^k - 1 \\ &= (1 + \nu)^k - 1. \end{aligned}$$

The polynomial $g(t) = (1+t)^k - 1 \in \mathbb{Z}[t]$ is divisible by $t$, so and we can write it as $g(t) = t \cdot f(t)$ with

$$f(t) = \frac{(1+t)^k - 1}{t} = kt + \binom{k}{2} t^2 + \cdots \in \mathbb{Z}[t].$$

As $\nu^2 = -2\nu \in \tilde{K}^0(\mathbb{RP}^n)$, we have $\psi^k(\nu) = g(\nu) = \nu \cdot f(\nu) = \nu \cdot f(-2)$, so

$$\psi^k(\nu) = \nu \cdot \frac{(1-2)^k - 1}{-2} = \begin{cases} \nu & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even.} \end{cases}$$

To see how $\psi^k$ acts on $\tilde{K}^{-1}(\mathbb{RP}^{2n+1}) \cong \mathbb{Z}$ we can consider the long exact sequence for the pair $(\mathbb{RP}^{2n+1}, \mathbb{RP}^{2n})$, where we have shown that the collapse map $c : \mathbb{RP}^{2n+1} \to \mathbb{RP}^{2n+1}/\mathbb{RP}^{2n} = S^{2n+1}$ is an isomorphism on $\tilde{K}^{-1}$. Thus $\psi^k$ acts by multiplication by $k^{n+1}$.

## 5.7 The Hopf invariant

Given a map $f : S^{4n-1} \to S^{2n}$, we may form a space $X = X_f = S^{2n} \cup_f D^{4n}$. Write $i : S^{2n} \to X$ for the inclusion, and $c : X \to S^{4n}$ for the map which collapses down $S^{2n} \subset X$. We have

$$H^i(X; \mathbb{Z}) = \begin{cases} \mathbb{Z}\{1\} & i = 0 \\ \mathbb{Z}\{a\} & i = 2n \\ \mathbb{Z}\{b\} & i = 4n \\ 0 & \text{else} \end{cases}$$

where, if $u_k \in H^k(S^k; \mathbb{Z})$ is the standard generator, $i^*(a) = u_{2n}$ and $b = c^*(u_{4n})$. We then have $a \smile a = h(f)b$ for some $h(f) \in \mathbb{Z}$ called the *Hopf invariant* of $f$.

**Example 5.7.1.** The spaces $\mathbb{CP}^2$ and $\mathbb{HP}^2$ are of the form $S^{2n} \cup_f D^{4n}$, and the fact that they are manifolds shows, via Poincaré duality, that $h(f) = \pm 1$ for these $f$'s.

We can mimic this construction in $K$-theory. The usual exact sequence on $K$-theory gives

$$0 \longrightarrow \tilde{K}^0(S^{4n}) \xrightarrow{c^*} \tilde{K}^0(X) \xrightarrow{i^*} \tilde{K}^0(S^{2n}) \longrightarrow 0,$$

as $\tilde{K}^{-1}(S^{2k}) = 0$. The standard generator of $\tilde{K}^0(S^{4n})$ is the exterior power $(H-1)^{\boxtimes 2n}$, and we can write $B \in \tilde{K}^0(X)$ for its image under $c^*$. The standard generator of $\tilde{K}^0(S^{2n})$ is the exterior power $(H-1)^{\boxtimes n}$, and we can write $A \in \tilde{K}^0(X)$ for *some choice of preimage* under $i^*$. As $i^*(A^2) = i^*(A)^2 = ((H-1)^{\boxtimes n})^2 = 0 \in \tilde{K}^0(S^{2n})$ we must have $A^2 = h^K(f)B$ for some $h^K(f) \in \mathbb{Z}$.

**Lemma 5.7.2.** *We have $h^K(f) = h(f)$. (In particular $h^K(f)$ does not depend on the choice of preimage $A$.)*

*Proof.* We have $ch(B) = b$ and $ch(A) = a + qb$ for some $q \in \mathbb{Q}$. Thus

$$ch(A^2) = ch(A)^2 = (a + qb)^2 = a^2 + 2qab + q^2b^2 = a^2 = h(f)b = ch(h(f)B)$$

using that $ab = b^2 = 0$ for degree reasons, and so $A^2 = h(f)B$ as $\tilde{K}^0(X)$ is torsion-free so $ch$ is injective. □

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**Theorem 5.7.3** (Hopf invariant 1 Theorem)**.** *If $f : S^{4n-1} \to S^{2n}$ has odd Hopf invariant then $2n = 2, 4, 8$.*

*Proof.* We apply Adams operations in $\tilde{K}^0(X)$. We have

$$\psi^k(B) = \psi^k(c^*((H-1)^{\boxtimes 2n})) = k^{2n}B$$

as $\psi^k$ acts by scalar multiplication by $k^{2n}$ on $\tilde{K}^0(S^{4n})$. Similarly, we have

$$\psi^k(A) = k^n A + \sigma(k)B$$

for some $\sigma(k) \in \mathbb{Z}$.

Compute

$$\begin{aligned}\psi^2\psi^3(A) &= \psi^2(3^n A + \sigma(3)B)\\ &= 3^n(2^n A + \sigma(2)B) + \sigma(3)2^{2n}B\end{aligned}$$

and

$$\begin{aligned}\psi^3\psi^2(A) &= \psi^3(2^n A + \sigma(2)B)\\ &= 2^n(3^n A + \sigma(3)B) + \sigma(2)3^{2n}B\end{aligned}$$

so as $\psi^2\psi^3 = \psi^6 = \psi^3\psi^2$ identifying coefficients of $B$ gives

$$2^n(2^n - 1)\sigma(3) = 3^n(3^n - 1)\sigma(2).$$

Now $h(f)B = A^2 \equiv_2 \psi^2(A) = 2^n A + \sigma(2)B$ so $\sigma(2) \equiv 1 \mod 2$ if $h(f)$ is odd. Then $\sigma(2)$ and $3^n$ are odd so it follows that $2^n \mid (3^n - 1)$. The following number-theoretic result concludes the argument. □

**Lemma 5.7.4.** *If $2^n \mid (3^n - 1)$ then $n = 1, 2, 4$.*

*Proof.* Let $n = 2^k m$ with $m$ odd. We will show that the largest power of 2 dividing $3^n - 1$ is $2^1$ if $k = 0$ and $2^{k+2}$ if $k > 0$. The claim then follows: if $k = 0$ then we get $n \leq 1$, and if $k > 0$ then we get $n = 2^k m \leq k + 2$ which by an easy estimate implies $n \leq 4$. The case $n = 3$ can be excluded manually.

If $k = 0$, then $3^n \equiv (-1)^n \equiv -1 \mod 4$, so $3^n - 1 \equiv 2 \mod 4$ so $3^n - 1$ is divisible by precisely $2^1$.

If $k = 1$ then $3^{2m} - 1 = (3^m - 1)(3^m + 1)$. As $3^2 = 9 \equiv 1 \mod 8$, we have $3^m \equiv 3 \mod 8$ so $3^m + 1 \equiv 4 \mod 8$, so $4 = 2^2$ is the largest power of 2 dividing the second factor; we already saw that $2^1$ is the largest power of 2 dividing $3^m - 1$, so $2^3$ is the largest power of 2 dividing the first factor.

If $k > 1$ then $3^{2^k m} - 1 = (3^{2^{k-1}m} - 1)(3^{2^{k-1}m} + 1)$. The largest power of 2 dividing the first factor is $2^{k+1}$ by inductive assumption. As $2^{k-1}m$ is even, we have $3^{2^{k-1}m} \equiv (-1)^{2^{k-1}m} \equiv 1 \mod 4$ so $3^{2^{k-1}m} + 1 \equiv 2 \mod 4$ so $2^1$ is the largest power of 2 dividing the second factor. □

## 5.8 Correction classes

If $\pi : E \to X$ is a $n$-dimensional complex vector bundle then we have produced a Thom class $\lambda_E \in \tilde{K}^0(Th(E))$ in $K$-theory. Furthermore, as a real vector bundle it is $\mathbb{Z}$-oriented so we also have a Thom class $u_E \in \tilde{H}^{2n}(Th(E);\mathbb{Z})$ in cohomology. We can make the following two constructions:

(i) Define a sequence of cohomology classes, the *total Todd class*,

$$Td(E) = Td_0(E) + Td_1(E) + \cdots \in H^*(X;\mathbb{Q})$$

by the formula $ch(\lambda_E) = Td(E) \cdot u_E \in \tilde{H}^*(Th(E);\mathbb{Q})$.

(ii) Define a $K$-theory class, the $k$th *cannibalistic class*,

$$\rho^k(E) \in K^0(X)$$

by the formula $\psi^k(\lambda_E) = \rho^k(E) \cdot \lambda_E \in \tilde{K}^0(Th(E))$.

In both cases these classes measure by how far some natural transformation of cohomology theories, $ch$ and $\psi^k$, fail to commute with the Thom isomorphism. In this section we will analyse how to compute these invariants.

### 5.8.1 The Todd class

We wish to find a formula for $Td(E) \in H^*(X;\mathbb{Q})$ in terms of the Chern classes of $E$. By considering the vector bundle $\mathbb{C}^n \to *$, where $Th(\mathbb{C}^n) = S^{2n}$ and the Thom class is $(H-1)^{\boxtimes n} \in \tilde{K}^0(S^{2n})$, we see that $Td_0(E)$ is non-zero for any vector bundle $E \to X$, so $Td(E)$ is a unit.

Recall that the cohomological Thom class $u_E$ satisfies $u_E \cdot u_E = e(E) \cdot u_E \in \tilde{H}^*(Th(E))$; the same argument shows that $\lambda_E \cdot \lambda_E = \Lambda_{-1}(\overline{E}) \cdot \lambda_E$. Taking the Chern character, this gives

$$Td(E) \cdot Td(E) \cdot u_E \cdot u_E = ch(\Lambda_{-1}(\overline{E})) \cdot Td(E) \cdot u_E$$

and so $Td(E)^2 \cdot e(E) = ch(\Lambda_{-1}(\overline{E})) \cdot Td(E) \in H^*(X;\mathbb{Q})$. As the total Todd class is a unit, we find that

$$Td(E) \cdot e(E) = ch(\Lambda_{-1}(\overline{E})) \in H^*(X;\mathbb{Q}).$$

We will use this, the splitting principle, and the fact that complex line bundles are pulled back from a complex projective space, to describe the Todd class in general. Let us write $Q(t) := \frac{1-\exp(-t)}{t} \in \mathbb{Q}[[t]]$.

**Lemma 5.8.1.** *The Todd class satisfies $Td(E \oplus E') = Td(E) \cdot Td(E')$, and if $L \to X$ is a complex line bundle then $Td(L) = Q(c_1(L)) \in H^*(X;\mathbb{Q})$.*

*Proof.* First consider the tautological line bundle $L = \gamma_{\mathbb{C}}^{1,N+1} \to \mathbb{CP}^N$. Then $\Lambda_{-1}(\overline{L}) = 1 - \overline{L}$, so the formula above becomes

$$Td(L) \cdot x = 1 - \exp(-x) \in H^*(\mathbb{CP}^N;\mathbb{Q}) = \mathbb{Q}[x]/(x^{N+1}).$$

It follows that

$$Td(L) = Q(x) + A \cdot x^N$$

for some $A \in \mathbb{Q}$. But the formula must be natural for inclusions $\mathbb{CP}^N \subset \mathbb{CP}^{N'}$, so we must have $A = 0$. This shows that $Td(L) = Q(c_1(L))$, so the same holds for any line bundle by naturality.

To verify the formula $Td(E \oplus E') = Td(E) \cdot Td(E')$ we may suppose without loss of generality that both $E$ and $E'$ are sums of line bundles. But then, by naturality, we may as well suppose that $X = (\mathbb{CP}^N)^{n+m}$ and that $E = L_1 \boxplus \cdots \boxplus L_n$ is the external direct sum of the tautological line bundles over the first $n$ factors, and $E' = L_{n+1} \boxplus \cdots \boxplus L_{n+m}$ external direct sum of the tautological line bundles over the last $m$ factors. Write $x_i = c_1(L_i)$.

The formula above shows that

$$Td(E \oplus E') \cdot x_1 \cdots x_{n+m} = ch(\prod_{i=1}^{n+m} (1 - \overline{L_i})) = \prod_{i=1}^{n+m} (1 - \exp(-x_i))$$

which is also $Td(E) \cdot Td(E') \cdot x_1 \cdots x_{n+m}$. Therefore as above the difference $Td(E \oplus E') - Td(E) \cdot Td(E')$ lies in the ideal $(x_1^N, \ldots, x_{n+m}^N)$, but by naturality with respect to $N$ it follows that it must be zero. □

For a sum of line bundles $L_i \to X$ with $x_i := c_1(L_i)$ we therefore have

$$Td(L_1 \oplus \cdots \oplus L_n) = Q(x_1) \cdots Q(x_n)$$

and in each cohomological degree the right-hand side is a symmetric polynomial in the $x_i$. Thus we may write

$$Td_k(L_1 \oplus \cdots \oplus L_n) = \tau_k(e_1(x_1, \ldots, x_n), \ldots, e_k(x_1, \ldots, x_n))$$

for a unique $\tau_k \in \mathbb{Q}[e_1, \ldots, e_k]$. Hence by the splitting principle for any $n$-dimensional vector bundle $E \to X$ we have

$$Td_k(E) = \tau_k(c_1(E), \ldots, c_k(E)).$$

The first few polynomials $\tau_k$ are

$$\begin{aligned} \tau_0 &= 1 \\ \tau_1 &= \frac{-e_1}{2} \\ \tau_2 &= \frac{2e_1^2 - e_2}{12} \\ \tau_3 &= \frac{e_1 e_2 - e_1^3}{24} \end{aligned}$$

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There is a further corollary of this discussion. If $\pi : E \to X$ and $\pi' : E' \to X'$ are complex vector bundles then choosing Hermitian metrics on $E$ and $E'$ induces one on $E \boxplus E' \to X \times X'$, and there is a homeomorphism

$$\mathbb{D}(E \boxplus E') \approx \mathbb{D}(E) \times \mathbb{D}(E').$$

Under this homeomorphism, there is an identification

$$\mathbb{S}(E \boxplus E') \approx (\mathbb{S}(E) \times \mathbb{D}(E')) \cup_{\mathbb{S}(E)\times\mathbb{S}(E')} (\mathbb{D}(E) \times \mathbb{S}(E')).$$

which gives a homeomorphism

$$Th(E \boxplus E') \xrightarrow{\sim} Th(E) \wedge Th(E')$$

We can then form $\lambda_E \boxtimes \lambda_{E'} \in \tilde{K}^0(Th(E \boxplus E'))$. It is easy to check that this is *a* Thom class, in the sense that it restricts to a generator of the $K$-theory of each fibre, but we want to know that it is precisely *the* Thom class $\lambda_{E\boxplus E'}$.

**Corollary 5.8.2.** *We have* $\lambda_E \boxtimes \lambda_{E'} = \lambda_{E\boxplus E'} \in \tilde{K}^0(Th(E \boxplus E'))$.

*Proof.* As in the proof of Lemma 5.8.1 it is enough to establish this formula when $E$ and $E'$ are external products of the tautological line bundle over $\mathbb{CP}^N$, in which case

$$Th(E \boxplus E') = Th(L_1) \wedge \cdots \wedge Th(L_{n+m}).$$

The normal bundle of $\mathbb{CP}^N$ inside $\mathbb{CP}^{N+1}$ is given by $\overline{L}$, the complex conjugate of the tautological bundle. As a real bundle this is of course identified with $L$, so collapsing the (contractible) complement of a tubular neighbourhood of $\mathbb{CP}^N$ inside $\mathbb{CP}^{N+1}$ gives a homotopy equivalence

$$h : \mathbb{CP}^{N+1} \xrightarrow{\sim} Th(\overline{L}) = Th(L).$$

This only has even-dimensional cells, so $Th(L_1) \wedge \cdots \wedge Th(L_{n+m})$ does too: thus its $K$-theory is torsion-free, so the Chern character

$$ch : \tilde{H}^0(Th(E \boxplus E')) \longrightarrow H^{2*}((\mathbb{CP}^N)^{n+m}; \mathbb{Q})$$

is injective. This means that it is enough to verify the identity $\lambda_E \boxtimes \lambda_{E'} = \lambda_{E\boxplus E'}$ after applying the Chern character, but then it does indeed hold by multiplicativity of the Todd class and of the cohomology Thom class. □

### 5.8.2 The cannibalistic classes

By the multiplicative property of the $K$-theory Thom class, and of the Adams operations, we also have $\rho^k(E \oplus E') = \rho^k(E) \cdot \rho^k(E')$.

**Lemma 5.8.3.** *If* $E \to X$ *is a complex line bundle then* $\rho^k(E) = 1 + \overline{E} + \cdots + \overline{E}^{k-1}$.

*Proof.* The Thom class $\lambda_E$ is defined to be the class which pulls back to

$$\Lambda_{-L}(\overline{E}) = 1 - L\overline{E} \in K^0(\mathbb{P}(E \oplus \mathbb{C})) = K^0(X)[L]/((1-L)\Lambda_{-L}(\overline{E}))$$

along the quotient map $q : \mathbb{P}(E \oplus \mathbb{C}) \to Th(E)$. Thus $\psi^k(\lambda_E)$ pulls back to

$$\psi^k(1 - L\overline{E}) = 1 - (L\overline{E})^k = (1 + (L\overline{E}) + (L\overline{E})^2 + \cdots + (L\overline{E})^{k-1}) \cdot (1 - L\overline{E}),$$

which by the relation $(1-L)\Lambda_{-L}(\overline{E}) = 0$ agrees with

$$(1 + \overline{E} + \overline{E}^2 + \cdots + \overline{E}^{k-1}) \cdot (1 - L\overline{E}).$$

The second term is the pullback of $\lambda_E$ from the Thom space, so $\rho^k(E) = 1 + \overline{E} + \overline{E}^2 + \cdots + \overline{E}^{k-1}$ as claimed. □

If $E = L_1 \oplus \cdots \oplus L_n$ is a sum of line bundles we therefore have

$$\rho^k(E) = \prod_{i=1}^{n}(1 + \overline{L_i} + \cdots + \overline{L_i}^{k-1})$$

a symmetric polynomial in the $\overline{L_i}$. It can therefore be uniquely expressed as a polynomial in $e_j(L_1, \ldots, L_n) = \Lambda^j(\overline{E})$, as $\rho^k(E) = q_k(\Lambda^1\overline{E}, \ldots, \Lambda^n\overline{E})$.

For example $\rho^2(E) = \prod_{i=1}^n(1 + \overline{L_i}) = \sum_{i=1}^n \Lambda^i(\overline{E}) = \Lambda_1(\overline{E})$. As another example, if $E$ is 3-dimensional then

$$\begin{aligned}\rho^3(E) = 1 &+ \Lambda^1(\overline{E}) - \Lambda^2(\overline{E}) - 2\Lambda^3(\overline{E}) + (\Lambda^3(\overline{E}))^2 + \Lambda^3(\overline{E})\Lambda^2(\overline{E})\\ &+ (\Lambda^2(\overline{E}))^2 - \Lambda^3(\overline{E})\Lambda^1(\overline{E}) + \Lambda^1(\overline{E})\Lambda^2(\overline{E}) + (\Lambda^1(\overline{E}))^3.\end{aligned}$$

## 5.9 Gysin maps and topological Grothendieck–Riemann–Roch

Let $f : M \to N$ be a smooth map of manifolds. A *complex orientation* for $f$ is a lift to an embedding $\hat{f} = (e, f) : M \to \mathbb{R}^K \times N$, along with a complex structure on the normal vector bundle $\nu_{\hat{f}}$. Complex orientations are equivalent if they differ by (i) stabilising the bundle by $\underline{\mathbb{C}}_N$, or (ii) changing $\hat{f}$ by a homotopy of embeddings (an "isotopy"). If $\hat{f} \subset U \subset \mathbb{R}^K \times N$ is a tubular neighbourhood, collapsing its complement gives a map

$$c : S^K \wedge N_+ \longrightarrow U^+ \cong Th(\nu_{\hat{f}}).$$

Using the Thom isomorphism and suspension isomorphism we can then form the map

$$f_!^K : K^i(M) \cong \widetilde{K}^i(Th(\nu_{\hat{f}})) \xrightarrow{c^*} \widetilde{K}^i(S^K \wedge N_+) \cong K^{i-K}(N),$$

the $K$-theory Gysin—or pushforward—map associated with $f$. The number $K$ is not very intrinsic to the situation, but it satisfies $\dim(N) + K \equiv \dim(M) \mod 2$ because $\nu_{\hat{f}}$ has a complex structure so is even-dimensional, so we can write the target as $K^{i+\dim(N)-\dim(M)}(N)$. Then $f_!^K$ depends on the complex orientation of the map $f$, but only up to equivalence of complex orientations.

Similarly, using the Thom and suspension isomorphisms in cohomology we obtain the cohomological Gysin map

$$\begin{array}{ccc} H^i(M;R) & \xrightarrow{f_!^H} & H^{i+\dim(N)-\dim(M)}(N;R) \\ \| & & \| \\ \widetilde{H}^{i+2\dim(\nu_{\hat{f}})}(Th(\nu_{\hat{f}});R) & \xrightarrow{c^*} & \widetilde{H}^{i+2\dim(\nu_{\hat{f}})}(S^K \wedge N_+;R). \end{array}$$

If $M$ and $N$ have $R$-orientations (compatible with that of $\nu_{\hat{f}}$) then $f_!^H$ can alternatively be expressed in terms of Poincaré duality on $M$ and $N$ as

$$\begin{array}{ccc} H^i(M;R) & \xrightarrow{f_!^H} & H^{i+\dim(N)-\dim(M)}(N;R) \\ \sim\downarrow [M]\frown- & & \sim\downarrow [N]\frown- \\ H_{\dim(M)-i}(M;R) & \xrightarrow{f_*} & H_{\dim(M)-i}(N;R). \end{array}$$

**Theorem 5.9.1** (Topological Grothendieck–Riemann–Roch)**.** *We have*

$$ch(f_!^K(x)) = f_!^H(ch(x) \cdot Td(\nu_{\hat{f}})) \in H^{2*}(N;\mathbb{Q}).$$

*Proof.* We simply chase through the isomorphisms in the definition. We have seen that the Chern character commutes with maps of spaces and the suspension isomorphism, so

$$ch(f_!^K(x)) = ch(c^*(x \cdot \lambda_{\nu_{\hat{f}}})) = c^*(ch(x) \cdot ch(\lambda_{\nu_{\hat{f}}})) = c^*(ch(x) \cdot Td(\nu_{\hat{f}}) \cdot u_{\nu_{\hat{f}}})$$

which is $f_!^H(ch(x) \cdot Td(\nu_{\hat{f}}))$ by definition. □

**Example 5.9.2.** If $M$ and $N$ have a given complex structure on their tangent bundles (for example, if they are complex manifolds), then a map $f : M \to N$ has a canonical complex orientation by requiring

$$\phi : TM \oplus E \longrightarrow \underline{\mathbb{C}}^k_M \oplus f^*(TN)$$

to be an isomorphism of complex bundles. In this case $Td(E) = \frac{f^*Td(TN)}{Td(TM)}$, so

$$ch(f_!^K(x)) = f_!^H\left(ch(x) \cdot \frac{f^*Td(TN)}{Td(TM)}\right) \in H^{2*}(N;\mathbb{Q}).$$

**Example 5.9.3.** If $M$ is a complex manifold of dimension $2n$ then this has important consequences even applied to the map $f : M \to \{*\}$. In this case $f_!^H : H^{2n}(M;\mathbb{Q}) \to H^0(\{*\};\mathbb{Q})$ is the only interesting cohomological Gysin map, and by the description above using Poincaré duality it is given by $\langle [M], - \rangle$, evaluating against the fundamental class. Thus for any complex vector bundle $\pi : V \to M$ we have

$$\langle [M], ch(V) \cdot Td(TM)^{-1} \rangle = ch_0(f_!^K(E)) \in H^0(\{*\};\mathbb{Q}).$$

But $ch_0 : K^0(*) = \mathbb{Z} \to H^0(\{*\};\mathbb{Q})$ takes integer values, so we find that

$$\langle [M], ch(V) \cdot Td(TM)^{-1} \rangle \in \mathbb{Z}.$$

*This is completely not obvious: the formulae for $ch$ and $Td$ have many denominators in them.*

**Example 5.9.4.** Let $M^4$ have a complex structure on its tangent bundle, with $c_i = c_i(TM) \in H^{2i}(M;\mathbb{Z})$ being its Chern classes. We have

$$Td(TM) = 1 - \frac{c_1}{2} + \frac{2c_1^2 - c_2}{12} + \cdots$$

so

$$Td(TM)^{-1} = 1 + \frac{c_1}{2} + \frac{c_2 + c_1^2}{12} + \cdots,$$

and hence

$$\langle [M], \frac{c_1(V)^2 - 2c_2(V)}{2} + \frac{c_1(V) \cdot c_1}{2} + \dim_{\mathbb{C}}(V) \cdot \frac{c_2 + c_1^2}{12} \rangle \in \mathbb{Z}.$$

Applying the above with $V = \underline{\mathbb{C}}^1_M$ shows that $\langle [M], \frac{c_2+c_1^2}{12} \rangle \in \mathbb{Z}$, so $\langle [M], c_2 + c_1^2 \rangle \in 12\mathbb{Z}$. As $c_2(TM) = e(TM)$, it follows that

$$\langle [M], c_1^2 \rangle = -\chi(M) + 12\mathbb{Z}.$$

Lecture 24

## 5.10 The $e$-invariant and stable homotopy groups of spheres

### 5.10.1 Homotopy groups

For any based space $(Y, y_0)$ and $p \geq 1$ the *p-th homotopy group* $\pi_p(Y, y_0)$ is the set of based homotopy classes of based maps $f : (S^p, *) \to (Y, y_0)$. It is a group similarly to the fundamental group: the operation $f \cdot g$ is given by $S^p \to S^p \vee S^p \stackrel{f \vee g}{\to} Y$ where the first map collapses the equator (so has degree 1 onto each wedge-summand). For $p \geq 2$ it is not hard to see that it is an abelian group, but we shall not need to use this.

If $q > p$ then any map $f : S^p \to S^q$ is homotopic to a map which is not surjective (for example by approximating it by a smooth map, or by a simplicial map with respect to some triangulations), which therefore lands in some $D^q \subset S^q$ and so is homotopic to a constant map. This shows that $\pi_p(S^q) = 0$ for $p < q$.

Assigning to a map $f : S^p \to S^p$ its *degree* defines a homomorphism

$$d : \pi_p(S^p) \longrightarrow \mathbb{Z}.$$

The fact that self-maps of spheres are determined up to homotopy by their degree, which you may have seen in Part III Algebraic Topology, is exactly the fact that this map is an isomorphism for all $p \geq 1$.

Assigning to a map $f : (S^p, *) \to (Y, y_0)$ its suspension $\Sigma f : (S^{p+1}, *) \to (\Sigma Y, y_0)$ provides a function $\pi_p(Y, y_0) \to \pi_{p+1}(\Sigma Y, y_0)$, which is easily checked to be a group homomorphism. The *p-th stable homotopy group* of $Y$ is given by the direct limit

$$\pi_p^s(Y, y_0) := \varinjlim_{k \to \infty} \pi_{p+k}(\Sigma^k Y, y_0).$$

Note that this makes sense even if $p$ is negative, as $p + k$ will be positive for $k \gg 0$. There are tautologically suspension isomorphisms $\pi_p^s(Y, y_0) \cong \pi_{p+1}^s(\Sigma Y, y_0)$.

We will be particularly interested in the case where $Y = S^q$ is a sphere. In this case $\pi_p^s(S^q) \cong \pi_{p-q}^s(S^0) =: \pi_{p-q}^s$, so there is really only one parameter in play. The discussion above shows that $\pi_k^s = 0$ for $k < 0$, and that $\pi_0^s \cong \mathbb{Z}$. Our goal in this section is to show that the higher (stable) homotopy groups of spheres must be quite intricate.

### 5.10.2 The $e$-invariant

We want to discuss an invariant of maps $f : S^{2n+2k-1} \to S^{2n}$ similar to the Hopf invariant, and especially similar to the proof of Lemma 5.7.2. Namely, we use $f$ to form the cell complex $X = X_f = S^{2n} \cup_f D^{2n+2k}$, which comes with an inclusion $i : S^{2n} \to X$ and a collapse map $c : X \to S^{2n+2k}$. On cohomology we have

$$H^i(X; \mathbb{Z}) = \begin{cases} \mathbb{Z}\{1\} & i = 0 \\ \mathbb{Z}\{a\} & i = 2n \\ \mathbb{Z}\{b\} & i = 2n + 2k \\ 0 & \text{else} \end{cases}$$

where $i^*(a) = u_{2n}$ and $b = c^*(u_{2n+2k})$. The usual exact sequence on reduced $K$-theory takes the form

$$0 \longrightarrow \tilde{K}^0(S^{2n+2k}) \xrightarrow{c^*} \tilde{K}^0(X) \xrightarrow{i^*} \tilde{K}^0(S^{2n}) \longrightarrow 0.$$

We write $B \in \tilde{K}^0(X)$ for the image of the standard generator $(H-1)^{\boxtimes n+k}$ under $c^*$, and $A \in \tilde{K}^0(X)$ for *some preimage* of the the standard generator $(H-1)^{\boxtimes n}$ under $i^*$. It follows that $ch(B) = b$, and $ch(A) = a + \lambda b$ for some $\lambda \in \mathbb{Q}$. If we re-choose $A$ as $A' = A + rB$ for some $r \in \mathbb{Z}$ then $ch(A') = a + \lambda b + rb$ so $\lambda' = \lambda + r$. Thus

$$e(f) := \lambda \mod \mathbb{Z}$$

is well-defined. This is the $e$-invariant of the map $f$.

**Lemma 5.10.1.** *The function $e : \pi_{2n+2k-1}(S^{2n}) \to \mathbb{Q}/\mathbb{Z}$ is a homomorphism.*
*Furthermore, the e-invariant of $\Sigma^2 f$ is the same as that of $f$.*

*Proof.* Let $X_{f,g} = (S^{2n} \cup_f D^{2n+2k}) \cup_g D^{2n+2k}$, so it contains $X_f$ and $X_g$ as subcomplexes. There is a map $\phi : X_{f\cdot g} \to X_{f,g}$ given by collapsing the equator $D^{2n+2k-1} \subset D^{2n+2k}$; this has degree 1 onto each $(2n+2k)$-cell. Collapsing the $2n$-cell gives a map $c_{f,g} : X_{f,g} \to S^{2n+2k} \vee S^{2n+2k}$,

$$0 \longrightarrow \tilde{K}^0(S^{2n+2k} \vee S^{2n+2k}) \xrightarrow{c^*_{f,g}} \tilde{K}^0(X_{f,g}) \xrightarrow{i^*} \tilde{K}^0(S^{2n}) \longrightarrow 0$$

from which we define classes $A_{f,g}, B_f, B_g \in \tilde{K}^0(X_{f,g})$ in the evident way, and similarly form classes $a_{f,g}, b_f, b_g$ in cohomology. Using naturality of the Chern character and the inclusions $X_f \to X_{f,g} \leftarrow X_g$ we find that $ch(B_f) = b_f$, $ch(B_g) = b_g$, and $ch(A_{f,g}) = a_{f,g} + \lambda_f b_f + \lambda_g b_g$. As $B_f$ and $B_g$ both pull back to $B_{f\cdot g}$ under $\phi$, and $A_{f,g}$ pulls back to a choice of $A_{f\cdot g}$, applying $\phi^*$ to this identity shows that

$$ch(A_{f\cdot g}) = a + (\lambda_f + \lambda_g)b \in H^*(X_{f\cdot g}; \mathbb{Q}),$$

so that $\lambda_{f\cdot g} = \lambda_f + \lambda_g$ for a certain choice of $A_{f\cdot g}$. As these represent the $e$-invariant mod $\mathbb{Z}$, the claim follows

For the second claim, one has $X_{\Sigma^2 f} \simeq \Sigma^2 X_f$. It then follows as the Chern character commutes with the Bott isomorphism. □

Combining the two parts of this lemma shows that the $e$-invariant extends to a homomorphism

$$e : \pi^s_{2k-1} \longrightarrow \mathbb{Q}/\mathbb{Z}$$

from the odd stable homotopy groups of spheres.

### 5.10.3 Evaluating the $e$-invariant

To evaluate the $e$-invariant we need a supply of maps $f : S^{2n+2k-1} \to S^{2n}$ for which we can understand the $K$-theory of $X_f = S^{2n} \cup_f D^{2n+2k}$. The best way to do so is to start with a cell complex $X$ that we understand some other way and which has just a 0-cell, a $2n$-cell, and a $(2n+2k)$-cell, and define $f$ to be the attaching map of the $(2n+2k)$-cell of $X$. For such an $f$ we tautologically have $X_f = X$.

**Lemma 5.10.2.** *If $\pi : E \to S^{2k}$ is a $n$-dimensional complex vector bundle, then the Thom space $Th(E)$ is homotopy equivalent to a cell complex with a 0-cell, a $2n$-cell, and a $(2n+2k)$-cell.*

*Proof.* Let $p : D^{2k} \to S^{2k}$ be the quotient map that identifies the boundary of the disc with the south pole $S$. The bundle $p^*\pi : p^*E \to D^{2k}$ may be trivialised, as the disc is contractible. We can therefore write $D(E)$ as the union of $S(E)$, a $2n$-cell given by the fibre $D(E_S)$ over the south pole, and a $(2n+2k)$-cell given by $D^{2k} \times D^{2n} \cong p^*D(E)$ attached along the map $\partial(D^{2k} \times D^{2n}) \cong (p^*D(E))|_{\partial D^{2k}} \cup p^*S(E) \to D(E_S) \cup S(E)$. Thus $Th(E) = D(E)/S(E)$ may be obtained from $D(E_S)/S(E_S) \cong S^{2n}$ by attaching a $(2n+2k)$-cell. □

Recall that $\tilde{K}^0(S^{2k}) = \mathbb{Z}$, generated by an element that we can denote as $(H-1)^{\boxtimes k}$ whose Chern character is $u_{2k} \in H^*(S^{2k};\mathbb{Q})$. We may represent this element as $[E] - n \in K^0(S^{2k})$ for $E$ an $n$-dimensional vector bundle and $n \gg 0$. Then $ch(E) = n + u_{2k} \in H^*(S^{2k};\mathbb{Q})$.

By the above lemma we have $Th(E) \simeq S^{2n} \cup_f D^{2n+2k}$ for some map $f : S^{2n+2k-1} \to S^{2n}$, and we want to calculate $e(f)$. To do so we have

$$0 \longrightarrow \tilde{K}^0(S^{2n+2k}) \xrightarrow{c^*} \tilde{K}^0(Th(E)) \xrightarrow{i^*} \tilde{K}^0(S^{2n}) \longrightarrow 0$$

and so get a class $B = c^*((H-1)^{\boxtimes n+k})$ and have to choose a class $A$ such that $i^*A = (H-1)^{\boxtimes n}$. A good choice of such a class is $A := \lambda_E$, the $K$-theory Thom class of $E$; by Theorem 5.4.1 (iii) this is indeed a choice of $A$. For this choice we have

$$ch(A) = ch(\lambda_E) = Td(E) \cdot u_E$$

for $u_E \in H^{2n}(Th(E);\mathbb{Q})$ the cohomological Thom class. We therefore need to calculate the Todd class $Td(E) \in H^*(S^{2k};\mathbb{Q})$. What we know is the Chern character of $E$, so want to determine the Todd class in terms of the Chern character classes.

**Lemma 5.10.3.** *Define rational numbers $d_j$ by the formal power series*

$$\log\left(\tfrac{1-\exp(-t)}{t}\right) = -\tfrac{1}{2}t + \tfrac{1}{24}t^2 - \tfrac{1}{2880}t^4 + \tfrac{1}{181440}t^6 + \cdots = \sum_{j\geq 1} d_j \tfrac{t^j}{j!}.$$

*Then for any vector bundle $\pi : E \to X$ there is an identity*

$$\log(Td(E)) = \sum_{j\geq 1} d_j ch_j(E) \in H^*(X;\mathbb{Q}).$$

*Proof.* By the splitting principle is suffices to establish this formula when $E = L_1 \oplus \cdots \oplus L_n$, in which case we write $x_i := c_1(L_i)$. Then $Td(E) = \prod_{i=1}^n \frac{1-\exp(-x_i)}{x_i}$ and so

$$\begin{aligned}\log(Td(E)) &= \sum_{i=1}^n \log\left(\tfrac{1-\exp(-x_i)}{x_i}\right)\\ &= \sum_{i=1}^n \sum_{j\geq 1} d_j \tfrac{x_i^j}{j!}\\ &= \sum_{j\geq 1} d_j ch_j(E)\end{aligned}$$

where the last equality is by definition of the Chern character classes. □

Applied to the $n$-dimensional vector bundle $\pi : E \to S^{2k}$ at hand, which has $ch(E) = n + u_{2k} \in H^*(S^{2k};\mathbb{Q})$, gives

$$\log(Td(E)) = d_k u_{2k}$$

and hence $Td(E) = \exp(d_k u_{2k}) = 1 + d_k u_{2k}$ as $u_{2k}^2 = 0$. Combined with the above, and using that $u_{2k} \cdot u_E \in H^{2n+2k}(Th(E);\mathbb{Q})$ is $b$, gives that

$$e(f) = d_k \mod \mathbb{Z}.$$

For example, using the first few $d_k$'s, and comparing with known calculations[1], we have

| $k$ | $d_k$ | $Im(e:\pi^s_{2k-1}\to\mathbb{Q}/\mathbb{Z})$ contains a group of order | $\pi^s_{2k-1}$ |
|---|---|---|---|
| 1 | $-\frac{1}{2}$ | 2 | $\mathbb{Z}/2$ |
| 2 | $\frac{1}{12}$ | 12 | $\mathbb{Z}/24$ |
| 4 | $-\frac{1}{120}$ | 120 | $\mathbb{Z}/240$ |
| 6 | $\frac{1}{252}$ | 252 | $\mathbb{Z}/504$ |
| 8 | $-\frac{1}{240}$ | 240 | $\mathbb{Z}/480 \oplus \mathbb{Z}/2$ |
| 10 | $\frac{1}{132}$ | 132 | $\mathbb{Z}/264 \oplus \mathbb{Z}/2$ |
| 12 | $-\frac{691}{32760}$ | 32760 | $\mathbb{Z}/65520 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$ |

so we see that the elements of $\pi^s_{2k-1}$ that we have (somewhat implicitly) constructed generate quite large subgroups in these degrees, but not usually everything.

Recall that the *Bernoulli numbers* $B_i \in \mathbb{Q}$ are defined by the formal power series

$$\frac{t}{\exp(t)-1} = \sum_{i\geq 0} B_i \frac{t^i}{i!} = 1 - \frac{1}{2}t + \frac{1}{12}t^2 - \frac{1}{720}t^4 + \cdots .$$

Only even powers of $t$ arise in higher terms, because $\frac{t}{\exp(t)-1} - (1 - \frac{1}{2}t)$ is easily checked to be an even function. Thus the odd Bernoulli numbers vanish except for $B_1 = -\frac{1}{2}$. Differentiating $\log(\frac{1-\exp(-t)}{t})$ with some care gives $\frac{1}{t}(\frac{t}{\exp(t)-1} - 1)$, which gives the identity

$$k \cdot d_k = B_k$$

for $k \geq 2$. The size of the cyclic group we have detected is therefore given by the denominator of $\frac{B_k}{k}$. A vast amount is known about Bernoulli numbers[2]: these denominators are completely known, and for $k$ even are

$$\mathrm{denom}(\tfrac{B_k}{k}) = \left( \prod_{\substack{\text{primes } p \\ \text{s.t. } p-1 \mid k}} p \right) \cdot \left( k / \prod_{\substack{\text{primes } p \\ \text{s.t. } p-1 \nmid k}} p^{\nu_p(k)} \right),$$

where $p^{\nu_p(k)}$ is the largest power of $p$ dividing $k$.

[1] https://en.wikipedia.org/wiki/Homotopy_groups_of_spheres

[2] https://www.bernoulli.org/