Groups, Rings, and Modules

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1 Groups

1.1 Basic concepts

We will begin by quickly recapping some definitions and results from IA Groups.

Definition (Group)**.** A *group* is a triple $(G, \cdot, e)$, where G is a set, $\cdot : G \times G \to G$ is a function and $e \in G$ is an element such that

(i) For all $a, b, c \in G$, we have $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. (associativity)

(ii) For all $a \in G$, we have $a \cdot e = e \cdot a = a$. (identity)

(iii) For all $a \in G$, there exists $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$. (inverse)

Some people add an axiom that says $g \cdot h \in G$ for all $g, h \in G$, but this is already implied by saying $\cdot$ is a function with codomain G, so right-thinking people omit it.

Lemma. The inverse of an element is unique.

Proof. Let a^{-1}, b be inverses of a. Then $b = b \cdot e = b \cdot a \cdot a^{-1} = e \cdot a^{-1} = a^{-1}$. □

Definition (Subgroup)**.** If $(G, \cdot, e)$ is a group and $H \subseteq G$ is a subset, then it is a *subgroup* if

(i) $e \in H$,

(ii) $a, b \in H$ implies $a \cdot b \in H$,

(iii) $\cdot : H \times H \to H$ makes $(H, \cdot, e)$ a group in its own right.

We write $H \leq G$ if H is a subgroup of G.

Note that the last condition in some sense encompasses the first two, but we need the first two conditions to hold before the last statement makes sense at all. The following lemma is the most convenient way to check that a subset forms a subgroup.

Lemma. A subset $H \subseteq G$ is a subgroup if H is non-empty and for any $h_1, h_2 \in H$, we have $h_1 h_2^{-1} \in H$. □

Definition (Abelian group)**.** A group G is *abelian* if $a \cdot b = b \cdot a$ for all $a, b \in G$.

Example. We have the following familiar examples of groups

(i) $(\mathbb{Z}, +, 0)$, $(\mathbb{Q}, +, 0)$, $(\mathbb{R}, +, 0)$, $(\mathbb{C}, +, 0)$.

(ii) We also have groups of symmetries:

  (a) The symmetric group S_n is the collection of all permutations of $\{1, 2, \cdots, n\}$.

  (b) The dihedral group D_{2n} is the symmetries of a regular n-gon.

  (c) The group $\mathrm{GL}_n(\mathbb{R})$ is the group of invertible $n \times n$ real matrices, which also is the group of invertible $\mathbb{R}$-linear maps from the vector space $\mathbb{R}^n$ to itself.

(iii) The alternating group $A_n \leq S_n$.

(iv) The cyclic group $C_n \leq D_{2n}$.

(v) The special linear group $\mathrm{SL}_n(\mathbb{R}) \leq \mathrm{GL}_n(\mathbb{R})$, the subgroup of matrices of determinant 1.

(vi) The Klein four-group $C_2 \times C_2$.

(vii) The quaternions $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ with $ij = k, ji = -k$, $i^2 = j^2 = k^2 = -1$, $(-1)^2 = 1$.

With groups and subgroups, we can talk about cosets.

Definition (Coset)**.** If $H \leq G$, $g \in G$, the *left coset* gH is the set

$$gH := \{x \in G : x = g \cdot h \text{ for some } h \in H\}.$$

For example, since H is a subgroup, we know $e \in H$. So for any $g \in G$, we must have $g \in gH$. The collection of H-cosets in G forms a partition of G, and furthermore, all H cosets gH are in bijection with H itself, via $h \mapsto gh : H \to gH$. An immediate consequence of this is

Theorem (Lagrange's theorem)**.** Let G be a finite group, and $H \leq G$. Then

$$|G| = |H||G : H|,$$

where $|G : H|$ is the number of H cosets in G. □

We can of course do exactly the same thing with right cosets and get the same conclusion. We have implicitly used the following notation:

Definition (Order of group)**.** The *order* of a group is the number of elements in G, written $|G|$.

Instead of order of the group, we can ask what the order of an element is.

Definition (Order of element)**.** The *order* of an element $g \in G$ is the smallest positive n such that $g^n = e$. (If there is no such n, we say g has infinite order.) We write $\text{ord}(g) = n$.

A basic consequence of Lagrange's theorem is the following.

Lemma. If G is a finite group and $g \in G$ has order n, then n divides $|G|$.

Proof. Consider the subset

$$H = \{e, g, g^2, \cdots, g^{n-1}\}$$

of G. This is a subgroup of G, because it is non-empty and $g^r g^{-s} = g^{r-s}$ lies in H (we might have to add n to the power of g to make it positive, but this is fine since $g^n = e$). Moreover, there are no repeats in the list: if $g^i = g^j$, and say $i \geq j$, then $g^{i-j} = e$. So $i - j < n$. By definition of n, we must have $i - j = 0$, i.e. $i = j$.

Hence Lagrange's theorem tells us that $n = |H|$ divides $|G|$. □

1.2 Normal subgroups, quotients, homomorphisms, isomorphisms

We all know what the definition of a normal subgroup is. However, instead of just stating the definition and proving things about it, lets motivate the definition by seeing how one could naturally come up with it.

Let $H \leq G$ be a subgroup. The objective is to try to make the collection of cosets

$$G/H := \{gH : g \in G\}$$

into a group.

Before we do that, we quickly come up with a criterion for when two cosets gH and $g'H$ are equal. Notice that if $gH = g'H$, then $g' \in gH$. So $g' = g \cdot h$ for some h. In other words,

$g^{-1} \cdot g' = h \in H$. So if two elements represent the same coset, their difference is in H. The argument is also reversible. Hence two elements $g, g' \in G$ represent the same H-coset if and only if $g^{-1}g' \in H$.

Suppose we try to make the set $G/H = \{gH : g \in G\}$ into a group by the formula

$$(g_1H) \cdot (g_2H) := g_1g_2H.$$

This is not necessarily well-defined: if we take a different representative for the same coset, we want to make sure that we get the same answer.

If $g_2H = g_2'H$, then we know $g_2' = g_2 \cdot h$ for some $h \in H$. So

$$(g_1H) \cdot (g_2'H) = g_1g_2'H = g_1g_2hH = g_1g_2H = (g_1H) \cdot (g_2H),$$

and we are safe. If $g_1H = g_1'H$, then $g_1' = g_1 \cdot h$ for some $h \in H$. So

$$(g_1'H) \cdot (g_2H) = g_1'g_2H = g_1hg_2H.$$

We need the equality

$$g_1hg_2H = g_1g_2H$$

to hold, which requires

$$(g_1g_2)^{-1}g_1hg_2 \in H.$$

This is equivalent to asking that

$$g_2^{-1}hg_2 \in H.$$

So for G/H to be a group under this operation we must have, for any $h \in H$ and $g \in G$, that $g^{-1}hg \in H$. This is not necessarily true for a given subgroup H.

Definition (Normal subgroup)**.** A subgroup $H \leq G$ is *normal* if for any $h \in H$ and $g \in G$, we have $g^{-1}hg \in H$. We write $H \lhd G$.

This allows us to make the following definition:

Definition (Quotient group)**.** If $H \lhd G$ is a normal subgroup, then the set G/H of left H-cosets forms a group with multiplication

$$(g_1H) \cdot (g_2H) := g_1g_2H.$$

with identity $eH = H$. This is known as the *quotient group*.

The quotient group is indeed a group: the definition of normal subgroup was chosen so that the multiplication is well-defined, it is associative as multiplication in G is associative, eH is easily seen to be an identity element and the inverse of gH is $g^{-1}H$.

So far, we have just been looking at individual groups, but we would also like to know how groups interact with each other. In other words, we want to study functions between groups. However, we should not consider *arbitrary* functions: since groups have some structure, we should only consider the functions which respect this structure. These are the *homomorphisms*.

Definition (Homomorphism)**.** If $(G, \cdot, e_G)$ and $(H, *, e_H)$ are groups, a function $\phi : G \to H$ is called a *homomorphism* if $\phi(e_G) = e_H$, and for $g, g' \in G$, we have

$$\phi(g \cdot g') = \phi(g) * \phi(g').$$

(One can derive $\phi(e_G) = e_H$ from the second condition, but it doesn't hurt to put it in as well.)

Lemma. If $\phi : G \to H$ is a homomorphism, then $\phi(g^{-1}) = \phi(g)^{-1}$.

Proof. We compute $\phi(g \cdot g^{-1})$ in two ways. On the one hand, we have $\phi(g \cdot g^{-1}) = \phi(e_G) = e_H$. On the other hand, we have $\phi(g \cdot g^{-1}) = \phi(g) * \phi(g^{-1})$. By the uniqueness of inverse, we must have $\phi(g^{-1}) = \phi(g)^{-1}$. □

Given a homomorphism, there are two groups we can associate to it.

Definition (Kernel)**.** The *kernel* of a homomorphism $\phi : G \to H$ is

$$\ker(\phi) := \{g \in G : \phi(g) = e_H\}.$$

Definition (Image)**.** The *image* of a homomorphism $\phi : G \to H$ is

$$\mathrm{im}(\phi) := \{h \in H : h = \phi(g) \text{ for some } g \in G\}.$$

Lemma. For a homomorphism $\phi : G \to H$, the kernel $\ker(\phi)$ is a *normal subgroup* of G, and the image $\mathrm{im}(\phi)$ is a subgroup of H.

Proof. To see $\ker(\phi)$ is a subgroup, let $g, h \in \ker \phi$. Then

$$\phi(g \cdot h^{-1}) = \phi(g) * \phi(h)^{-1} = e_H * e_H^{-1} = e_H.$$

So $gh^{-1} \in \ker(\phi)$. Also, $\phi(e_G) = e_H$, so $\ker(\phi)$ is non-empty. Hence it is a subgroup.

To show it is normal, let $g \in \ker(\phi)$ and $x \in G$. We want to show $x^{-1}gx \in \ker(\phi)$. We have

$$\phi(x^{-1}xg) = \phi(x^{-1}) * \phi(g) * \phi(x) = \phi(x^{-1}) * \phi(x) = \phi(x^{-1}x) = \phi(e_G) = e_H.$$

So $x^{-1}gx \in \ker(\phi)$ and hence $\ker(\phi)$ is normal.

If $\phi(g), \phi(h) \in \mathrm{im}(\phi)$ then

$$\phi(g) * \phi(h)^{-1} = \phi(gh^{-1}) \in \mathrm{im}(\phi).$$

Furthermore, $e_H = \phi(e_G) \in \mathrm{im}(\phi)$, so $\mathrm{im}(\phi)$ is non-empty. Hence $\mathrm{im}(\phi)$ is a subgroup. □

Definition (Isomorphism)**.** An *isomorphism* is a homomorphism which is also a bijection.

If a function $\phi : G \to H$ is an isomorphism, then the inverse function $\phi^{-1} : H \to G$ is too.

Definition (Isomorphic group)**.** Two groups G and H are *isomorphic* if there is an isomorphism between them. We write $G \cong H$.

Often, we consider isomorphic groups as being "the same", and do not distinguish between them. We are being careless when we do this, and should be aware that we are being careless.

It is often helpful to be able to break groups apart into smaller pieces. The following three "isomorphism theorems" allow us to do this in various ways. The first relates the kernel and image of a homomorphism.

Theorem (First isomorphism theorem)**.** Let $\phi : G \to H$ be a homomorphism. Then $\ker(\phi)$ is a normal subgroup of G and

$$\frac{G}{\ker(\phi)} \cong \mathrm{im}(\phi).$$

Proof. We have already shown that $\ker(\phi)$ is a normal subgroup. We now have to construct a homomorphism $f : G/\ker(\phi) \to \mathrm{im}(\phi)$, and prove it is an isomorphism.

We define f as follows:

$$\begin{aligned} f : \frac{G}{\ker(\phi)} &\longrightarrow \mathrm{im}(\phi) \\ g\ker(\phi) &\longmapsto \phi(g). \end{aligned}$$

As the function f is defined using a coset representative, we must show it is well-defined. If $g\ker(\phi) = g'\ker(\phi)$, then $g^{-1} \cdot g' \in \ker(\phi)$ and so $\phi(g^{-1} \cdot g') = e_H$. Thus

$$e_H = \phi(g^{-1} \cdot g') = \phi(g)^{-1} * \phi(g'),$$

and so multiplying by $\phi(g)$ gives $\phi(g) = \phi(g')$, so the function is well-defined.

To show that f is a homomorphism, we calculate

$$\begin{aligned} f(g\ker(\phi) \cdot g'\ker(\phi)) &= f(gg'\ker(\phi)) \\ &= \phi(gg') \\ &= \phi(g) * \phi(g') \\ &= f(g\ker(\phi)) * f(g'\ker(\phi)). \end{aligned}$$

Finally, we show f is a bijection. To show that it is surjective, let $h \in \mathrm{im}(\phi)$. Then $h = \phi(g)$ for some g. So $h = f(g\ker(\phi))$ is in the image of f. To show that it is injective, suppose that $f(g\ker(\phi)) = f(g'\ker(\phi))$. Then $\phi(g) = \phi(g')$, so $\phi(g^{-1} \cdot g') = e_H$. Hence $g^{-1} \cdot g' \in \ker(\phi)$, and hence $g\ker(\phi) = g'\ker(\phi)$, as required. □

Before we move on to further isomorphism theorems, lets seen an example of how this one can be used to identify otherwise mysterious groups.

Example. Consider the function $\phi : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ given by $z \mapsto e^z$. As $e^{z+w} = e^z e^w$, the function ϕ defines a homomorphism $\phi : (\mathbb{C}, +, 0) \to (\mathbb{C} \setminus \{0\}, \times, 1)$.

The existence of log shows that ϕ is surjective, and so $\mathrm{im}\,\phi = \mathbb{C} \setminus \{0\}$. The kernel is given by

$$\ker(\phi) = \{z \in \mathbb{C} : e^z = 1\} = 2\pi i\mathbb{Z},$$

i.e. the set of all integer multiples of $2\pi i$. The conclusion is that

$$(\mathbb{C}/(2\pi i\mathbb{Z}), +, 0) \cong (\mathbb{C} \setminus \{0\}, \times, 1).$$

The second isomorphism theorem has a slightly more complicated statement.

Theorem (Second isomorphism theorem)**.** Let $H \leq G$ and $K \lhd G$. Then $HK := \{h \cdot k : h \in H, k \in K\}$ is a subgroup of G, and $H \cap K$ is a normal subgroup of H. Moreover,

$$\frac{HK}{K} \cong \frac{H}{H \cap K}.$$

Proof. Let $hk, h'k' \in HK$. Then

$$h'k'(hk)^{-1} = h'k'k^{-1}h^{-1} = (h'h^{-1})(hk'k^{-1}h^{-1}).$$

The first term is in H, while the second term is $k'k^{-1} \in K$ conjugated by h, which also has to be in K be normality. So this is something in H times something in K, and hence lies in HK. The set HK also contains e_G, and is hence a subgroup.

To see that $H \cap K$ is normal in H and to prove the second isomorphism theorem, we apply the first isomorphism theorem to a certain homomorphism. Define

$$\begin{aligned} \phi : H &\longrightarrow G/K \\ h &\longmapsto hK \end{aligned}$$

This is a homomorphism. The image of ϕ is the set of K-cosets which may be represented by an element of H, i.e.

$$\mathrm{im}(\phi) = \frac{HK}{K}.$$

The kernel of ϕ is

$$\ker(\phi) = \{h \in H : hK = eK\} = \{h \in H : h \in K\} = H \cap K;$$

as it is the kernel of a homomorphism, it is normal in H. By the first isomorphism theorem we have

$$\frac{H}{H \cap K} \cong \frac{HK}{K}. \qquad \square$$

Before moving on to the third isomorphism theorem, notice that if $K \lhd G$, then there is a bijection between subgroups of G/K and subgroups of G containing K, given by

$$\begin{aligned} \{\text{subgroups of } G/K\} &\longleftrightarrow \{\text{subgroups of } G \text{ which contain } K\} \\ X \leq \frac{G}{K} &\longmapsto \{g \in G : gK \in X\} \\ \frac{L}{K} \leq \frac{G}{K} &\longleftarrow\!\shortmid K \lhd L \leq G. \end{aligned}$$

This specialises to a bijection between *normal* subgroups, too:

$$\{\text{normal subgroups of } G/K\} \longleftrightarrow \{\text{normal subgroups of } G \text{ which contain } K\}.$$

We will often use this correspondence.

Theorem (Third isomorphism theorem)**.** Let $K \leq L \leq G$ be normal subgroups of G. Then

$$\frac{G}{K} \Big/ \frac{L}{K} \cong \frac{G}{L}.$$

Proof. Define a homomorphism

$$\begin{aligned} \phi : G/K &\longrightarrow G/L \\ gK &\longmapsto gL \end{aligned}$$

As always, we have to check this is well-defined. If $gK = g'K$, then $g^{-1}g' \in K \subseteq L$, and so $gL = g'L$. It is a homomorphism since

$$\phi(gK \cdot g'K) = \phi(gg'K) = gg'L = (gL) \cdot (g'L) = \phi(gK) \cdot \phi(g'K).$$

The function ϕ is surjective, since $gL = \phi(gK)$, so the image of ϕ is G/L. The kernel of ϕ is

$$\ker(\phi) = \{gK \in G/K : gL = L\} = \{gK \in G/K : g \in L\} = L/K.$$

So the conclusion follows by the first isomorphism theorem. $\square$

The idea of all three of these theorems is to take a group, find a normal subgroup, and then quotient it out: hopefully the normal subgroup and the quotient group will be simpler. However, such a simplification is not always possible.

Definition (Simple group)**.** A (non-trivial) group G is *simple* if it has no normal subgroups except $\{e\}$ and G.

In general, simple groups are complicated. However, if we only look at abelian groups, then it is not difficult to describe all simple groups. Note that by commutativity, the normality condition is always trivially satisfied, so *any* subgroup is normal. Hence an abelian group can be simple only if it has no non-trivial subgroups at all.

Lemma. An abelian group is simple if and only if it is isomorphic to the cyclic group C_p for some prime number p.

Proof. By Lagrange's theorem, any subgroup of C_p has order dividing $|C_p| = p$. Hence, if p is prime, any subgroup must have order 1 or p, i.e. be either $\{e\}$ or C_p. In particular any normal subgroup must be $\{e\}$ or C_p, so it is simple.

Now suppose that G is abelian and simple. Let $e \neq g \in G$ be a non-trivial element, and consider the subset $H = \{\cdots, g^{-2}, g^{-1}, e, g, g^2, \cdots\}$ of G, which is a subgroup. Since G is abelian every subgroup is normal, so H is a normal subgroup. As G is simple, $H = \{e\}$ or $H = G$. Since it contains $g \neq e$ it is non-trivial, so we must have $H = G$, and so G is cyclic.

If G is infinite cyclic, then it is isomorphic to $\mathbb{Z}$. But $\mathbb{Z}$ is not simple, since $2\mathbb{Z} \lhd \mathbb{Z}$. So G must be a finite cyclic group, i.e. $G \cong C_m$ for some finite m.

If $n \mid m$, then $g^{m/n}$ generates a (normal) subgroup of G of order n, so for G to be simple n must be m or 1. Hence G cannot simple unless m has no divisors except 1 and m, i.e. m is a prime number. □

Simple finite groups are the building blocks for all finite groups, in the following sense.

Theorem. If G is a finite group, then there are subgroups

$$G = H_1 \rhd H_2 \rhd H_3 \rhd H_4 \rhd \cdots \rhd H_n = \{e\}$$

such that each quotient H_{i+1}/H_i is simple.

We only claim that H_i is normal in H_{i+1}, not necessarily in G.

Proof. If G is simple, let $H_2 = \{e\}$. Then we are done.

If G is not simple, let H_2 be a proper normal subgroup of G of maximal order. We claim that G/H_2 is simple.

If G/H_2 is not simple, then it contains a proper non-trivial normal subgroup $L \lhd G/H_2$ such that $L \neq \{e\}, G/H_2$. However, by the correspondence between normal subgroups of G/H_2 and normal subgroups of G containing H_2, L must be K/H_2 for some $K \lhd G$ such that $K \geq H_2$. Moreover, since L is non-trivial and not G/H_2, we know K is not G or H_2. So K is a larger normal subgroup than H_2, which is a contradiction.

So we have found an $H_2 \lhd G$ such that G/H_2 is simple. Iterating this process with H_2 gives the desired result. Note that this process eventually stops, as $H_{i+1} < H_i$, and hence $|H_{i+1}| < |H_i|$, and all these numbers are finite. □

1.3 Actions and permutations

Recall that the *symmetric group* S_n is the group of all permutations of $\{1, 2, \dots, n\}$, i.e. of bijections from this set to itself. We have seen in IA Groups that it is convenient to write permutations in disjoint cycle form, e.g. (1 2 3)(4 5)(6) (1-cycles are usually omitted). We have also seen that every permutation can be written as a product of transpositions: we say that a permutation is *even* (or *odd*) if it can be written as a product of an even (or odd) number of transpositions, called its sign; we have seen that this is well-defined. The even permutations form a subgroup $A_n \leq S_n$, the alternating group. As the function

$$\begin{aligned} \text{sgn} : S_n &\longrightarrow (\{\pm 1\}, \times, 1) \\ \sigma &\longmapsto \begin{cases} +1 & \sigma \text{ is even} \\ -1 & \sigma \text{ is odd} \end{cases} \end{aligned}$$

is a homomorphism, and is onto for $n \geq 2$, it follows from the first isomorphism theorem that $A_n = \ker(\text{sgn})$ is a normal subgroup of S_n, and has index 2.

More generally, for a set X, we can define its symmetric group as follows:

Definition (Symmetric group of X)**.** Let X be a set. We write $\text{Sym}(X)$ for the group of all permutations of X.

However, we don't always want the whole symmetric group. Sometimes, we just want some subgroups of symmetric groups, as in our initial motivation. So we make the following definition.

Definition (Permutation group)**.** A group G is called a *permutation group* if it is a subgroup of $\text{Sym}(X)$ for some X, i.e. it is given by some—but not necessarily all—permutations of a set.

We say G is a *permutation group of degree* n if $|X| = n$.

This is not really a good or interesting definition, since, as we will soon see, every group is (isomorphic to) a permutation group. However, in some cases, thinking of a group as a permutation group of some object gives us better intuition about it.

Example. S_n and A_n are obviously permutation groups (of degree n). Also, the dihedral group D_{2n} is a permutation group of degree n, viewing it as a permutation of the vertices of a regular n-gon.

If $G \leq \text{Sym}(X)$, then each $g \in G$ gives us a permutation of X, in a way that is compatible with the group structure. We say the group G *acts* on X. In general, we make the following definition:

Definition (Group action)**.** An *action* of a group $(G, \cdot, e)$ on a set X is a function

$$- * - : G \times X \longrightarrow X$$

such that

(i) $g_1 * (g_2 * x) = (g_1 \cdot g_2) * x$ for all $g_1, g_2 \in G$ and $x \in X$.

(ii) $e * x = x$ for all $x \in X$.

There is another way of defining group actions, which is arguably a better way of thinking about them.

Lemma. An action of G on a set X is equivalent to a homomorphism $\phi : G \to \operatorname{Sym} X$.

The statement of this lemma by itself is useless, since it does not say how to translate between the homomorphism and a group action. The important part is the proof.

Proof. Let $* : G \times X \to X$ be an action. Define $\phi : G \to \operatorname{Sym} X$ by sending g to the function $\phi(g) = (g * - : X \to X)$. This is indeed a permutation, as the function $g^{-1} * -$ is an inverse:

$$\phi(g^{-1})(\phi(g)(x)) = g^{-1} * (g * x) = (g^{-1} \cdot g) * x = e * x = x,$$

and a similar argument shows that $\phi(g) \circ \phi(g^{-1}) = \mathrm{id}_X$. So ϕ is at least a well-defined function.

To show that ϕ is a homomorphism, we calculate

$$\phi(g_1)(\phi(g_2)(x)) = g_1 * (g_2 * x) = (g_1 \cdot g_2) * x = \phi(g_1 \cdot g_2)(x).$$

Since this is true for all $x \in X$, it follows that $\phi(g_1) \circ \phi(g_2) = \phi(g_1 \cdot g_2)$. Also, $\phi(e)(x) = e * x = x$. So $\phi(e) = \mathrm{id}_X$. Hence ϕ is a homomorphism.

We now show how to construct a group action from a homomorphism $\phi : G \to \operatorname{Sym} X$. Given ϕ, define a function $- * - : G \times X \to X$ by $g * x = \phi(g)(x)$. We now check that this indeed defines a group action. Using the definitions of a homomorphism, we know

(i) $g_1 * (g_2 * x) = \phi(g_1)(\phi(g_2)(x)) = (\phi(g_1) \circ \phi(g_2))(x) = \phi(g_1 \cdot g_2)(x) = (g_1 \cdot g_2) * x.$

(ii) $e * x = \phi(e)(x) = \mathrm{id}_X(x) = x.$

So ϕ gives a group action. These two operations are clearly inverses to each other, showing that group actions of G on X are the same as homomorphisms $G \to \mathrm{Sym}(X)$. □

Definition (Permutation representation)**.** A *permutation representation* of a group G is a homomorphism $G \to \mathrm{Sym}(X)$.

The lemma above has shown that a permutation representation is the same as a group action. The good thing about thinking of group actions as homomorphisms is that we can use all we know about homomorphisms on them.

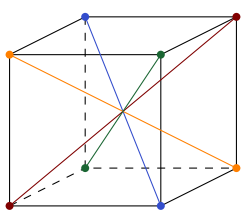
Notation. For an action of G on X given by $\phi : G \to \mathrm{Sym}(X)$, we write $G^X = \mathrm{im}(\phi)$ and $G_X = \ker(\phi)$.

The first isomorphism theorem theorem immediately gives

Proposition. $G_X \lhd G$ and $G/G_X \cong G^X$. □

In particular, if $G_X = \{e\}$ is trivial, then $G \cong G^X \leq \mathrm{Sym}(X)$.

Example. Let G be the group of symmetries of a cube. Let X be the set of diagonals of the cube.



Then G acts on X, and so we get $\phi : G \to \mathrm{Sym}(X)$. What is its kernel? To preserve the diagonals, it either does nothing to the diagonal, or flips the two vertices. So

$$G_X = \ker(\phi) = \{\mathrm{id}, \text{symmetry that sends each vertex to its opposite}\} \cong C_2.$$

How about the image? We have $G^X = \mathrm{Im}(\phi) \leq \mathrm{Sym}(X) \cong S_4$. It is an easy exercise to show that $\mathrm{im}(\phi) = \mathrm{Sym}(X)$, i.e. it is surjective. Then the first isomorphism theorem tells us $G^X \cong G/G_X$. So

$$|G| = |G^X||G_X| = 4! \cdot 2 = 48.$$

This is an example of how we can use group actions to count elements in a group.

Example (Cayley's theorem)**.** For any group G, we have an action of the group G on the set G via

$$g * g_1 = gg_1.$$

This is indeed an action. This gives a group homomorphism $\phi : G \to \mathrm{Sym}(G)$. What is its kernel? If $g \in \ker(\phi)$, then it acts trivially on every element. In particular, it acts trivially on the identity. So $g * e = e$, which means $g = e$. So $\ker(\phi) = \{e\}$. By the first isomorphism theorem, we get

$$G \cong G/\{e\} \cong \mathrm{im}\, \phi \leq \mathrm{Sym}(G).$$

So we see that every group is (isomorphic to) a subgroup of a symmetric group.

Example. Let H be a subgroup of G, and $X = G/H$ be the set of left cosets of H. We let G act on X via

$$g * g_1 H = gg_1 H.$$

This is well-defined and is indeed a group action. So we get $\phi : G \to \mathrm{Sym}(X)$.

Now consider $G_X = \ker(\phi)$. If $g \in G_X$, then for every $g_1 \in G$, we have $g * g_1 H = g_1 H$. This means $g_1^{-1} g g_1 \in H$. In other words, we have $g \in g_1 H g_1^{-1}$. This has to hold for *all* $g_1 \in G$, so

$$G_X \subseteq \bigcap_{g_1 \in G} g_1 H g_1^{-1}.$$

This argument is completely reversible: if $g \in \bigcap_{g_1 \in G} g_1 H g_1^{-1}$, then for each $g_1 \in G$, we know that $g_1^{-1} g g_1 \in H$ and hence $gg_1 H = g_1 H$. So $g * g_1 H = g_1 H$ and hence $g \in G_X$. Thus we have

$$\ker(\phi) = G_X = \bigcap_{g_1 \in G} g_1 H g_1^{-1}.$$

Since this is a kernel, it is a normal subgroup of G, and is contained in H. Starting with an arbitrary subgroup H, this allows us to generate a normal subgroup. (If we think about the construction, we see that this is the largest normal subgroup of G that is contained in H.)

We can use this construction to prove the following theorem.

Theorem. Let G be a finite group, and $H \leq G$ a subgroup of index n. Then there is a normal subgroup $K \lhd G$ with $K \leq H$ such that G/K is isomorphic to a subgroup of S_n. Hence $|G/K| \mid n!$ and $|G/K| \geq n$.

Proof. We apply the previous example, giving $\phi : G \to \mathrm{Sym}(G/H)$, and let K be the kernel of this homomorphism. We have already shown that $K \leq H$. Then the first isomorphism theorem gives

$$G/K \cong \mathrm{im}\, \phi \leq \mathrm{Sym}(G/H) \cong S_n.$$

Then by Lagrange's theorem, we know $|G/K| \mid |S_n| = n!$, and we also have $|G/K| \geq |G/H| = n$. □

Corollary. Let G be a non-abelian simple group. Let $H \leq G$ be a proper subgroup of index $n > 1$. Then G is isomorphic to a subgroup of A_n. Moreover, we must have $n \geq 5$, ie. G cannot have a subgroup of index less than 5.

Proof. The action of G on $X = G/H$ gives a homomorphism $\phi : G \to \mathrm{Sym}(X)$. Then $\ker(\phi) \lhd G$. Since G is simple, $\ker(\phi)$ is either G or $\{e\}$. We first show that it cannot be G. If $\ker(\phi) = G$, then every element of G acts trivially on $X = G/H$. But if $g \in G \setminus H$, which exists since H is a *proper* subgroup, then $g * H = gH \neq H$ and so g does not act trivially. So the kernel cannot be the whole of G. Hence, and hence it must be $\{e\}$.

By the first isomorphism theorem, we have

$$G \cong \mathrm{im}(\phi) \leq \mathrm{Sym}(X) \cong S_n.$$

We now need to show that G is in fact a subgroup of A_n.

We know that $A_n \lhd S_n$, so $\mathrm{im}(\phi) \cap A_n \lhd \mathrm{im}(\phi) \cong G$. As G is simple, $\mathrm{im}(\phi) \cap A_n$ is either $\{e\}$ or $G = \mathrm{im}(\phi)$. We want to show that the second case occurs, i.e. that the intersection is not the trivial group. We use the second isomorphism theorem: if we had $\mathrm{im}(\phi) \cap A_n = \{e\}$, then

$$\mathrm{im}(\phi) \cong \frac{\mathrm{im}(\phi)}{\mathrm{im}(\phi) \cap A_n} \cong \frac{\mathrm{im}(\phi)A_n}{A_n} \leq \frac{S_n}{A_n} \cong C_2.$$

So $G \cong \mathrm{im}(\phi)$ is a subgroup of C_2, so abelian: this is a contradiction. So we must have $\mathrm{im}(\phi) \cap A_n = \mathrm{im}(\phi)$, i.e. $\mathrm{im}(\phi) \leq A_n$.

The last part follows from the fact that S_1, S_2, S_3, S_4 have no non-abelian simple subgroups, which one may check by going to a quiet room and listing all their subgroups. □

Let us recall some definitions from IA Groups.

Definition (Orbit)**.** If G acts on a set X, the *orbit* of $x \in X$ is

$$G \cdot x = \{g * x \in X : g \in G\}.$$

Definition (Stabiliser)**.** If G acts on a set X, the *stabiliser* of $x \in X$ is

$$G_x = \{g \in G : g * x = x\}.$$

The main theorem relating these concepts is the orbit-stabiliser theorem.

Theorem (Orbit-stabiliser theorem)**.** Let G act on X. Then for any $x \in X$, there is a bijection between $G \cdot x$ and G/G_x, given by $g \cdot x \leftrightarrow g \cdot G_x$.

In particular, if G is finite then $|G| = |G_x||G \cdot X|$. □

It takes some work to show that this is well-defined and a bijection, but it has been done in IA Groups. In that course, you perhaps saw just the second statement instead, but the first statement remains is true for not necessarily finite groups.

1.4 Conjugacy classes, centralisers, and normalisers

We have seen that every group G acts on the set G by multiplying on the left. A group G can also act on the set G in a different way, by conjugation:

$$g * g_1 := gg_1g^{-1}.$$

Let $\phi : G \to \mathrm{Sym}(G)$ be the associated permutation representation. We know, by definition, that $\phi(g)$ is a bijection from G to G as sets. However, here G is not an arbitrary set, but is a group. A natural question to ask is whether $\phi(g)$ is a homomorphism or not. Indeed, we have

$$\phi(g)(g_1 \cdot g_2) = gg_1g_2g^{-1} = (gg_1g^{-1})(gg_2g^{-1}) = \phi(g)(g_1)\phi(g)(g_2).$$

So $\phi(g)$ is a homomorphism from G to G. Since $\phi(g)$ is bijective (as in *any* group action), it is in fact a group isomorphism.

We can take the collection of *all* isomorphisms of G, and form a new group out of it.

Definition (Automorphism group)**.** The *automorphism group* of G is

$$\mathrm{Aut}(G) := \{f \in \mathrm{Sym}\, G : f \text{ is a group isomorphism}\}.$$

This is a group under composition, with the identity map as the identity element.

This is a subgroup of $\mathrm{Sym}(G)$, and the homomorphism $\phi : G \to \mathrm{Sym}(G)$ given by conjugation has image in $\mathrm{Aut}(G)$.

Definition (Conjugacy class)**.** The *conjugacy class* of $g \in G$ is

$$\mathrm{ccl}_G(g) := \{hgh^{-1} \in G : h \in G\},$$

i.e. the orbit of $g \in G$ under the conjugation action.

Definition (Centraliser)**.** The *centraliser* of $g \in G$ is

$$C_G(g) := \{h \in G : hgh^{-1} = g\},$$

i.e. the stabiliser of g under the conjugation action. Alternatively it is the set of all $h \in G$ which commute with g.

Definition (Centre)**.** The *centre* of a group G is

$$Z(G) := \{h \in G : hgh^{-1} = g \text{ for all } g \in G\} = \bigcap_{g \in G} C_G(g) = \ker(\phi),$$

i.e. the set of all $h \in G$ which commute with all $g \in G$.

Proposition. Let G be a finite group. Then

$$|\,\mathrm{ccl}(x)| = |G : C_G(x)| = |G|/|C_G(x)|.$$

In particular, the size of each conjugacy class divides the order of the group.

Proof. Apply the orbit-stabiliser theorem to the action of G on itself by conjugation, giving a bijection $\mathrm{ccl}(x) \leftrightarrow G/C_G(x)$. □

Definition (Normaliser)**.** Let $H \leq G$. The *normaliser* of H in G is

$$N_G(H) = \{g \in G : g^{-1}Hg = H\}.$$

We certainly have $H \leq N_G(H)$. Even better, $H \lhd N_G(H)$, essentially by definition: the normaliser is the biggest subgroup of G in which H is normal.

We are now going to look at conjugacy classes of S_n. Recall from IA Groups that permutations in S_n are conjugate if and only if they have the same cycle type when written as a product of disjoint cycles. We can think of the cycle types as partitions of n. For example, the partition $2, 2, 1$ of 5 corresponds to the conjugacy class of $(1\ 2)(3\ 4)(5)$. So the conjugacy classes of S_n are exactly the partitions of n.

Example. In S_5 we have the following conjugacy classes

Cycle type	1^5	$2 \cdot 1^3$	$2^2 \cdot 1$	$3 \cdot 1^2$	$3 \cdot 2$	$4 \cdot 1$	5
# elements	1	10	15	20	20	30	24

Theorem. The alternating groups A_n are simple for $n \geq 5$ (also for $n = 2, 3$).

The cases in brackets follow from a direct check since $A_2 \cong \{e\}$ and $A_3 \cong C_3$, all of which are simple. We can also check manually that A_4 has non-trivial normal subgroups, and hence not simple.

Recall we also proved that A_5 is simple in IA Groups by brute force — we listed all its conjugacy classes, and see they cannot be put together to make a normal subgroup. This obviously cannot be easily generalized to higher values of n. Hence we need to prove this with a different approach.

Proof. We start with the following claim:

Claim. A_n is generated by 3-cycles.

As any element of A_n is a product of evenly-many transpositions, it suffices to show that every product of two transpositions is also a product of 3-cycles.

There are three possible cases: let a, b, c, d be distinct. Then

(i) $(a\ b)(a\ b) = e$.

(ii) $(a\ b)(b\ c) = (a\ b\ c)$.

(iii) $(a\ b)(c\ d) = (a\ c\ b)(a\ c\ d)$.

So we have shown that every possible product of two transpositions is a product of three-cycles.

Claim. Let $H \lhd A_n$. If H contains a 3-cycle, then $H = A_n$.

We show that if H contains any 3-cycle, then it contains every 3-cycle: then we are done since A_n is generated by 3-cycles. For concreteness, suppose we know $(a\ b\ c) \in H$, and we want to show $(1\ 2\ 3) \in H$.

Since they have the same cycle type, there is a $\sigma \in S_n$ such that $(a\ b\ c) = \sigma(1\ 2\ 3)\sigma^{-1}$. If σ is even, i.e. $\sigma \in A_n$, then we have that $(1\ 2\ 3) \in \sigma^{-1}H\sigma = H$, by the normality of H and we are done. If σ is odd, replace it by $\bar{\sigma} = \sigma \cdot (4\ 5)$. This is where we first use the fact that $n \geq 5$ (we will use it again later). Then we have

$$\bar{\sigma}(1\ 2\ 3)\bar{\sigma}^{-1} = \sigma(4\ 5)(1\ 2\ 3)(4\ 5)\sigma^{-1} = \sigma(1\ 2\ 3)\sigma^{-1} = (a\ b\ c),$$

using the fact that $(1\ 2\ 3)$ and $(4\ 5)$ commute. Now $\bar{\sigma}$ is even, so $(1\ 2\ 3) \in H$ as above.

So far we have shown that if $H \lhd A_n$ contains *any* 3-cycle, then it is A_n. Finally, we have to show that every normal subgroup must contain at least one 3-cycle.

Claim. Let $H \lhd A_n$ be non-trivial. Then H contains a 3-cycle.

We separate this into many cases.

(i) Suppose H contains an element which can be written in disjoint cycle notation

$$\sigma = (1\ 2\ 3 \cdots r)\tau,$$

for $r \geq 4$. We now let $\delta = (1\ 2\ 3) \in A_n$. Then by normality of H, we have $\delta^{-1}\sigma\delta \in H$, and so $\sigma^{-1}\delta^{-1}\sigma\delta \in H$ too. As τ does not contain $1, 2, 3$ it commutes with δ, and also by assumption with $(1\ 2\ 3\ \cdots\ r)$. We can expand this to obtain

$$\sigma^{-1}\delta^{-1}\sigma\delta = (r\ \cdots\ 2\ 1)(1\ 3\ 2)(1\ 2\ 3\ \cdots\ r)(1\ 2\ 3) = (2\ 3\ r),$$

which is a 3-cycle, so we are done.

The same argument goes through if $\sigma = (a_1\ a_2\ \cdots\ a_r)\tau$ for any $a_1, \cdots, a_r$ and $r \geq 4$.

(ii) Suppose H contains an element consisting of at least two 3-cycles in disjoint cycle notation, say

$$\sigma = (1\ 2\ 3)(4\ 5\ 6)\tau$$

We now let $\delta = (1\ 2\ 4)$, and again calculate

$$\sigma^{-1}\delta^{-1}\sigma\delta = (1\ 3\ 2)(4\ 6\ 5)(1\ 4\ 2)(1\ 2\ 3)(4\ 5\ 6)(1\ 2\ 4) = (1\ 2\ 4\ 3\ 6).$$

This is a 5-cycle, which is necessarily in H. By the previous case, we get a 3-cycle in H too, and hence $H = A_n$.

(iii) Suppose H contains $\sigma = (1\ 2\ 3)\tau$ in disjoint cycle notation, with τ a product of 2-cycles (if τ contains anything longer, then it would fit in one of the previous two cases). Then $\sigma^2 = (1\ 2\ 3)^2 = (1\ 3\ 2)$ is a 3-cycle.

(iv) Suppose H contains $\sigma = (1\ 2)(3\ 4)\tau$, where τ is a product of 2-cycles. We first let $\delta = (1\ 2\ 3)$ and calculate

$$u = \sigma^{-1}\delta^{-1}\sigma\delta = (1\ 2)(3\ 4)(1\ 3\ 2)(1\ 2)(3\ 4)(1\ 2\ 3) = (1\ 4)(2\ 3),$$

which is again in H. We landed in the same case, but instead of two transpositions times a mess, we just have two transpositions, which is nicer. Now let

$$v = (1\ 5\ 2)u(1\ 2\ 5) = (1\ 3)(4\ 5) \in H.$$

Note that we have used the assumption $n \geq 5$ again here. We have yet again landed in the same case. Notice however, that these are not the same transpositions. We multiply

$$uv = (1\ 4)(2\ 3)(1\ 3)(4\ 5) = (1\ 2\ 3\ 4\ 5) \in H.$$

This is covered by the first case, so we are done. □

1.5 Finite p-groups

When studying the orders of groups and subgroups we always talk about divisibility, since that is what Lagrange's theorem tells us about; we never talk about things like the sum of the orders to two subgroups. From this point of view, the simplest groups are those whose order is prime, but we have classified all such groups already: they are cyclic. The next simplest groups are perhaps those whose order is a power of a prime.

Definition (p-group)**.** A finite group G is a *p-group* if $|G| = p^n$ for some prime number p and $n \geq 1$.

Theorem. If G is a finite p-group, then its centre $Z(G) = \{x \in G : xg = gx \text{ for all } g \in G\}$ is non-trivial.

Proof. Let G act on itself by conjugation. Each orbit of this action (which are precisely the conjugacy classes) has size dividing $|G| = p^n$, so is either a singleton, or has size divisible by p.

Since the conjugacy classes partition G, we know the sum of the size of the conjugacy classes is $|G|$. In particular,

$$|G| = \#\{\text{conjugacy class of size } 1\} + \sum \text{order of all other conjugacy classes.}$$

By the above discussion, the second term is divisible by p, as is $|G| = p^n$. Hence the number of conjugacy classes of size 1 is divisible by p. We know that $\{e\}$ is a conjugacy class of size 1, so there must be at least p conjugacy classes of size 1. Since $p \geq 2$, there is a conjugacy class $\{x\} \neq \{e\}$.

But if $\{x\}$ is a conjugacy class of size 1, then by definition $g^{-1}xg = x$ for all $g \in G$, i.e. $xg = gx$ for all $g \in G$. So $x \in Z(G)$. So $Z(G)$ is non-trivial. □

As the centre of a group is normal this immediately tells us that, for $n \geq 2$, a p-group is never simple. This will allow us to prove interesting things about p-groups by induction on their order—by considering the smaller p-group $G/Z(G)$. One way to do this is via the below lemma.

Lemma. For any group G, if $G/Z(G)$ is cyclic, then G is abelian.

In other words, if $G/Z(G)$ is cyclic, then it is in fact trivial, since the centre of an abelian group is the abelian group itself.

Proof. Let the coset $gZ(G)$ be a generator of the cyclic group $G/Z(G)$, so every coset of $Z(G)$ is of the form $g^r Z(G)$. It follows that every element $x \in G$ must be of the form $g^r z$ for some $z \in Z(G)$ and $r \in \mathbb{Z}$. To show that G is abelian, let $\bar{x} = g^{\bar{r}}\bar{z}$ be another element, with $\bar{z} \in Z(G), \bar{r} \in \mathbb{Z}$. As z and $\bar{z}$ are in the centre they commute with every element, so we have

$$x\bar{x} = g^r z g^{\bar{r}}\bar{z} = g^r g^{\bar{r}} z\bar{z} = g^{\bar{r}} g^r \bar{z}z = g^{\bar{r}}\bar{z}g^r z = \bar{x}x.$$

So x and $\bar{x}$ commute, and hence G is abelian. □

This lemma holds for all groups, but is particularly useful when applied to p-groups.

Corollary. If p is prime and $|G| = p^2$, then G is abelian.

Proof. Since $Z(G) \leq G$, its order must be 1, p or p^2. Since it is not trivial, it can only be p or p^2. If it has order p^2, then it is the whole group and the group is abelian. Otherwise, $G/Z(G)$ has order $p^2/p = p$. But then it must be cyclic, and hence G must be abelian by the above lemma. □

Theorem. Let G be a group of order p^a, where p is a prime number. Then G has a subgroup of order p^b for any $0 \leq b \leq a$.

This means that G has a subgroup of every possible order. This is not true for general groups. For example, A_5 (of order 60) has no subgroup of order 30 (as such a subgroup would have to be normal).

Proof. We induct on a. If $a = 1$, then $\{e\}$ and G give subgroups of order p^0 and p^1, so we are done.

Now suppose that $a > 1$, and we want to construct a subgroup of order p^b. If $b = 0$, then this is trivial, as $\{e\} \leq G$ has order 1.

Otherwise, we know $Z(G)$ is non-trivial. So let $x \neq e \in Z(G)$. Since $\text{ord}(x) \mid |G|$, its order is a power of p. If it in fact has order p^c, then $x^{p^{c-1}}$ has order p. So we can suppose, by renaming, that x has order p. We have thus generated a subgroup $\langle x \rangle$ of order exactly p. Moreover, since x is in the centre, $\langle x \rangle$ commutes with everything in G. So $\langle x \rangle$ is in fact a normal subgroup of G. (This is the point of choosing x to be in the centre.) Therefore $G/\langle x \rangle$ has order p^{a-1}.

Since this is a strictly smaller group, we may suppose by induction that $G/\langle x \rangle$ has a subgroup of any order. In particular, it has a subgroup L of order p^{b-1}. By the subgroup correspondence, there is some $K \leq G$ such that $\langle x \rangle \lhd K$ and $L = K/\langle x \rangle$. But then K has order p^b, as required. □

1.6 Finite abelian groups

In this short section we will state the classification of finite abelian groups, which we will prove in Chapter 3 as a special case of the classification of modules over certain rings.

Theorem (Classification of finite abelian groups)**.** Let G be a finite abelian group. Then there exists some $d_1, \cdots, d_r$ such that

$$G \cong C_{d_1} \times C_{d_2} \times \cdots \times C_{d_r}.$$

Moreover, we can choose the d_i so that $d_{i+1} \mid d_i$ for each i, in which case this expression is unique.

Example. The abelian groups of order 8 are C_8, $C_4 \times C_2$, $C_2 \times C_2 \times C_2$.

Sometimes the decomposition given by this theorem is not the most useful form. To get a nicer decomposition, we can use the following lemma:

Lemma. If n and m are coprime, then $C_{mn} \cong C_m \times C_n$.

This is essentially the Chinese remainder theorem, and this formulation is how you should think of that theorem.

Proof. It suffices to find an element of order nm in $C_m \times C_n$. Then, since $C_n \times C_m$ has order nm, it must be cyclic and hence isomorphic to C_{nm}.

Let $g \in C_m$ have order m and $h \in C_n$ have order n, and consider the element $(g, h) \in C_m \times C_n$. Suppose the order of (g, h) is k. Then $(g, h)^k = (e, e)$. Hence $(g^k, h^k) = (e, e)$. So the order of g and h divide k, ie. $m \mid k$ and $n \mid k$. As m and n are coprime, this means that $mn \mid k$. As $k = \text{ord}(g, h)$ and $(g, h) \in C_m \times C_n$ is a group of order mn, we must have $k \mid nm$. So $k = nm$. □

Corollary. For any finite abelian group G, we have

$$G \cong C_{d_1} \times C_{d_2} \times \cdots \times C_{d_r},$$

where each d_i is a prime power.

Proof. From the classification theorem, iteratively apply the previous lemma to break each component up into products of prime powers. □

We shall return to this result, and generalisations of it, in Chapter 3.

1.7 Sylow's theorems

Definition. Let G be a finite group of order $p^a \cdot m$, with p prime and $p \nmid m$. A *Sylow p-subgroup* of G is a subgroup $P \leq G$ of order p^a.

The main theorem of this part of the course is as follows.

Theorem (Sylow's theorems)**.** Let G be a finite group of order $p^a \cdot m$, with p prime and $p \nmid m$. Then

(i) The set

$$\text{Syl}_p(G) := \{P \leq G : |P| = p^a\}$$

of Sylow p-subgroups of G is non-empty. In other words, G has a Sylow p-subgroup.

(ii) All elements of $\mathrm{Syl}_p(G)$ are conjugate in G.

(iii) The number $n_p = |\mathrm{Syl}_p(G)|$ of Sylow p-subgroups satisfies $n_p \equiv 1 \pmod{p}$ and $n_p \mid |G|$ (in fact $n_p \mid m$, since p is coprime to n_p).

These are sometimes known as Sylow's first, second, and third theorem respectively. We will not prove them immediately, but first look at how they can be applied.

Lemma. If there is a unique Sylow p-subgroup, i.e. $n_p = 1$, then it is normal in G.

Proof. Let P be the unique Sylow p-subgroup, and let $g \in G$, and consider the conjugate subgroup $g^{-1}Pg$. Since this is isomorphic to P, we must have $|g^{-1}Pg| = p^a$, i.e. it is also a Sylow p-subgroup. Since there is only one, we must have $P = g^{-1}Pg$, so P is normal. □

Corollary. Let G be a non-abelian simple group. Then for every prime number p such that $p \mid |G|$ we have $|G| \mid \frac{n_p!}{2}$, and $n_p \geq 5$.

Proof. The group G acts on $\mathrm{Syl}_p(G)$ by conjugation, giving a permutation representation $\phi : G \to \mathrm{Sym}(\mathrm{Syl}_p(G)) \cong S_{n_p}$. We know $\ker\phi \lhd G$. But G is simple, so $\ker(\phi)$ is either $\{e\}$ or G.

If $G = \ker(\phi)$, then $g^{-1}Pg = P$ for all $g \in G$ and so P is a normal subgroup of G. As G is simple, we must then have $P = \{e\}$ or $P = G$. The group P cannot be trivial since $p \mid |G|$, but if $G = P$ then G is a p-group, so has a non-trivial centre, and hence G is not non-abelian simple. So we must have $\ker(\phi) = \{e\}$.

Then, by the first isomorphism theorem, we have $G \cong \mathrm{im}\,\phi \leq S_{n_p}$ which proves the theorem without the divide-by-two part. To prove the full result we will show that in fact $\mathrm{im}(\phi) \leq A_{n_p}$. To see this we consider the composition

$$G \xrightarrow{\phi} S_{n_p} \xrightarrow{\mathrm{sgn}} \{\pm 1\}.$$

If this is surjective, then $\ker(\mathrm{sgn}\circ\phi) \lhd G$ has index 2 (since the index is the size of the image) so is not the whole of G, and hence G is not simple (the case $|G| = C_2$ is ruled out since it is abelian).

It follows that the kernel of $\mathrm{sgn}\circ\phi$ must be the whole of G, so $\mathrm{sgn}(\phi(g)) = +1$ and hence $\phi(g) \in A_{n_p}$. So in fact we have $G \cong \mathrm{im}(\phi) \leq A_{n_p}$, and $|G| \mid \frac{n_p!}{2}$. For the final statement, one can check that all non-abelian subgroups of A_2, A_3, and A_4 are not simple, so $n_p \geq 5$. □

Example. Let us show that if $|G| = 1000$ then $|G|$ is not simple.

We have $1000 = 2^3 \cdot 5^3$. Choose $p = 5$. Then by Sylow's theorem $n_5 \equiv 1 \pmod{5}$, and $n_5 \mid 2^3 = 8$. The only number that satisfies this is $n_5 = 1$, so the Sylow 5-subgroup is normal, and hence G is not simple.

Example. Let us show that if $|G| = 132 = 2^2 \cdot 3 \cdot 11$ then G is not simple. For a contradiction we suppose it is.

We start by looking at $p = 11$. We know $n_{11} \equiv 1 \pmod{11}$ and $n_{11} \mid 12$. As G is simple we cannot have $n_{11} = 1$ so must have $n_{11} = 12$.

Now we look at $p = 3$. We know $n_3 \equiv 1 \pmod{3}$ and $n_3 \mid 44$. As G is simple the possible values of n_3 are 4 and 22. If $n_3 = 4$, then the corollary above says $|G| \mid \frac{4!}{2} = 12$, which is impossible, so we must have $n_3 = 22$ instead.

At this point, we count how many elements of each order there are. (This is particularly useful when $p \mid |G|$ but $p^2 \nmid |G|$, i.e. the Sylow p-subgroups have order p and hence are cyclic.) As any two Sylow 11-subgroups intersect only in $\{e\}$, we know there are $12 \cdot (11 - 1) = 120$ elements of order 11. By the same argument with the Sylow 3-subgroups we find that there are $22 \cdot (3 - 1) = 44$ elements of order 3. But this gives more elements than there are in the group, a contradiction. So G must be simple.

We will now prove Sylow's theorem. This involves a non-trivial amount of trickery. Let G be a finite group with $|G| = p^a \cdot m$, and $p \nmid m$.

Proof of Sylow's first theorem. We need to show that $\mathrm{Syl}_p(G) \neq \emptyset$, i.e. we need to find some subgroup of order p^a. As always, we find something clever for G to act on. Let

$$\Omega := \{X \text{ subset of } G : |X| = p^a\}.$$

(We indeed mean sub*set* here, not sub*group*.) Let G act on Ω by

$$g * \{g_1, g_2, \cdots, g_{p^a}\} := \{gg_1, gg_2, \cdots, gg_{p^a}\},$$

and let $\Sigma \leq \Omega$ be some orbit.

First note that if $\{g_1, \cdots, g_{p^a}\} \in \Sigma$, then by the definition of an orbit, for every $g \in G$,

$$gg_1^{-1} * \{g_1, \cdots, g_{p^a}\} = \{g, gg_1^{-1}g_2, \cdots, gg_1^{-1}g_{p^a}\} \in \Sigma,$$

so Σ contains a set which contains g. Since each set X has size p^a, we must have

$$|\Sigma| \geq \frac{|G|}{p^a} = m.$$

Suppose that $|\Sigma| = m$. Then the orbit-stabiliser theorem says the stabiliser of any $\{g_1, \cdots, g_{p^a}\} \in \Sigma$ has index m, so has order p^a, and thus is a Sylow p-subgroup.

So we will be finished if we show that not every orbit Σ can have size $> m$. Again, by the orbit-stabiliser theorem, the size of any orbit divides the order of the group, $|G| = p^a m$. So if $|\Sigma| > m$, then $p \mid |\Sigma|$. If we can show that $p \nmid |\Omega|$ then not every orbit Σ can have size $> m$, since Ω is the disjoint union of all the orbits, and we will be done.

So we have to show $p \nmid |\Omega|$. This is just some basic counting: we have

$$|\Omega| = \binom{|G|}{p^a} = \binom{p^a m}{p^a} = \prod_{j=0}^{p^a-1} \frac{p^a m - j}{p^a - j}.$$

As $j < p^a$, the largest power of p dividing $p^a m - j$ is the largest power of p dividing j. Similarly, the largest power of p dividing $p^a - j$ is also the largest power of p dividing j. So we have the same power of p on top and bottom for each term in the product, so they cancel and the result is not divisible by p. □

This proof is not straightforward. We first needed the clever idea of letting G act on Ω, but even if we are given this set, the obvious thing to do would be to find an element of Ω which also happens to be a group. This is not what we do! Instead, we find an orbit whose stabilizer is a Sylow p-subgroup.

Proof of Sylow's second theorem. We instead prove something stronger: if $Q \leq G$ is a p-subgroup (ie. $|Q| = p^b$, for b not necessarily equal to a), and $P \leq G$ is a Sylow p-subgroup, then there is a $g \in G$ such that $g^{-1}Qg \leq P$. (Applying this to the case where Q is another Sylow p-subgroup says there is a g such that $g^{-1}Qg \leq P$, but since $g^{-1}Qg$ has the same size as P, they must be equal.)

We let Q act on the set of cosets G/P via

$$q * gP := qgP.$$

We know the orbits of this action have size dividing $|Q|$, so either 1 or divisible by p. They cannot all be divisible by p, since $|G/P|$ is coprime to p, so at least one of them have size 1, say $\{gP\}$. In other words, for every $q \in Q$, we have $qgP = gP$, which means that $g^{-1}qg \in P$. This holds for every element $q \in Q$ so we have found a g such that $g^{-1}Qg \leq P$. □

Proof of Sylow's third theorem. We need to show that $n_p \equiv 1 \pmod{p}$ and that $n_p \mid |G|$, where $n_p = |\operatorname{Syl}_p(G)|$.

The second part is easier — by Sylow's second theorem, the action of G on $\operatorname{Syl}_p(G)$ by conjugation has one orbit. By the orbit-stabiliser theorem the size of the orbit, which is $|\operatorname{Syl}_p(G)| = n_p$, divides $|G|$. This proves the second part.

For the first part, let $P \in \operatorname{Syl}_p(G)$. Consider the action by conjugation of P on $\operatorname{Syl}_p(G)$. Again by the orbit-stabiliser theorem, the orbits each have size 1 or divisible by p. There is one orbit of size 1, namely $\{P\}$ itself, so to show that $n_p \equiv 1 \pmod{p}$ it is enough to show there are no other orbits of size 1.

Suppose $\{Q\}$ is an orbit of size 1. This means for every $p \in P$, we get

$$p^{-1}Qp = Q.$$

In other words, $P \leq N_G(Q)$. Now $N_G(Q)$ is itself a group, and we can look at its Sylow p-subgroups. We know that $Q \leq N_G(Q) \leq G$, so $p^a \mid |N_G(Q)| \mid p^a m$. Thus p^a is the biggest power of p that divides $|N_G(Q)|$, so Q is a Sylow p-subgroup of $N_G(Q)$.

Now $P \leq N_G(Q)$ is *also* a Sylow p-subgroup of $N_G(Q)$, so by Sylow's second theorem Q and P must be conjugate in $N_G(Q)$. But conjugating Q by something in $N_G(Q)$ does nothing, by definition of the normaliser $N_G(Q)$, so we must have $P = Q$. So the only orbit of size 1 is $\{P\}$ itself. □

Example. Let $G = \mathrm{GL}_n(\mathbb{Z}/p)$, i.e. the set of invertible $n \times n$ matrices with entries in $\mathbb{Z}/p$, the integers modulo p, a prime number. (When we discuss rings in the next chapter, we will study this more extensively.)

First of all, we would like to know the order of this group. Giving a matrix $A \in \mathrm{GL}_n(\mathbb{Z}/p)$ is the same as giving n linearly independent vectors in the vector space $(\mathbb{Z}/p)^n$. We can pick the first vector to be anything except zero, so there are $p^n - 1$ ways of choosing the first vector. Next, we need to pick the second vector, which can be anything that is not in the span of the first vector, so there are $p^n - p$ ways of choosing the second vector. Continuing in this way we have

$$|\mathrm{GL}_n(\mathbb{Z}/p)| = (p^n - 1)(p^n - p)(p^n - p^2)\cdots(p^n - p^{n-1}).$$

We can factorise this as

$$|\mathrm{GL}_n(\mathbb{Z}/p)| = (1 \cdot p \cdot p^2 \cdot \cdots \cdot p^{n-1})((p^n - 1)(p^{n-1} - 1)\cdots(p - 1)),$$

so the largest power of p which divides $|\mathrm{GL}_n(\mathbb{Z}/p)|$ is $p^{\binom{n}{2}}$.

To give a Sylow p-subgroup of $\mathrm{GL}_n(\mathbb{Z}/p)$, we consider the subgroup of matrices of the following form

$$U := \left\{ \begin{pmatrix} 1 & * & * & \cdots & * \\ 0 & 1 & * & \cdots & * \\ 0 & 0 & 1 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \in \mathrm{GL}_n(\mathbb{Z}/p) \right\}.$$

We have $|U| = p^{\binom{n}{2}}$, as each $*$ can be chosen to be anything in $\mathbb{Z}/p$, and there are $\binom{n}{2}$ $*$'s. (There is not a unique Sylow p-subgroup: we could also take the lower triangular matrices, or other things.)

Example. Let's be less ambitious and consider $\mathrm{GL}_2(\mathbb{Z}/p)$, with

$$|G| = p(p^2 - 1)(p - 1) = p(p - 1)^2(p + 1).$$

Suppose that ℓ is another prime number such that $\ell \mid p-1$ and $\ell^3 \nmid |G|$.

Let us find an explicit Sylow ℓ-subgroup. First, we find elements of order ℓ. From IA Numbers and Sets we know that

$$(\mathbb{Z}/p)^\times = \{x \in \mathbb{Z}/p : (\exists y)\, s.t.\, xy \equiv 1 \pmod p\} \cong C_{p-1},$$

so as $\ell \mid p-1$, there is a subgroup $C_\ell \leq C_{p-1} \cong (\mathbb{Z}/p)^\times$. We immediately find a subgroup of order ℓ^2: we have

$$C_\ell \times C_\ell \leq (\mathbb{Z}/p)^\times \times (\mathbb{Z}/p)^\times \leq \mathrm{GL}_2(\mathbb{Z}/p),$$

where the second inclusion is the diagonal matrices, identifying

$$(a, b) \leftrightarrow \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

So this is a Sylow ℓ-subgroup.

2 Rings

2.1 Definitions and examples

We now move on to something completely different — rings. In a ring, we are allowed to add, subtract, multiply but not divide. Our canonical example of a ring would be $\mathbb{Z}$, the integers, as studied in IA Numbers and Sets.

In this course we are only going to consider rings in which multiplication is commutative, since these rings behave like "number systems", in which we can often ask the usual questions of number theory. However, many commutative rings do not behave much like $\mathbb{Z}$. Thus one major goal of this part is to understand the special properties of $\mathbb{Z}$, whether they are present in arbitrary rings, and how these different properties relate to one another.

Definition (Ring)**.** A *ring* is a quintuple $(R, +, \cdot, 0_R, 1_R)$ where $0_R, 1_R \in R$, and $+, \cdot : R \times R \to R$ are binary operations such that

(i) $(R, +, 0_R)$ is an abelian group.

(ii) The operation $\cdot : R \times R \to R$ satisfies associativity, i.e.

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c,$$

and identity:

$$1_R \cdot r = r \cdot 1_R = r.$$

(iii) Multiplication distributes over addition, i.e.

$$r_1 \cdot (r_2 + r_3) = (r_1 \cdot r_2) + (r_1 \cdot r_3)$$
$$(r_1 + r_2) \cdot r_3 = (r_1 \cdot r_3) + (r_2 \cdot r_3).$$

Notation. If R is a ring and $r \in R$, we write $-r$ for the inverse to r in the group $(R, +, 0_R)$. This satisfies $r + (-r) = 0_R$, and we write $r - s$ to mean $r + (-s)$ and so on.

Some authors do not insist on the existence of the multiplicative identity, but we do.

Since we can add and multiply two elements, by induction, we can add and multiply any finite number of elements. However, the notions of infinite sum and product are not defined: it does not make sense to ask if an infinite sum converges.

Definition (Commutative ring)**.** We say a ring R is *commutative* if $a \cdot b = b \cdot a$ for all $a, b \in R$.

From now onwards, all rings in this course are commutative.

Definition (Subring)**.** Let $(R, +, \cdot, 0_R, 1_R)$ be a ring, and $S \subseteq R$ is a subset. We say S is a *subring* of R if $0_R, 1_R \in S$, and the operations $+, \cdot$ make S into a ring in its own right. In this case we write $S \leq R$.

Example. The familiar number systems are all rings: we have $\mathbb{Z} \leq \mathbb{Q} \leq \mathbb{R} \leq \mathbb{C}$, under the usual $0, 1, +, \cdot$.

Example. The set $\mathbb{Z}[i] := \{a + ib : a, b \in \mathbb{Z}\} \leq \mathbb{C}$ of *Gaussian integers* is a subring of the complex numbers. The set $\mathbb{Q}[\sqrt{2}] := \{a + b\sqrt{2} \in \mathbb{R} : a, b \in \mathbb{Q}\} \leq \mathbb{R}$ is a subring of the real numbers.

We will use this square brackets notation quite frequently; it should be clear what it means, but we will define it properly later.

Definition (Unit)**.** An element $u \in R$ is a *unit* if there is another element $v \in R$ such that $u \cdot v = 1_R$. We call v the inverse of u.

It is important that this depends on R, not just on u. For example, $2 \in \mathbb{Z}$ is not a unit, but $2 \in \mathbb{Q}$ is a unit (since $\frac{1}{2}$ is its inverse).

Definition (Field)**.** A *field* is a non-zero ring in which every non-zero element is a unit.

Example. $\mathbb{Z}$ is not a field, but $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are all fields. Similarly, $\mathbb{Z}[i]$ is not a field, while $\mathbb{Q}[\sqrt{2}]$ is.

Example. Let R be a ring. Then $0_R + 0_R = 0_R$, since this is true in the group $(R, +, 0_R)$. Then for any $r \in R$, we get

$$r \cdot (0_R + 0_R) = r \cdot 0_R.$$

We now use the fact that multiplication distributes over addition. So

$$r \cdot 0_R + r \cdot 0_R = r \cdot 0_R.$$

Adding $(-r \cdot 0_R)$ to both sides give

$$r \cdot 0_R = 0_R.$$

This is true for any element $r \in R$. From this, it follows that if $R \neq \{0\}$, then $1_R \neq 0_R$ — if they were equal, then choose any $r \neq 0_R$ and calculate $r = r \cdot 1_R = r \cdot 0_R = 0_R$, which is a contradiction.

Note, however, that $\{0\}$ forms a ring (with the only possible operations and identities), the zero ring, albeit a boring one. (However, it is often a counterexample to incautious claims about rings.)

Definition (Product of rings)**.** Let R, S be rings. Then the *product* $R \times S$ is a ring via

$$(r, s) + (r', s') = (r + r', s + s'), \quad (r, s) \cdot (r', s') = (r \cdot r', s \cdot s').$$

The zero element is $(0_R, 0_S)$ and the one element is $(1_R, 1_S)$.

One can (and should) check that these are indeed rings.

Definition (Polynomial)**.** Let R be a ring. Then a *polynomial* with coefficients in R is an expression

$$f = a_0 + a_1 X + a_2 X^2 + \cdots + a_n X^n,$$

with $a_i \in R$. The X^i are formal symbols.

We identify polynomials f and $f + 0_R \cdot X^{n+1}$ as the same.

Definition (Degree of polynomial)**.** The *degree* of a polynomial f is the largest m such that $a_m \neq 0$.

Definition (Monic polynomial)**.** Let f have degree m. If $a_m = 1$, then f is called *monic*.

Definition (Polynomial ring)**.** We write $R[X]$ for the set of all polynomials with coefficients in R. The operations are performed in the obvious way, ie. if $f = a_0 + a_1 X + \cdots + a_n X$ and $g = b_0 + b_1 X + \cdots + b_k X^k$ are polynomials, then

$$f + g = \sum_{r=0}^{\max\{n,k\}} (a_i + b_i) X^i,$$

and

$$f \cdot g = \sum_{i=0}^{n+k} \left(\sum_{j=0}^{i} a_j b_{i-j} \right) X^i,$$

We identify R with the subring of constant polynomials, i.e. polynomials $\sum a_i X^i$ with $a_i = 0$ for $i > 0$. In particular, $0_R \in R$ and $1_R \in R$ are the zero and one of $R[x]$.

One can (and should) check that these are indeed rings.

Remark. A polynomial with coefficients in R is just a sequence of elements of R, interpreted as the coefficients of some formal symbols. While it does indeed induce a function from R to R in the obvious way, we shall not identify the polynomial with the function it induces, since different polynomials can give rise to the same function.

For example, in $\mathbb{Z}/2\mathbb{Z}[X]$, $f = X^2 + X$ is not the zero polynomial, since its coefficients are not zero. However, $f(0) = 0$ and $f(1) = 0$, so the function induced by f is identically zero.

Definition (Power series)**.** We write $R[[x]]$ for the ring of *(formal) power series* with coefficients in R, i.e.

$$f = a_0 + a_1 X + a_2 X^2 + \cdots,$$

where each $a_i \in R$. This has addition and multiplication the same as for polynomials, but without upper limits.

A power series is also very much not a function. We do not ask whether the sum converges or not, because *it is not a sum*: it is a formal symbol which can be manipulated similarly to a convergent infinite sum.

Example. Is $1 - X \in R[X]$ a unit? For any $g = a_0 + \cdots + a_n X^n$ (with $a_n \neq 0$), we get

$$(1 - X)g = a_0 + (a_1 - a_0)X + \cdots + (a_n - a_{n-1})X^n - a_n X^{n+1},$$

which is not 1 as the coefficient of X^{n+1} is not zero. So g cannot be the inverse of $1 - X$, and hence $1 - X$ is not a unit in $R[X]$.

However, $1 - x \in R[[X]]$ *is* a unit, since

$$(1 - X)(1 + X + X^2 + X^3 + \cdots) = 1.$$

Definition (Laurent polynomials)**.** We write $R[X, X^{-1}]$ for the set of *Laurent polynomials* with coefficients in R, i.e.

$$f = \sum_{i \in \mathbb{Z}} a_i X^i$$

where $a_i \in R$ and only finitely many a_i are non-zero. The operations of addition and multiplication are the obvious ones.

We can also think of Laurent series, but we have to be careful: we allow infinitely many positive coefficients, but only finitely many negative ones. Or else, in the formula for multiplication, we will have an infinite sum of elements in R, which is not defined.

Example. Let X be a set, and R be a ring. Then the set of all R-valued functions on X, i.e. functions $f : X \to R$, is a ring given by

$$(f + g)(x) = f(x) + g(x), \quad (f \cdot g)(x) = f(x) \cdot g(x).$$

Here zero is the constant function 0 and one is the constant function 1.

Usually, we do not want to consider all functions $X \to R$ but instead certain subrings of this. For example, we can consider the ring of all continuous functions $\mathbb{R} \to \mathbb{R}$. This contains, for example, the polynomial functions, which is just $\mathbb{R}[X]$ (since over $\mathbb{R}$, polynomials *are* functions).

2.2 Homomorphisms, ideals, quotients and isomorphisms

Just like groups, we will come up with analogues of homomorphisms, normal subgroups (which are now known as ideals), and quotients.

Definition (Homomorphism of rings)**.** Let R and S be rings. A function $\phi : R \to S$ is a *ring homomorphism* if it satisfies

(i) $\phi(r_1 + r_2) = \phi(r_1) + \phi(r_2)$,

(ii) $\phi(0_R) = 0_S$,

(iii) $\phi(r_1 \cdot r_2) = \phi(r_1) \cdot \phi(r_2)$,

(iv) $\phi(1_R) = 1_S$.

Definition (Isomorphism of rings)**.** If a homomorphism $\phi : R \to S$ is a bijection, we call it an *isomorphism*. The inverse function $\phi^{-1} : S \to R$ is then also a ring homomorphism.

Definition (Kernel)**.** The *kernel* of a homomorphism $\phi : R \to S$ is

$$\ker(\phi) := \{r \in R : \phi(r) = 0_S\}.$$

Definition (Image)**.** The *image* of a homomorphism $\phi : R \to S$ is

$$\mathrm{im}(\phi) := \{s \in S : s = \phi(r) \text{ for some } r \in R\}.$$

Lemma. A homomorphism $\phi : R \to S$ is injective if and only if $\ker \phi = \{0_R\}$.

Proof. A ring homomorphism is in particular a homomorphism $\phi : (R, +, 0_R) \to (S, +, 0_S)$ of (abelian) groups, so this follows from the case of groups. □

In the group scenario, we had groups, subgroups and *normal* subgroups, which are special subgroups. Here, we have a special kind of subsets of a ring that act like normal subgroups, known as *ideals*.

Definition (Ideal)**.** A subset $I \subseteq R$ is an *ideal*, written $I \lhd R$, if

(i) It is a subgroup of $(R, +, 0_R)$. (additive closure)

(ii) If $a \in I$ and $b \in R$, then $a \cdot b \in I$. (strong closure)

We say I is a *proper* ideal if $I \neq R$.

Multiplicative closure is stronger than what we required for subrings — for subrings, it has to be closed under multiplication by its own elements; for ideals, it has to be closed under multiplication by everything. This is similar to how normal subgroups not only have to be closed under internal multiplication, but also conjugation by external elements.

Lemma. If $\phi : R \to S$ is a homomorphism, then $\ker(\phi) \lhd R$.

Proof. Since $\phi : (R, +, 0_R) \to (S, +, 0_R)$ is a group homomorphism, $\ker(\phi)$ is a subgroup of $(R, +, 0_R)$. For the second part, let $a \in \ker(\phi)$, $b \in R$. We need to show that their product is in the kernel. We have

$$\phi(a \cdot b) = \phi(a) \cdot \phi(b) = 0 \cdot \phi(b) = 0.$$

So $a \cdot b \in \ker(\phi)$. □

Example. Suppose $I \lhd R$ is an ideal, and $1_R \in I$. Then for any $r \in R$, we have $1_R \cdot r \in I$. But $1_R \cdot r = r$. So if $1_R \in I$, then $I = R$. In other words, a proper ideal does not contain 1. In particular, a proper ideal is definitely not a subring, since a subring must contain 1.

We are starting to diverge from the situation with groups: in groups, a normal subgroup is a subgroup, but here an ideal is not a subring.

Example. We can generalize the above example a bit. Suppose $I \lhd R$ and $u \in I$ is a unit, i.e. there is some $v \in R$ such that $uv = 1_R$. Then by strong closure, $1_R = u \cdot v \in I$. So $I = R$. Hence proper ideals cannot contain any unit at all.

Example. Consider the ring $\mathbb{Z}$ of integers. Then every ideal of $\mathbb{Z}$ is of the form

$$n\mathbb{Z} := \{\cdots, -2n, -n, 0, n, 2n, \cdots\} \subseteq \mathbb{Z}.$$

It is easy to see this is indeed an ideal. To show these are all the ideals, let $I \lhd \mathbb{Z}$. If $I = \{0\}$, then $I = 0\mathbb{Z}$. Otherwise, let $n \in \mathbb{N}$ be the smallest positive element of I. We want to show that $I = n\mathbb{Z}$; certainly $n\mathbb{Z} \subseteq I$ by strong closure.

Now let $m \in I$. By the Euclidean algorithm, we can write

$$m = q \cdot n + r$$

with $0 \leq r < n$. Now $n, m \in I$. So by strong closure, $m, qn \in I$. So $r = m - q \cdot n \in I$. As n is the smallest positive element of I, and $r < n$, we must have $r = 0$. So $m = q \cdot n \in n\mathbb{Z}$, and hence $I \subseteq n\mathbb{Z}$.

The key to proving this was that we can perform the Euclidean algorithm on elements of $\mathbb{Z}$. Thus, for any ring R in which we can "do the Euclidean algorithm", every ideal must be of the form $aR = \{a \cdot r : r \in R\}$ for some $a \in R$. We will make this notion precise later.

Definition (Generator of ideal)**.** For an element $a \in R$, we write

$$(a) := aR := \{a \cdot r : r \in R\} \lhd R,$$

and call it the *ideal generated by* a.

In general, for $a_1, a_2, \cdots, a_k \in R$, we write

$$(a_1, a_2, \cdots, a_k) = \{a_1 r_1 + \cdots + a_k r_k : r_1, \cdots, r_k \in R\}.$$

This is the *ideal generated by* $a_1, \cdots, a_k$.

We can also have ideals generated by infinitely many elements of a ring, but we have to be a little careful since we cannot use infinite sums.

Definition (Generator of ideal)**.** For $A \subseteq R$ a subset, the *ideal generated by* A is

$$(A) = \left\{ \sum_{a \in A} r_a \cdot a : r_a \in R, \text{ only finitely-many } r_a \text{ non-zero} \right\}.$$

Definition (Principal ideal)**.** An ideal I is a *principal ideal* if $I = (a)$ for some $a \in R$.

What we have proved for the ring $\mathbb{Z}$ is that all its ideals are principal. Not all rings have this property, these are very special and we will study then in more depth later.

Example. Consider the subset

$$\{f \in \mathbb{R}[X] : \text{ the constant coefficient of } f \text{ is } 0\} \subset \mathbb{R}[X].$$

This is an ideal, which can be checked manually (alternatively, it is the kernel of the homomorphism which sends a polynomial to its value at 0). One can easily show that it is the ideal (X), and hence principal.

We have said ideals are like normal subgroups: the key property is that we can divide out by ideals.

Definition (Quotient ring)**.** Let $I \lhd R$. The *quotient ring* R/I consists of the set of (additive) cosets $r + I$ with the zero and one as $0_R + I$ and $1_R + I$, and operations

$$(r_1 + I) + (r_2 + I) = (r_1 + r_2) + I$$
$$(r_1 + I) \cdot (r_2 + I) = r_1 r_2 + I.$$

Proposition. The quotient ring is a ring, and the function

$$R \longrightarrow R/I$$
$$r \longmapsto r + I$$

is a ring homomorphism.

Proof. We know the group $(R/I, +, 0_{R/I})$ is well-defined, since I is a (normal) subgroup of R. So we only have to check that multiplication is well-defined.

Suppose $r_1 + I = r_1' + I$ and $r_2 + I = r_2' + I$. Then $r_1' - r_1 = a_1 \in I$ and $r_2' - r_2 = a_2 \in I$. So

$$r_1' r_2' = (r_1 + a_1)(r_2 + a_2) = r_1 r_2 + r_1 a_2 + r_2 a_1 + a_1 a_1.$$

By the strong closure property, the last three terms are in I. So $r_1' r_2' + I = r_1 r_2 + I$.

It is easy to check that $0_R + I$ and $1_R + I$ are indeed the zero and one, and the function given is a homomorphism. □

Example. We have ideals $n\mathbb{Z} \lhd \mathbb{Z}$, and so quotient rings $\mathbb{Z}/n\mathbb{Z}$. The elements of $\mathbb{Z}/n\mathbb{Z}$ are of the form $m + n\mathbb{Z}$, so are

$$0 + n\mathbb{Z}, 1 + n\mathbb{Z}, 2 + n\mathbb{Z}, \cdots, (n-1) + n\mathbb{Z}.$$

Addition and multiplication is just what we are used to — addition and multiplication modulo n.

Example. Consider $(X) \lhd \mathbb{C}[X]$. What is the ring $\mathbb{C}[X]/(X)$?

Elements are represented as

$$a_0 + a_1 X + a_2 X^2 + \cdots + a_n X^n + (X),$$

but everything except the first term is in (X), so this element is equivalent to $a_0 + (X)$. This representation is unique, so in fact $\mathbb{C}[X]/(X) \cong \mathbb{C}$, via the ring isomorphism $a_0 + (X) \leftrightarrow a_0$.

If we want to carefully prove things like this, we have to convince ourselves that "this representation is unique"'. We can do that by hand in this case, but in general we want to be able to do this systematically.

Proposition (Euclidean algorithm for polynomials)**.** Let $\mathbb{F}$ be a field and $f, g \in \mathbb{F}[X]$. Then there are $r, q \in \mathbb{F}[X]$ such that

$$f = gq + r,$$

with $\deg r < \deg g$.

This is very much like the usual Euclidean algorithm, except that instead of absolute value, we use the degree to measure how "big" a polynomial is.

Proof. Let $\deg(f) = n$. So

$$f = \sum_{i=0}^{n} a_i X^i,$$

and $a_n \neq 0$. Similarly, if $\deg(g) = m$, then

$$g = \sum_{i=0}^{m} b_i X^i,$$

with $b_m \neq 0$. If $n < m$, we let $q = 0$ and $r = f$, and done.

Otherwise, suppose $n \geq m$, and proceed by induction on n.

Let

$$f_1 = f - a_n b_m^{-1} X^{n-m} g.$$

This is possible since $b_m \neq 0$, and $\mathbb{F}$ is a field so every non-zero element is a unit (i.e. has a multiplicative inverse). Then by construction, the coefficients of X^n cancel out. So $\deg(f_1) < n$.

If $n = m$, then $\deg(f_1) < n = m$, so we can write

$$f = (a_n b_m^{-1} X^{n-m})g + f_1,$$

and $\deg(f_1) < \deg(f)$, so we are done. Otherwise, if $n > m$, then because $\deg(f_1) < n$, we may by induction find r_1, q_1 such that

$$f_1 = gq_1 + r_1,$$

and $\deg(r_1) < \deg g = m$. Then

$$f = a_n b_m^{-1} X^{n-m} g + q_1 g + r_1 = (a_n b_m^{-1} X^{n-m} + q_1)g + r_1. \qquad \square$$

Now that we have a Euclidean algorithm for polynomials we can show that every ideal of $\mathbb{F}[X]$ is generated by one polynomial. We will not do this specifically here, but later we will show that in any ring where the Euclidean algorithm is possible, all ideals are principal.

We now look at some applications of the Euclidean algorithm.

Example. Consider the ring $\mathbb{R}[X]$ and the principal ideal $(X^2 + 1) \lhd \mathbb{R}[X]$. Let $R = \mathbb{R}[X]/(X^2 + 1)$.

Elements of R are polynomials

$$\underbrace{a_0 + a_1 X + a_2 X^2 + \cdots + a_n X^n}_{f} + (X^2 + 1).$$

By the Euclidean algorithm, we have

$$f = q(X^2 + 1) + r,$$

with $\deg(r) < 2$, i.e. $r = b_0 + b_1 X$. Thus $f + (X^2+1) = r + (X^2+1)$. So every element of $\mathbb{R}[X]/(X^2+1)$ is representable as $a + bX$ for some $a, b \in R$.

Is this representation unique? If $a + bX + (X^2+1) = a' + b'X + (X^2+1)$, then the difference $(a-a') + (b-b')X \in (X^2+1)$. So it is $(X^2+1)q$ for some q. This is possible only if $q = 0$, since for non-zero q, we know $(X^2+1)q$ has degree at least 2. So we must have $(a-a') + (b-b')X = 0$. So $a + bX = a' + b'X$. So the representation is unique.

What we have shown is that every element in R is uniquely of the form $a + bX$, and we know that $X^2 + 1 = 0$ so $X^2 = -1$. This sounds like the complex numbers, just that we are writing X instead of i.

To prove this, we define the function

$$\begin{aligned} \phi : \mathbb{R}[x]/(X^2+1) &\longrightarrow \mathbb{C} \\ a + bX + (X^2+1) &\longmapsto a + bi. \end{aligned}$$

This is well-defined and a bijection, and is also clearly additive. To prove it is a ring isomorphism, we must show it is multiplicative. We check this manually, via

$$\begin{aligned} &\phi((a + bX + (X^2+1))(c + dX + (X^2+1))) \\ &= \phi(ac + (ad+bc)X + bdX^2 + (X^2+1)) \\ &= \phi((ac-bd) + (ad+bc)X + (X^2+1)) \\ &= (ac-bd) + (ad+bc)i \\ &= (a+bi)(c+di) \\ &= \phi(a + bX + (X^2+1))\phi(c + dX + (X^2+1)). \end{aligned}$$

So is is indeed a ring isomorphism.

This is pretty tedious. Fortunately, there are some helpful results we can use, namely the isomorphism theorems. These are exactly analogous to those for groups.

Theorem (First isomorphism theorem)**.** Let $\phi : R \to S$ be a ring homomorphism. Then $\ker(\phi)$ is an ideal of R, and

$$\frac{R}{\ker(\phi)} \cong \operatorname{im}(\phi) \leq S.$$

Proof. We have already seen that $\ker(\phi)$ is an ideal. Now define

$$\begin{aligned} \Phi : R/\ker(\phi) &\longrightarrow \operatorname{im}(\phi) \\ r + \ker(\phi) &\longmapsto \phi(r). \end{aligned}$$

We do not have to check this is well-defined, bijective or additive, since that comes for free from the (proof of the) first isomorphism theorem of groups. So we just have to check it is multiplicative. To show Φ is multiplicative, we have

$$\begin{aligned} \Phi((r + \ker(\phi))(t + \ker(\phi))) &= \Phi(rt + \ker(\phi)) \\ &= \phi(rt) \\ &= \phi(r)\phi(t) \\ &= \Phi(r + \ker(\phi))\Phi(t + \ker(\phi)). \end{aligned}$$

□

Theorem (Second isomorphism theorem)**.** Let $R \leq S$ and $J \lhd S$. Then $J \cap R \lhd R$, and

$$\frac{R+J}{J} = \{r + J : r \in R\} \leq \frac{S}{J}$$

is a subring, and

$$\frac{R}{R \cap J} \cong \frac{R+J}{J}.$$

Proof. Define the function

$$\begin{aligned}\phi : R &\longrightarrow S/J \\ r &\longmapsto r + J.\end{aligned}$$

Since this is the quotient map, it is a ring homomorphism. The kernel is

$$\ker(\phi) = \{r \in R : r + J = 0, \text{ ie. } r \in J\} = R \cap J.$$

Then the image is

$$\mathrm{im}(\phi) = \{r + J : r \in R\} = \frac{R + J}{J}.$$

Then by the first isomorphism theorem, we know $R \cap J \lhd R$, and $\frac{R+J}{J} \leq S$, and

$$\frac{R}{R \cap J} \cong \frac{R + J}{J}.$$ □

Before we get to the third isomorphism theorem, recall we had the subgroup correspondence for groups. Analogously, for an ideal $I \lhd R$, there is a correspondence

$$\begin{aligned}\{\text{subrings of } R/I\} &\longleftrightarrow \{\text{subrings of } R \text{ which contain } I\} \\ L \leq \frac{R}{I} &\longmapsto \{x \in R : x + I \in L\} \\ \frac{S}{I} \leq \frac{R}{I} &\longmapsto I \lhd S \leq R.\end{aligned}$$

This is exactly the same formula as for groups.

For groups, we also had a correspondence for normal subgroups. Here, we have a correspondence between ideals

$$\{\text{ideals of } R/I\} \longleftrightarrow \{\text{ideals of } R \text{ which contain } I\}$$

It is important to note here quotienting in groups and rings have somewhat different flavours. In (finite) groups, we often take quotients so that we have a simpler group to work with. On the other hand in rings, we often take quotients to get more interesting rings: For example, $\mathbb{R}[X]$ is quite boring, but $\mathbb{R}[X]/(X^2 + 1) \cong \mathbb{C}$ is rather interesting. Thus this ideal correspondence allows us to occasionally get interesting ideals from less interesting ones.

Theorem (Third isomorphism theorem)**.** Let $I \lhd R$ and $J \lhd R$, and $I \subseteq J$. Then $J/I \lhd R/I$ and

$$\left(\frac{R}{I}\right) / \left(\frac{J}{I}\right) \cong \frac{R}{J}.$$

Proof. We define the map

$$\begin{aligned}\phi : R/I &\longrightarrow R/J \\ r + I &\longmapsto r + J.\end{aligned}$$

This is well-defined and surjective by the groups case. Also it is a ring homomorphism since multiplication in both R/I and R/J is given by multiplication (in R) of coset representatives. The kernel is

$$\ker(\phi) = \{r + I : r + J = 0, \text{ i.e. } r \in J\} = \frac{J}{I}.$$

So the result follows from the first isomorphism theorem. □

For any ring R there is a unique ring homomorphism $\mathbb{Z} \to R$, given by

$$\begin{aligned} \iota : \mathbb{Z} &\longrightarrow R \\ n \geq 0 &\longmapsto \underbrace{1_R + 1_R + \cdots + 1_R}_{n \text{ times}} \\ n \leq 0 &\longmapsto -(\underbrace{1_R + 1_R + \cdots + 1_R}_{-n \text{ times}}) \end{aligned}$$

Any homomorphism $\mathbb{Z} \to R$ must be given by this formula, since it must send $1_\mathbb{Z}$ to 1_R, and we can show this is indeed a homomorphism by distributivity. So the ring homomorphism is unique.

We then have an ideal $\ker(\iota) \lhd \mathbb{Z}$, so must have $\ker(\iota) = n\mathbb{Z}$ for some n.

Definition (Characteristic of ring)**.** Let R be a ring, and $\iota : \mathbb{Z} \to R$ be the unique ring homomorphism. The *characteristic* of R is the unique non-negative n such that $\ker(\iota) = n\mathbb{Z}$.

Example. The rings $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ all have characteristic 0. The ring $\mathbb{Z}/n\mathbb{Z}$ has characteristic n. In particular, any natural number can be the characteristic of some ring.

The notion of the characteristic will not be used much in this course. However, fields of non-zero characteristic often provide interesting examples and counterexamples.

2.3 Integral domains, field of fractions, maximum and prime ideals

Rings can be very different to $\mathbb{Z}$. For example, in $\mathbb{Z}$ we know that if $a, b \neq 0$ then $ab \neq 0$. However, in, say, $\mathbb{Z}/6\mathbb{Z}$, we have $2 + (6), 3 + (6) \neq 0$, but their product is zero. Also, in $\mathbb{Z}$ every ideal is principal, and every integer has an (essentially) unique factorization. We will try to organise rings according to which of these properties they have.

We start with the most fundamental property that the product of two non-zero elements are non-zero.

Definition (Integral domain)**.** A non-zero ring R is an *integral domain* if for all $a, b \in R$, if $a \cdot b = 0_R$, then $a = 0_R$ or $b = 0_R$.

An element that violates this property is known as a *zero divisor*.

Definition (Zero divisor)**.** An element $x \in R$ is a *zero divisor* if $x \neq 0$ and there is a $y \neq 0$ such that $xy = 0 \in R$.

In other words, a ring is an integral domain if it has no zero divisors.

Example. All fields are integral domains, since if $a \cdot b = 0$, and $b \neq 0$, then $a = a \cdot (b \cdot b^{-1}) = 0$. Similarly, if $a \neq 0$, then $b = 0$.

Example. A subring of an integral domain is an integral domain, since a zero divisor in the smaller ring would also be a zero divisor in the larger ring.

Example. $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are integral domains, since $\mathbb{C}$ is a field, and these are subrings of it. Also, $\mathbb{Z}[i] \leq \mathbb{C}$ is an integral domain.

These are the rings one should consider in number theory, since there we can sensibly talk about factorization. It turns out that finite integral domains are especially simple.

Lemma. Let R be a finite ring which is an integral domain. Then R is a field.

Proof. Let $a \in R$ be non-zero, and consider the group homomorphism

$$\begin{aligned} a \cdot - : (R, +, 0_R) &\to (R, +, 0_R) \\ b &\mapsto a \cdot b \end{aligned}$$

We want to show this is injective, for which it suffices to show that the kernel is trivial. If $r \in \ker(a \cdot -)$, then $a \cdot r = 0$. So $r = 0$ since R is an integral domain. So the kernel is trivial.

Since R is finite, $a \cdot -$ must also be surjective. In particular, there is an element $b \in R$ such that $a \cdot b = 1_R$, so a has an inverse. Since a was arbitrary, R is a field. □

So far, we know fields are integral domains, and subsets of integral domains are integral domains. We have another good source of integral domain as follows:

Lemma. If R is an integral domain, then $R[X]$ is too.

Proof. We need to show the product of two non-zero elements are non-zero. Let $f, g \in R[X]$ be non-zero, say

$$\begin{aligned} f &= a_0 + a_1 X + \cdots + a_n X^n \in R[X] \\ g &= b_0 + b_1 X + \cdots + b_m X^m \in R[X], \end{aligned}$$

with $a_n, b_m \neq 0$. Then the coefficient of X^{n+m} in fg is $a_n b_m$. This is non-zero since R is an integral domain, so fg is non-zero. So $R[X]$ is an integral domain. □

So, for instance, $\mathbb{Z}[X]$ is an integral domain. We can iterate the above.

Notation. Write $R[X, Y]$ for $(R[X])[Y]$, the polynomial ring of R in two variables. In general, write $R[X_1, \cdots, X_n] = (\cdots((R[X_1])[X_2])\cdots)[X_n]$.

If R is an integral domain,it follows from the lemma above that $R[X_1, \cdots, X_n]$ is too.

We now want to mimic the familiar construction of $\mathbb{Q}$ from $\mathbb{Z}$. For any integral domain R, we want to construct a field F that consists of "fractions" of elements in R. Recall that the subring of any field is an integral domain. This construction will prove the converse: every integral domain is a subring of a field.

Definition (Field of fractions)**.** Let R be an integral domain. A *field of fractions* F of R is a field with the following properties

(i) $R \leq F$

(ii) Every element of F may be written as $a \cdot b^{-1}$ for $a, b \in R$, where b^{-1} means the multiplicative inverse to $b \neq 0$ in F.

For example, $\mathbb{Q}$ is a field of fractions of $\mathbb{Z}$.

Theorem. Every integral domain has a field of fractions.

Proof. The construction is exactly how we construct the rationals from the integers — as equivalence classes of pairs of integers. Let

$$S = \{(a, b) \in R \times R : b \neq 0\}.$$

We think of $(a, b) \in S$ as the fraction $\frac{a}{b}$. We define an equivalence relation $\sim$ on S by

$$(a, b) \sim (c, d) \Leftrightarrow ad = bc.$$

We need to show this is indeed a equivalence relation. Symmetry and reflexivity are obvious. To show transitivity, suppose

$$(a, b) \sim (c, d), \quad (c, d) \sim (e, f),$$

so $ad = bc$ and $cf = de$. We multiply the first equation by f and the second by b, to obtain

$$adf = bcf, \quad bcf = bed.$$

Rearranging, we get

$$d(af - be) = 0.$$

Since d is a denominator, $d \neq 0$. Since R is an integral domain, we must therefore have $af - be = 0$, i.e. $af = be$. So $(a, b) \sim (e, f)$. (This is where being an integral domain is important.)

Now let

$$F = S/\sim$$

be the set of equivalence classes. We write $\frac{a}{b} = [(a, b)] \in F$, and define addition and multiplication operations by

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$
$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

This is well-defined, and makes $(F, +, \cdot, \frac{0}{1}, \frac{1}{1})$ into a ring. There are many things to check, but they are straightforward, and we will not do them. Finally, we wish to show F is a field so need to show every non-zero element has an inverse. Let $\frac{a}{b} \neq 0_F$, i.e. $\frac{a}{b} \neq \frac{0}{1}$, or $a \cdot 1 \neq b \cdot 0 \in R$, i.e. $a \neq 0$. Then $\frac{b}{a} \in F$ is defined, and

$$\frac{b}{a} \cdot \frac{a}{b} = \frac{ba}{ba} = 1.$$

So $\frac{a}{b}$ has a multiplicative inverse, and hence F is a field.

We now need to construct a subring of F that is isomorphic to R. To do so, we need to define an injective homomorphism $\phi : R \to F$. This is given by

$$\phi : R \to F$$
$$r \mapsto \frac{r}{1}.$$

This is a ring homomorphism, as one can check easily. The kernel is the set of all $r \in R$ such that $\frac{r}{1} = 0$, so is trivial and ϕ is injective. By the first isomorphism theorem, $R \cong \mathrm{im}(\phi) \subseteq F$.

Finally, we need to show everything is a quotient of two things in R. We have

$$\frac{a}{b} = \frac{a}{1} \cdot \frac{1}{b} = \frac{a}{1} \cdot \left(\frac{b}{1}\right)^{-1},$$

as required. □

This gives us a very useful tool: since it gives us a field from an integral domain, it allows us to use field techniques to study integral domains. Moreover, we can also use it to construct new interesting fields from integral domains.

Lemma. A non-zero ring R is a field if and only if its only ideals are $\{0\}$ and R.

Proof. ($\Rightarrow$) Let $I \lhd R$ and R be a field. Suppose $x \neq 0 \in I$. Then as x is a unit, $I = R$.

($\Leftarrow$) Suppose $x \neq 0 \in R$. Then (x) is an ideal of R which is not $\{0\}$ since it contains x. So $(x) = R$ and so $1_R \in (x)$. Thus there is some $u \in R$ such that $x \cdot u = 1_R$, so x is a unit. Since x was arbitrary, R is a field. □

This is another reason why fields are special. They have the simplest possible ideal structure. It motivates the following definition.

Definition (Maximal ideal)**.** An ideal I of a ring R is *maximal* if $I \neq R$ and for any ideal J with $I \leq J \leq R$, either $J = I$ or $J = R$.

The relation with what we have discussed above is quite simple: there is an easy way to recognize if an ideal is maximal.

Lemma. An ideal $I \lhd R$ is maximal if and only if R/I is a field.

Proof. R/I is a field if and only if $\{0\}$ and R/I are the only ideals of R/I. By the ideal correspondence, this is equivalent to saying I and R are the only ideals of R which contains I, i.e. I is maximal. □

This result characterises a property of an ideal I in terms of a property of the quotient R/I. Here is another one:

Definition (Prime ideal)**.** An ideal I of a ring R is *prime* if $I \neq R$ and whenever $a, b \in R$ are such that $a \cdot b \in I$, then $a \in I$ or $b \in I$.

This is like the converse of the property of being an ideal — being an ideal means if we have something in the ideal and something outside, the product is always in the ideal. This does the converse: if the product of two elements is in the ideal, then one of them must be from the ideal.

Example. A non-zero ideal $n\mathbb{Z} \lhd \mathbb{Z}$ is prime if and only if n is a prime number.

To show this, first suppose $n = p$ is a prime number, and $a \cdot b \in p\mathbb{Z}$. So $p \mid a \cdot b$. So $p \mid a$ or $p \mid b$, i.e. $a \in p\mathbb{Z}$ or $b \in p\mathbb{Z}$.

For the other direction, suppose $n = pq$ is a composite number $(p, q \neq 1)$. Then $n \in n\mathbb{Z}$ but $p \notin n\mathbb{Z}$ and $q \notin n\mathbb{Z}$, since $0 < p, q < n$.

We prove a characterisation similar to the lemma above.

Lemma. An ideal $I \lhd R$ is prime if and only if R/I is an integral domain.

Proof. Let I be prime. Let $a + I, b + I \in R/I$, and suppose that $(a + I)(b + I) = 0_{R/I}$. By definition, $(a + I)(b + I) = ab + I$. So we must have $ab \in I$. As I is prime, either $a \in I$ or $b \in I$. So $a + I = 0_{R/I}$ or $b + I = 0_{R/I}$, and hence R/I is an integral domain.

Conversely, suppose R/I is an integral domain. Let $a, b \in R$ be such that $ab \in I$. Then $(a + I)(b + I) = ab + I = 0_{R/I} \in R/I$. Since R/I is an integral domain, either $a + I = 0_{R/I}$ or $b + I = 0_{R/i}$, i.e. $a \in I$ or $b \in I$. So I is a prime ideal. □

Prime ideals and maximal ideals are the main types of ideals we shall be interested in. Every field is an integral domain, so we immediately have the following.

Proposition. Every maximal ideal is a prime ideal.

Proof. $I \lhd R$ is maximal implies R/I is a field implies R/I is an integral domain implies I is prime. □

The converse is not true. For example, $\{0\} \subseteq \mathbb{Z}$ is prime but not maximal. Less stupidly, $(X) \in \mathbb{Z}[X, Y]$ is prime but not maximal (since $\mathbb{Z}[X, Y]/(X) \cong \mathbb{Z}[Y]$ is an integral domain but is not a field).

Lemma. Let R be an integral domain. Then its characteristic is either 0 or a prime number.

Proof. Consider the unique map $\phi : \mathbb{Z} \to R$, and $\ker(\phi) = n\mathbb{Z}$. Then n is the characteristic of R by definition. By the first isomorphism theorem, $\mathbb{Z}/n\mathbb{Z} = \text{im}(\phi) \leq R$. So $\mathbb{Z}/n\mathbb{Z}$ is an integral domain. So $n\mathbb{Z} \lhd \mathbb{Z}$ is a prime. So $n = 0$ or a prime number. □

2.4 Factorization in integral domains

We now move on to tackle the problem of factorization in rings. We suppose throughout the section that R is an integral domain.

Definition (Unit)**.** An element $a \in R$ is a *unit* if there is an $b \in R$ such that $ab = 1_R$. Equivalently, if $(a) = R$.

Definition (Division)**.** For elements $a, b \in R$, we say a *divides* b, written $a \mid b$, if there is a $c \in R$ such that $b = ac$. Equivalently, if $(b) \subseteq (a)$.

Definition (Associates)**.** We say $a, b \in R$ are *associates* if $a = bc$ for some unit c. Equivalently, if $(a) = (b)$. Equivalently, if $a \mid b$ and $b \mid a$.

In integers, this can only happen if a and b differ by a sign, but in more interesting rings, more interesting things can happen. When considering division in rings, we often consider two associates to be "the same". For example, in $\mathbb{Z}$, we can factorize 6 as

$$6 = 2 \cdot 3 = (-2) \cdot (-3),$$

but this does not violate unique factorization, since 2 and -2 are associates (and so are 3 and -3), and we consider these two factorizations to be "the same".

Definition (Irreducible)**.** We say $a \in R$ is *irreducible* if $a \neq 0$, a is not a unit, and if $a = xy$, then x or y is a unit.

For integers, being irreducible is the same as being a prime number. However, "prime" means something different in general rings.

Definition (Prime)**.** We say $a \in R$ is *prime* if a is non-zero, not a unit, and whenever $a \mid xy$, either $a \mid x$ or $a \mid y$.

It is important to note all these properties depend on the ring, not the element itself.

Example. $2 \in \mathbb{Z}$ is a prime, but $2 \in \mathbb{Q}$ is not (since it is a unit).

Similarly, the polynomial $2X \in \mathbb{Q}[X]$ is irreducible (since 2 is a unit), but $2X \in \mathbb{Z}[X]$ not irreducible.

We have two things called prime, so they had better be related.

Lemma. A principal ideal (r) is a prime ideal in R if and only if $r = 0$ or r is prime.

Proof. ($\Rightarrow$) Let (r) be a prime ideal. If $r = 0$, then done. Otherwise, as prime ideals are proper, ie. not the whole ring, r is not a unit. Now suppose $r \mid a \cdot b$. Then $a \cdot b \in (r)$. But (r) is prime. So $a \in (r)$ or $b \in (r)$. So $r \mid a$ or $r \mid b$. So r is prime.

($\Leftarrow$) If $r = 0$, then $(0) = \{0\} \lhd R$, which is prime since R is an integral domain. Otherwise, let $r \neq 0$ be prime. Suppose $a \cdot b \in (r)$. This means $r \mid a \cdot b$. So $r \mid a$ or $r \mid b$. So $a \in (r)$ and $b \in (r)$. So (r) is prime. □

Note that in $\mathbb{Z}$, prime numbers are exactly the irreducibles, but prime numbers are also prime (surprise!). In general, it is not true that irreducibles are the same as primes. However, one direction is always true.

Lemma. Let $r \in R$ be prime. Then it is irreducible.

Proof. Let $r \in R$ be prime, and suppose $r = ab$. Since $r \mid r = ab$, and r is prime, we must have $r \mid a$ or $r \mid b$. wlog, $r \mid a$. So $a = rc$ for some $c \in R$. So $r = ab = rcb$. Since we are in an integral domain, we must have $1 = cb$. So b is a unit. □

We now do a long interesting example.

Example. Let

$$R = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\} \leq \mathbb{C}.$$

By definition, it is a subring of a field. So it is an integral domain. What are the units of the ring? There is a nice trick we can use, when things are lying inside $\mathbb{C}$. Consider the function

$$\begin{aligned} N : R &\longrightarrow \mathbb{Z}_{\geq 0} \\ a + b\sqrt{-5} &\longmapsto a^2 + 5b^2, \end{aligned}$$

called the *norm*. It is convenient to think of this as $z \mapsto z\bar{z} = |z|^2$. It satisfies $N(z \cdot w) = N(z)N(w)$. This is a desirable thing to have for a ring, since it immediately implies all units have norm 1 — if $r \cdot s = 1$, then $1 = N(1) = N(rs) = N(r)N(s)$. So $N(r) = N(s) = 1$.

So to find the units, we need to solve $a^2 + 5b^2 = 1$, for a and b integers. The only solutions are ± 1. So only $\pm 1 \in R$ can potentially be units, and these obviously are units. So these are all the units.

Next, we claim that $2 \in R$ is irreducible. We again use the norm. Suppose $2 = ab$. Then $4 = N(2) = N(a)N(b)$. Now note that nothing has norm 2: $a^2 + 5b^2$ can never be 2 for integers $a, b \in \mathbb{Z}$. So one of a and b must have norm 1, and so must be a unit. Similarly, we see that $3, 1 + \sqrt{-5}, 1 - \sqrt{-5}$ are irreducible (since there is also no element of norm 3).

We have four irreducible elements in this ring. Are they prime? No! Note that

$$(1 + \sqrt{-5})(1 - \sqrt{-5}) = 6 = 2 \cdot 3.$$

We now claim 2 does not divide $1 + \sqrt{-5}$ or $1 - \sqrt{-5}$. So 2 is not prime.

To see this, suppose $2 \mid 1 + \sqrt{-5}$. Then $N(2) \mid N(1 + \sqrt{-5})$. But $N(2) = 4$ and $N(1 + \sqrt{-5}) = 6$, and $4 \nmid 6$. Similarly, $N(1 - \sqrt{-5}) = 6$ as well. So $2 \nmid 1 \pm \sqrt{-5}$.

There are several lessons here. First is that primes and irreducibles are not the same thing in general. The second one is that factorization into irreducibles is not necessarily unique, since $2 \cdot 3 = 6 = (1 + \sqrt{-5})(1 - \sqrt{-5})$ are two factorizations into irreducibles.

However, there is a situation when unique factorizations holds. This is when we have a Euclidean algorithm available.

Definition (Euclidean domain)**.** An integral domain R is a *Euclidean domain* (ED) if there is a *Euclidean function* $\phi : R \setminus \{0\} \to \mathbb{Z}_{\geq 0}$ such that

(i) $\phi(a \cdot b) \geq \phi(b)$ for all $a, b \neq 0$

(ii) If $a, b \in R$, with $b \neq 0$, then there are $q, r \in R$ such that

$$a = b \cdot q + r,$$

and either $r = 0$ or $\phi(r) < \phi(b)$.

What are examples? Every time in this course where we said "Euclidean algorithm", we have an example.

Example. $\mathbb{Z}$ is a Euclidean domain with $\phi(n) = |n|$.

Example. For any field $\mathbb{F}$, $\mathbb{F}[X]$ is a Euclidean domain with $\phi(f) = \deg(f)$.

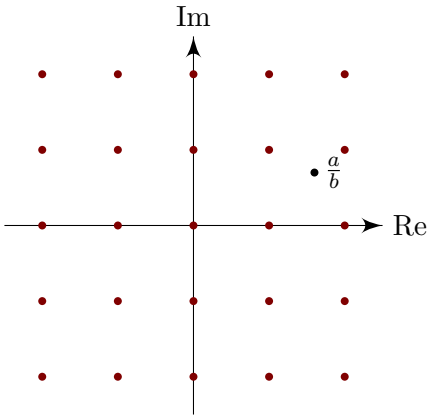
Example. The Gaussian integers $R = \mathbb{Z}[i] \leq \mathbb{C}$ is a Euclidean domain with $\phi(z) = N(z) = |z|^2$. We now check this:

(i) We have $\phi(zw) = \phi(z)\phi(w) \geq \phi(z)$, since $\phi(w)$ is a positive integer.

(ii) Given $a, b \in \mathbb{Z}[i]$, $b \neq 0$. We consider the complex number

$$\frac{a}{b} \in \mathbb{C}.$$

Consider the complex plane, where the red dots are points in $\mathbb{Z}[i]$.



By looking at the picture, we know that there is some $q \in \mathbb{Z}[i]$ such that $\left|\frac{a}{b} - q\right| < 1$. So we can write

$$\frac{a}{b} = q + c$$

with $|c| < 1$. Then we have

$$a = b \cdot q + \underbrace{b \cdot c}_{r}.$$

We know $r = a - bq \in \mathbb{Z}[i]$, and $\phi(r) = N(bc) = N(b)N(c) < N(b) = \phi(b)$. So done.

This is not just true for the Gaussian integers. All we really needed was that $R \leq \mathbb{C}$, and for any $x \in \mathbb{C}$, there is some point in R that is less than distance 1 from x. If we draw some more pictures, we will see this is not true for $\mathbb{Z}[\sqrt{-5}]$.

Before we move on to prove unique factorization, we first derive something we've previously mentioned. Recall we showed that every ideal in $\mathbb{Z}$ is principal, and we proved this by the Euclidean algorithm. So we might expect this to be true in an arbitrary Euclidean domain.

Definition (Principal ideal domain)**.** A ring R is a *principal ideal domain* (PID) if it is an integral domain, and every ideal is a principal ideal, i.e for all $I \lhd R$, there is some a such that $I = (a)$.

Example. $\mathbb{Z}$ is a principal ideal domain.

Proposition. Let R be a Euclidean domain. Then R is a principal ideal domain.

We have already proved this, just that we did it for a particular Euclidean domain $\mathbb{Z}$. Nonetheless, we shall do it again.

Proof. Let R have a Euclidean function $\phi : R \setminus \{0\} \to \mathbb{Z}_{\geq 0}$. We let $I \lhd R$ be a non-zero ideal, and let $b \in I \setminus \{0\}$ be an element with $\phi(b)$ minimal. Then for any $a \in I$, we write

$$a = bq + r,$$

with $r = 0$ or $\phi(r) < \phi(b)$. However, any such r must be in I since $r = a - bq \in I$. So we cannot have $\phi(r) < \phi(b)$. So we must have $r = 0$. So $a = bq$. So $a \in (b)$. Since this is true for all $a \in I$, we must have $I \subseteq (b)$. On the other hand, since $b \in I$, we must have $(b) \subseteq I$. So we must have $I = (b)$. □

This is exactly the same proof as we gave for the integers, except we replaced the absolute value with ϕ.

Example. $\mathbb{Z}$ is a Euclidean domain, and hence a principal ideal domain. Also, for any field $\mathbb{F}$, $\mathbb{F}[X]$ is a Euclidean domain, hence principal ideal domain.

Also, $\mathbb{Z}[i]$ is a Euclidean domain, and hence a principal ideal domain.

Example. What is a non-example of principal ideal domains? In $\mathbb{Z}[X]$, the ideal $(2, X) \lhd \mathbb{Z}[X]$ is not a principal ideal. Suppose it were. Then $(2, X) = (f)$. Since $2 \in (2, X) = (f)$, we know $2 \in (f)$, ie. $2 = f \cdot g$ for some g. So f has degree zero, and hence is constant. So $f \in \{\pm 1, \pm 2\}$.

If $f = \pm 1$, since ± 1 are units, then $(f) = \mathbb{Z}[X]$. But $(2, X) \neq \mathbb{Z}[X]$ (convince yourself of this). If $f = \pm 2$, then since $X \in (2, X) = (f)$, we must have $\pm 2 \mid X$, but this is false. So $(2, X)$ cannot be a principal ideal.

Example. Let $A \in M_{n\times n}(\mathbb{F})$ be an $n \times n$ matrix over a field $\mathbb{F}$. We consider the following set

$$I = \{f \in \mathbb{F}[X] : f(A) = 0\}.$$

This is an ideal: if $f, g \in I$ then $(f + g)(A) = f(A) + g(A) = 0$, and if $f \in I$ and $h \in \mathbb{F}[X]$, then $(fg)(A) = f(A)g(A) = 0$.

But we know $\mathbb{F}[X]$ is a principal ideal domain. So there must be some $m \in \mathbb{F}[X]$ such that $I = (m)$ for some m.

Suppose $f \in \mathbb{F}[X]$ such that $f(A) = 0$, ie. $f \in I$. Then $m \mid f$. So m is a polynomial which divides all polynomials that kill A, ie. m is a *minimal polynomial* of A.

We have just proved that all matrices have minimal polynomials, and that the minimal polynomial divides all other polynomials that kill A. Furthermore, a minimal polynomial is unique up to multiplication by units (it is usually taken to be monic, to get rid of this ambiguity: then we call it *the* minimal polynomial).

For a general ring, we cannot factorize things into irreducibles uniquely. However, in some rings, this is possible.

Definition (Unique factorization domain)**.** An integral domain R is a *unique factorization domain* (UFD) if

(i) Every non-unit may be written as a product of irreducibles;

(ii) If $p_1 p_2 \cdots p_n = q_1 \cdots q_m$ with p_i, q_j irreducibles, then $n = m$, and they can be reordered such that p_i is an associate of q_i.

This is a really nice property, and here we can do things we are familiar with in number theory. So how do we know if something is a unique factorization domain?

Our goal is to show that all principal ideal domains are unique factorization domains. To do so, we are going to prove several lemmas that give us some really nice properties of principal ideal domains.

Recall that every prime is an irreducible, but in $\mathbb{Z}[\sqrt{-5}]$, for example, there are some irreducibles that are not prime. However, this cannot happen in principal ideal domains.

Lemma. Let R be a principal ideal domain. If $p \in R$ is irreducible, then it is prime.

Proof. Let $p \in R$ be irreducible, and suppose $p \mid a \cdot b$. Also, suppose $p \nmid a$. We need to show $p \mid b$.

Consider the ideal $(p, a) \lhd R$. Since R is a principal ideal domain, there is some $d \in R$ such that $(p, a) = (d)$. So $d \mid p$ and $d \mid a$.

Since $d \mid p$, there is some q_1 such that $p = q_1 d$. As p is irreducible, either q_1 or d is a unit.

If q_1 is a unit, then $d = q_1^{-1} p$, and this divides a. So $a = q_1^{-1} p x$ for some x. This is a contradiction, since $p \nmid a$.

Therefore d is a unit. So $(p, a) = (d) = R$. In particular, $1_R \in (p, a)$. So suppose $1_R = rp + sa$, for some $r, s \in R$. We now take the whole thing and multiply by b. Then

$$b = rpb + sab.$$

We observe that ab is divisible by p, and so is p. So b is divisible by p. So done. □

This is similar to the argument for integers. For integers, we would say if $p \nmid a$, then p and a are coprime. Therefore there are some r, s such that $1 = rp + sa$. Then we continue the proof as above. Hence what we did in the middle is to do something similar to showing p and a are "coprime".

Another nice property of principal ideal domains is the following:

Lemma. Let R be a principal ideal domain. Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ be a chain of ideals. Then there is some $N \in \mathbb{N}$ such that $I_n = I_{n+1}$ for some $n \geq N$.

So in a principal ideal domain, we cannot have an infinite chain of bigger and bigger ideals. Rings with this property are called *Noetherian*, and we will study them in more detail later.

Proof. The obvious thing to do when we have an infinite chain of ideals is to take the union of them. We let

$$I = \bigcup_{n \geq 1}^{\infty} I_n,$$

which is again an ideal. Since R is a principal ideal domain, $I = (a)$ for some $a \in R$. We know $a \in I = \bigcup_{n=0}^{\infty} I_n$. So $a \in I_N$ for some N. Then we have $(a) \subseteq I_N \subseteq I = (a)$. So we must have $I_N = I$. So $I_n = I_N = I$ for all $n \geq N$. □

Finally, we have done the setup, and we can prove the proposition promised.

Proposition. Let R be a principal ideal domain. Then R is a unique factorization domain.

Proof. We first need to show any (non-unit) $r \in R$ is a product of irreducibles.

Suppose $r \in R$ cannot be factored as a product of irreducibles. Then it is certainly not irreducible. So we can write $r = r_1 s_1$, with r_1, s_1 both non-units. Since r cannot be factored as a product of irreducibles, without loss of generality we may suppose that r_1 cannot be factored as a product of irreducibles (if both r_1 and s_1 can, then r would be a

product of irreducibles). So we can write $r_1 = r_2 s_2$, with r_2, s_2 not units. Again, without loss of generality we may suppose that r_2 cannot be factored as a product of irreducibles. We continue in this way.

By assumption, the process does not end, and then we have the following chain of ideals:

$$(r) \subseteq (r_1) \subseteq (r_2) \subseteq \cdots \subseteq (r_n) \subseteq \cdots$$

But then we have an ascending chain of ideals. By the previous Lemma, these are all eventually equal, i.e. there is some n such that $(r_n) = (r_{n+1}) = (r_{n+2}) = \cdots$. In particular, since $(r_n) = (r_{n+1})$, and $r_n = r_{n+1}s_{n+1}$, then s_{n+1} is a unit. But this is a contradiction, since s_{n+1} is not a unit. So r must be a product of irreducibles.

To show uniqueness, we let $p_1 p_2 \cdots p_n = q_1 q_2 \cdots q_m$, with p_i, q_i irreducible. So in particular $p_1 \mid q_1 \cdots q_m$. Since p_1 is irreducible, it is prime. So p_1 divides some q_i. We reorder and suppose $p_1 \mid q_1$. So $q_1 = p_1 \cdot a$ for some a. But since q_1 is irreducible, a must be a unit. So p_1, q_1 are associates. Since R is a principal ideal domain, hence integral domain, we can cancel p_1 to obtain

$$p_2 p_3 \cdots p_n = (aq_2) q_3 \cdots q_m.$$

We now rename aq_2 as q_2, so that we in fact have

$$p_2 p_3 \cdots p_n = q_2 q_3 \cdots q_m.$$

We can then continue to show that p_i and q_i are associates for all i. This also shows that $n = m$, or else if $n = m + k$, saw, then $p_{k+1} \cdots p_n = 1$, which is a contradiction. □

We can now use this to define other familiar notions from number theory.

Definition (Greatest common divisor)**.** d is a *greatest common divisor* (gcd) of $a_1, a_2, \cdots, a_n$ if $d \mid a_i$ for all i, and if any other d' satisfies $d' \mid a_i$ for all i, then $d' \mid d$.

Definition (Least common multiple)**.** m is a *least common multiple* (lcm) of $a_1, a_2, \cdots, a_n$ if $a_i \mid m$ for all i, and if any other m' satisfies $a_i | m'$ for all i, then $m \mid m'$.

This is a definition that says what it means to be a gcd or lcm. However, it does not always have to exist.

Lemma. Let R be a unique factorization domain. Then gcd's and lcm's exist, and are unique up to associates.

Proof. We construct the greatest common divisor using prime factorization.

We let $p_1, p_2, \cdots, p_m$ be a list of all irreducible factors of a_i, such that no two of these are associates of each other. We now write

$$a_i = u_i \prod_{j=1}^{m} p_j^{n_{ij}},$$

where $n_{ij} \in \mathbb{N}$ and u_i are units. We let

$$m_j = \min_i \{n_{ij}\},$$

and choose

$$d = \prod_{j=1}^{m} p_j^{m_j}.$$

As, by definition, $m_j \leq n_{ij}$ for all i, we know $d \mid a_i$ for all i.

Finally, if $d' \mid a_i$ for all i, then we let

$$d' = v \prod_{j=1}^{m} p_j^{t_j}.$$

Then we must have $t_j \leq n_{ij}$ for all i, j. So we must have $t_j \leq m_j$ for all j. So $d' \mid d$.

Uniqueness is immediate since any two greatest common divisors have to divide each other.

The argument for least common multiples is similar. □

2.5 Factorization in polynomial rings

Recall that for F a field, we know $F[X]$ is a Euclidean domain, hence a principal ideal domain, hence a unique factorization domain. Therefore we know

(i) If $I \lhd F[X]$, then $I = (f)$ for some $f \in F[X]$.

(ii) If $f \in F[X]$, then f is irreducible if and only if f is prime.

(iii) Let f be irreducible, and suppose $(f) \subseteq J \subseteq F[X]$. Then $J = (g)$ for some g. Since $(f) \subseteq (g)$, we must have $f = gh$ for some h. But f is irreducible. So either g or h is a unit. If g is a unit, then $(g) = F[X]$. If h is a unit, then $(f) = (g)$. So (f) is a maximal ideal. Note that this argument is valid for any PID, not just polynomial rings.

(iv) Let (f) be a prime ideal. Then f is prime. So f is irreducible. So (f) is maximal. But we also know in complete generality that maximal ideals are prime. So in $F[X]$, prime ideals are the same as maximal ideals. Again, this is true for all PIDs in general.

(v) Thus f is irreducible if and only if $F[X]/(f)$ is a field.

To use the last item, we can first show that $F[X]/(f)$ is a field, and then use this to deduce that f is irreducible. But we can also do something more interesting — find an irreducible f, and then generate an interesting field $F[X]/(f)$.

So we want to understand (ir)reducibility, i.e. we want to know whether we can factorize a polynomial f. Firstly, we want to get rid of the trivial case where we just factor out a scalar, eg. $2X^2 + 2 = 2(X^2 + 1) \in \mathbb{Z}[X]$ is a boring factorization.

Definition (Content)**.** Let R be a UFD and $f = a_0 + a_1 X + \cdots + a_n X^n \in R[X]$. The *content* $c(f)$ of f is

$$c(f) = \gcd(a_0, a_1, \cdots, a_n) \in R.$$

Again, since the gcd is only defined up to a unit, so is the content.

Definition (Primitive polynomial)**.** A polynomial is *primitive* if $c(f)$ is a unit, i.e. the a_i are coprime.

Note that this is the best we can do. We cannot ask for $c(f)$ to be exactly 1, since the gcd is only well-defined up to a unit.

Lemma (Gauss' lemma)**.** Let R be a UFD, and $f \in R[X]$ be a primitive polynomial. Then f is reducible in $R[X]$ if and only if f is reducible $F[X]$, where F is the field of fractions of R.

We can't do this right away. We first need some preparation. Before that, we do some examples.

Example. Consider $X^3 + X + 1 \in \mathbb{Z}[X]$. This has content 1 so is primitive. We show it is not reducible in $\mathbb{Z}[X]$, and hence not reducible in $\mathbb{Q}[X]$.

Suppose f *is* reducible in $\mathbb{Q}[X]$. Then by Gauss' lemma it is reducible in $\mathbb{Z}[X]$, so

$$X^3 + X + 1 = gh,$$

for some polynomials $g, h \in \mathbb{Z}[X]$, with g, h not units. But if g and h are not units, then they cannot be constant, so they have degree at least 1. Since the degrees add up to 3, we may suppose that g has degree 1 and h has degree 2. So suppose

$$g = b_0 + b_1 X, \quad h = c_0 + c_1 X + c_2 X^2.$$

Multiplying out and equating coefficients, we get

$$b_0 c_0 = 1$$
$$c_2 b_1 = 1$$

So b_0 and b_1 must be ± 1. So g is either $1 + X, 1 - X, -1 + X$ or $-1 - X$, and hence has ± 1 as a root. But this is a contradiction, since ± 1 is not a root of $X^3 + X + 1$. So f is not reducible in $\mathbb{Q}[X]$. In particular f has no root in $\mathbb{Q}$, and $\mathbb{Q}[X]/(X^3 + X + 1)$ is a field.

We see the advantage of using Gauss' lemma — if we worked in $\mathbb{Q}$ instead, we could have got to the step $b_0 c_0 = 1$, and then we can do nothing, since b_0 and c_0 can be many things if we work over $\mathbb{Q}$.

Now we start working towards proving Gauss' lemma, with the following preparatory tool.

Lemma. Let R be a UFD. If $f, g \in R[X]$ are primitive, then so is fg.

Proof. We let

$$f = a_0 + a_1 X + \cdots + a_n X^n,$$
$$g = b_0 + b_1 X + \cdots + b_m X^m,$$

where $a_n, b_m \neq 0$, and f, g are primitive. We want to show that the content of fg is a unit.

If fg is not primitive then $c(fg)$ is not a unit. Since R is a UFD, we can find an irreducible $p \in R$ which divides $c(fg)$.

By assumption, $c(f)$ and $c(g)$ are units, so $p \nmid c(f)$ and $p \nmid c(g)$. So suppose $p \mid a_0$, $p \mid a_1$, $\ldots$, $p \mid a_{k-1}$ but $p \nmid a_k$. (Note it is possible that $k = 0$.) Similarly, suppose $p \mid b_0, p \mid b_1, \cdots, p \mid b_{\ell-1}, p \nmid b_\ell$.

We look at the coefficient of $X^{k+\ell}$ in fg. It is given by

$$\sum_{i+j=k+\ell} a_i b_j = a_{k+\ell} b_0 + \cdots + a_{k+1} b_{\ell-1} + a_k b_\ell + a_{k-1} b_{\ell+1} + \cdots + a_0 b_{\ell+k}.$$

By assumption, this is divisible by p. So

$$p \mid \sum_{i+j=k+\ell} a_i b_j.$$

However, the terms $a_{k+\ell} b_0 + \cdots + a_{k+1} b_{\ell-1}$, is divisible by p, as $p \mid b_j$ for $j < \ell$. Similarly, $a_{k-1} b_{\ell+1} + \cdots + a_0 b_{\ell+k}$ is divisible by p, as $p \mid a_i$ for $i < k$. So we must have $p \mid a_k b_\ell$. As p is irreducible, and hence prime, we must have $p \mid a_k$ or $p \mid b_\ell$. This is a contradiction. So $c(fg)$ must be a unit. $\square$

Corollary. Let R be a UFD. Then for $f, g \in R[X]$, we have that $c(fg)$ is an associate of $c(f)c(g)$.

Proof. We can write $f = c(f)f_1$ and $g = c(g)g_1$, with f_1 and g_1 irreducible. Then

$$fg = c(f)c(g)f_1g_1.$$

Since f_1g_1 is primitive, $c(f)c(g)$ is a gcd of the coefficients of fg; so is $c(fg)$, by definition. Thus they are associates. □

Finally, we can prove Gauss' lemma.

Lemma (Gauss' lemma)**.** Let R be a UFD, and $f \in R[X]$ be a primitive polynomial. Then f is reducible in $R[X]$ if and only if f is reducible $F[X]$, where F is the field of fractions of R.

Proof. We will show that a primitive $f \in R[X]$ is reducible in $R[X]$ if and only if f is reducible in $F[X]$.

One direction is almost immediately obvious. Let $f = gh$ be a product in $R[X]$ with g, h not units. As f is primitive, so are g and h. So both have degree > 0. So g, h are not units in $F[X]$. So f is reducible in $F[X]$.

The other direction is less obvious. We let $f = gh$ in $F[X]$, with g, h not units. So g and h have degree > 0, since F is a field. So we can clear denominators by finding $a, b \in R$ such that $(ag), (bh) \in R[X]$ (eg. let a be the product of denominators of coefficients of g). Then we get

$$abf = (ag)(bh),$$

and this is a factorization in $R[X]$. Here we have to be careful — (ag) is an element of $R[X]$, and is not necessarily a product in $R[X]$, since g might not be in $R[X]$. So we should just treat it as a single symbol.

We now write

$$(ag) = c(ag)g_1,$$
$$(bh) = c(bh)h_1,$$

where g_1, h_1 are primitive. So we have

$$ab = c(abf) = c((ag)(bh)) = u \cdot c(ag)c(bh),$$

where $u \in R$ is a unit, by the previous corollary. But also we have

$$abf = c(ag)c(gh)g_1h_1 = u^{-1}abg_1h_1.$$

So cancelling ab gives

$$f = u^{-1}g_1h_1 \in R[X].$$

So f is reducible in $R[X]$. □

We will do another proof performed in a similar manner.

Proposition. Let R be a UFD, and F be its field of fractions. Let $g \in R[X]$ be primitive. We let

$$J = (g) \lhd R[X], \quad I = (g) \lhd F[X].$$

Then

$$J = I \cap R[X].$$

In other words, if $f \in R[X]$ is divisible by g in $F[X]$, then it is divisible by it in $R[X]$.

Proof. We certainly have $J \subseteq I \cap R[X]$. Now let $f \in I \cap R[X]$. So we can write

$$f = gh,$$

with $h \in F[X]$. So we can choose $b \in R$ such that $bh \in R[X]$. Then we know

$$bf = g(bh) \in R[X].$$

We let

$$(bh) = c(bh)h_1,$$

for $h_1 \in R[X]$ primitive. Thus

$$bf = c(bh)gh_1.$$

Since g is primitive, so is gh_1. So $c(bh) = uc(bf)$ for u a unit. But bf is a product in $R[X]$, so

$$c(bf) = c(b)c(f) = bc(f).$$

This gives

$$bf = ubc(f)gh_1.$$

Cancelling b gives

$$f = g \cdot (uc(f)h_1).$$

So $g \mid f$ in $R[X]$, and hence $f \in J$. □

From this we can get ourselves a large class of UFDs.

Theorem. If R is a UFD, then $R[X]$ is a UFD.

In particular, if R is a UFD, then $R[X_1, \cdots, X_n]$ is also a UFD.

Proof. Let $f \in R[X]$. We can write $f = c(f)f_1$, with f_1 primitive. Firstly, as R is a UFD, we may factor

$$c(f) = p_1 p_2 \cdots p_n,$$

for $p_i \in R$ irreducible (so also irreducible in $R[X]$). Now we want to deal with f_1.

If f_1 is not irreducible, then we can write

$$f_1 = f_2 f_3,$$

with f_2, f_3 both not units. Since f_1 is primitive, f_2, f_3 also cannot be constants. So we must have $\deg f_2, \deg f_3 > 0$. Also, since $\deg f_2 + \deg f_3 = \deg f_1$, we must have $\deg f_2, \deg f_3 < \deg f_1$. If f_2, f_3 are irreducible, then done. Otherwise, keep on going. We will eventually stop since the degrees have to keep on decreasing. So we can write

$$f_1 = q_1 \cdots q_m,$$

with q_i irreducible. Then

$$f = p_1 p_2 \cdots p_n q_1 q_2 \cdots q_m$$

is a product of irreducibles.

For uniqueness, we first deal with the p's. We note that

$$c(f) = p_1 p_2 \cdots p_n$$

is a unique factorization of the content, up to reordering and associates, as R is a UFD. So cancelling the content, we only have to show that primitives can be factored uniquely.

Suppose we have two factorizations

$$f_1 = q_1 q_2 \cdots q_m = r_1 r_2 \cdots r_\ell.$$

Note that each q_i and each r_i is a factor of the primitive polynomial f_1, so are also primitive. Now we do (perhaps) an unexpected thing. We let F be the field of fractions of R, and consider $q_i, r_i \in F[X]$. Since F is a field, $F[X]$ is a Euclidean domain, hence a principal ideal domain, hence a unique factorization domain.

By Gauss' lemma, since the q_i and r_i are irreducible in $R[X]$, they are also irreducible in $F[X]$. As $F[X]$ is a UFD, we find that $\ell = m$, and after reordering, r_i and q_i are associates, say

$$r_i = u_i q_i,$$

with $u_i \in F[X]$ a unit. What we want to say is that r_i is a unit times q_i in $R[X]$. Firstly, note that $u_i \in F$ as it is a unit. Clearing denominators, we can write

$$a_i r_i = b_i q_i \in R[X].$$

Taking contents, since r_i, q_i are primitives, we know a_i and b_i are associates, say

$$b_i = v_i a_i,$$

with $v_i \in R$ a unit. Cancelling a_i on both sides, we know $r_i = v_i q_i$ as required. □

The key idea is to use Gauss' lemma to say the reducibility in $R[X]$ is the same as reducibility in $F[X]$, as long as we are primitive. The first part about contents is just to turn everything into primitives.

Note that the last part of the proof is just our previous proposition. We could have applied it, but we decide to spell it out in full for clarity.

Example. We know $\mathbb{Z}[X]$ is a UFD, and if R is a UFD, then $R[X_1, \cdots, X_n]$ is also a UFD.

This is a useful thing to know. In particular, it gives us examples of UFDs that are not PIDs. However, in such rings, we would also like to have an easy to determine whether something is reducible. Fortunately, we have the following criterion:

Proposition (Eisentein's criterion)**.** Let R be a UFD, and let

$$f = a_0 + a_1 X + \cdots + a_n X^n \in R[X]$$

be primitive with $a_n \neq 0$. Let $p \in R$ be an irreducible (hence a prime) such that

(i) $p \nmid a_n$;

(ii) $p \mid a_i$ for all $0 \leq i < n$;

(iii) $p^2 \nmid a_0$.

Then f is irreducible in $R[X]$, and hence in $F[X]$ (where F is the field of fractions of F).

It is important that we work in $R[X]$ all the time, until the end where we apply Gauss' lemma. Otherwise, we cannot possibly apply Eisentein's criterion since there are no primes in F.

Proof. Suppose we have a factorization $f = gh$ with

$$g = r_0 + r_1 X + \cdots + r_k X^k$$
$$h = s_0 + s_1 X + \cdots + s_\ell X^\ell,$$

for $r_k, s_\ell \neq 0$.

We know $r_k s_\ell = a_n$. Since $p \nmid a_n$, so $p \nmid r_k$ and $p \nmid s_\ell$. We can also look at bottom coefficients. We know $r_0 s_0 = a_0$. We know $p \mid a_0$ and $p^2 \nmid a_0$. So p divides exactly one of r_0 and s_0. Let us suppose that $p \mid r_0$ and $p \nmid s_0$.

Now let j be such that

$$p \mid r_0, \quad p \mid r_1, \cdots, \quad p \mid r_{j-1}, \quad p \nmid r_j.$$

We now look at a_j. This is, by definition,

$$a_j = r_0 s_j + r_1 s_{j-1} + \cdots + r_{j-1} s_1 + r_j s_0.$$

We know $r_0, \cdots, r_{j-1}$ are all divisible by p. So

$$p \mid r_0 s_j + r_1 s_{j-1} + \cdots + r_{j-1} s_1.$$

Also, since $p \nmid r_j$ and $p \nmid s_0$, we know $p \nmid r_j s_0$, using the fact that p is prime. So $p \nmid a_j$. So we must have $j = n$.

We also know that $j \leq k \leq n$. So we must have $j = k = n$. So $\deg g = n$ and $\deg h = 0$, i.e. h is a constant. But we also know that f is primitive, so h must be a unit. So this was not a proper factorization. □

Example. Consider the polynomial $X^n - p \in \mathbb{Z}[X]$ for p a prime number. Apply Eisentein's criterion with $p \in \mathbb{Z}$, and observe all the conditions hold. This is certainly primitive, since this is monic. So $X^n - p$ is irreducible in $\mathbb{Z}[X]$, hence in $\mathbb{Q}[X]$. In particular, $X^n - p$ has no rational roots, ie. $\sqrt[n]{p}$ is irrational (for $n > 1$).

Example. Consider the polynomial

$$f = X^{p-1} + X^{p-2} + \cdots + X^2 + X + 1 \in \mathbb{Z}[X],$$

where p is a prime number. If we look at this, we notice Eisentein's criteria does not apply. What should we do? We observe that

$$f = \frac{X^p - 1}{X - 1}.$$

So it might be a good idea to let $Y = X - 1$. Then we get a new polynomial

$$\hat{f} = \hat{f}(Y) = \frac{(Y+1)^p - 1}{Y} = Y^{p-1} + \binom{p}{1} Y^{p-2} + \binom{p}{2} Y^{p-3} + \cdots + \binom{p}{p-1}.$$

When we look at it hard enough, we notice Eisentein's criteria can be applied — we know $p \mid \binom{p}{i}$ for $1 \leq i \leq p-1$, but $p^2 \nmid \binom{p}{p-1} = p$. So $\hat{f}$ is irreducible in $\mathbb{Z}[Y]$.

Now if we had a factorization

$$f(X) = g(X)h(X) \in \mathbb{Z}[X],$$

then we get

$$\hat{f}(Y) = g(Y+1)h(Y+1)$$

in $\mathbb{Z}[Y]$. So f is irreducible.

Hence none of the roots of f are rational (but we already know that — they are not even real!).

2.6 Gaussian integers

We've mentioned the Gaussian integers already.

Definition (Gaussian integers)**.** The *Gaussian integers* is the subring

$$\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\} \leq \mathbb{C}.$$

We have already seen that the norm $N(a + ib) = a^2 + b^2$, which is multiplicative, is a Euclidean function for $\mathbb{Z}[i]$. So $\mathbb{Z}[i]$ is a Euclidean domain, hence a principal ideal domain, hence a unique factorization domain. In particular, primes and irreducibles are the same in $\mathbb{Z}[i]$. The units in $\mathbb{Z}[i]$ are $\pm 1, \pm i$, as these are the only elements of norm 1.

We have

$$2 = (1 + i)(1 - i),$$

so 2 is not a prime. However, 3 is a prime, as follows. We have $N(3) = 9$. So if $3 = uv$, with u, v not units, then $9 = N(u)N(v)$, and neither $N(u)$ nor $N(v)$ are 1. So $N(u) = N(v) = 3$. However, $3 = a^2 + b^2$ has no solutions with $a, b \in \mathbb{Z}$, so there is nothing of norm 3. Thus 3 is irreducible, hence a prime. Also, 5 is not a prime, since

$$5 = (1 + 2i)(1 - 2i).$$

The argument above shows that 7 is still a prime in $\mathbb{Z}[i]$.

How can we understand which prime numbers stay primes in the Gaussian integers?

Proposition. A prime number $p \in \mathbb{Z}$ is prime in $\mathbb{Z}[i]$ if and only if $p \neq a^2 + b^2$ for $a, b \in \mathbb{Z} \setminus \{0\}$.

Proof. If $p = a^2 + b^2$, then $p = (a + ib)(a - ib)$. So p is not irreducible.

Now suppose $p = uv$, with u, v not units. Taking norms, we get $p^2 = N(u)N(v)$. So if u and v are not units, then $N(u) = N(v) = p$. Writing $u = a + ib$, then this says $a^2 + b^2 = p$. □

This tells us about some primes in $\mathbb{Z}[i]$; we want to classify all of them. We will need the following helpful lemma:

Lemma. Let p be a prime number. Let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ be the field with p elements. Let $\mathbb{F}_p^\times = \mathbb{F}_p \setminus \{0\}$ be the group of invertible elements under multiplication. Then $\mathbb{F}_p^\times \cong C_{p-1}$.

Proof. Certainly $\mathbb{F}_p^\times$ has order $p - 1$, and is abelian. We know from the classification of finite abelian groups that if $\mathbb{F}_p^\times$ is not cyclic, then it must contain a subgroup $C_m \times C_m$ for $m > 1$.

We consider the polynomial $X^m - 1 \in \mathbb{F}_p[x]$, which is a UFD. At best, this factors into m linear factors. So $X^m - 1$ has at most m distinct roots. But if $C_m \times C_m \leq \mathbb{F}_p^\times$, then we can find m^2 elements of order diving m. So there are m^2 elements of $\mathbb{F}_p$ which are roots of $X^m - 1$. This is a contradiction. □

This is a funny proof, since we have not explicitly found any element of order $p - 1$.

Proposition. The primes in $\mathbb{Z}[i]$ are, up to associates,

(i) prime numbers $p \in \mathbb{Z} \leq \mathbb{Z}[i]$ such that $p \equiv 3 \pmod 4$,

(ii) Gaussian integers $z \in \mathbb{Z}[i]$ with $N(z) = z\bar{z} = p$ for some prime number p such that $p = 2$ or $p \equiv 1 \pmod 4$.

Proof. We first show these are primes. If $p \equiv 3 \pmod 4$, then $p \neq a^2 + b^2$, since a square number mod 4 is always 0 or 1. So these are primes in $\mathbb{Z}[i]$.

On the other hand, if $N(z) = p$, and $z = uv$, then $N(u)N(v) = p$. So $N(u)$ is 1 or $N(v)$ is 1. So u or v is a unit. Note that we did not use the condition that $p \not\equiv 3 \pmod 4$. This is not needed, since $N(z)$ is always a sum of squares, and hence $N(z)$ cannot be a prime that is 3 mod 4.

Now let $z \in \mathbb{Z}[i]$ be irreducible, hence prime. Then $\bar{z}$ is also irreducible. So $N(z) = z\bar{z}$ is a factorization of $N(z)$ into irreducibles. Let $p \in \mathbb{Z}$ be a prime number dividing $N(z)$, which exists since $N(z) \neq 1$.

Now if $p \equiv 3 \pmod 4$, then p itself is prime in $\mathbb{Z}[i]$ by the first part of the proof. So $p \mid N(z) = z\bar{z}$. So $p \mid z$ or $p \mid \bar{z}$. Note that if $p \mid \bar{z}$, then $p \mid z$ by taking complex conjugates. So we get $p \mid z$. Since both p and z are both irreducible, they must be equal up to associates.

Otherwise, we get $p = 2$ or $p \equiv 1 \pmod 4$. If $p \equiv 1 \pmod 4$, then $p - 1 = 4k$ for some $k \in \mathbb{Z}$. As $\mathbb{F}_p^\times \cong C_{p-1} = C_{4k}$, there is a unique element of order 2 (this is true for any cyclic group of even order), which must be $[-1] \in \mathbb{F}_p$. Now let $a \in \mathbb{F}_p^\times$ be an element of order 4. Then a^2 has order 2, so $[a^2] = [-1]$.

This is a complicated way of saying we can find an a such that $p \mid a^2 + 1$. Thus $p \mid (a+i)(a-i)$. In the case where $p = 2$, we know by checking directly that $2 = (1+i)(1-i)$.

In either case, we deduce that p is not prime (hence irreducible), since it clearly does not divide $a \pm i$ (or $1 \pm i$). So we can write $p = z_1 z_2$, for $z_1, z_2 \in \mathbb{Z}[i]$ not units. Now we get

$$p^2 = N(p) = N(z_1)N(z_2).$$

As the z_i are not units, we know $N(z_1) = N(z_2) = p$. By definition, this means $p = z_1\bar{z}_1 = z_2\bar{z}_2$. But also $p = z_1 z_2$. So we must have $\bar{z}_1 = z_2$.

Finally, we have $p = z_1\bar{z}_1 \mid N(z) = z\bar{z}$. All these z, z_i are irreducible. So z must be an associate of z_1 (or maybe $\bar{z}_1$). So in particular $N(z) = p$. □

Corollary. An integer $n \in \mathbb{Z}_{\geq 0}$ may be written as $x^2 + y^2$ (as the sum of two squares) if and only if "when we write $n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ as a product as distinct primes, then $p_i \equiv 3 \pmod 4$ implies n_i is even".

We have proved this in the case when n is a prime.

Proof. If $n = x^2 + y^2$, then we have

$$n = (x + iy)(x - iy) = N(x + iy).$$

Let $z = x + iy$. So we can write $z = \alpha_1 \cdots \alpha_q$ as a product of irreducibles in $\mathbb{Z}[i]$. By the proposition, each α_i have either $\alpha_i = p$ (a genuine prime number with $p \equiv 3 \pmod 4$), or $N(\alpha_i) = p$ is a prime number which is either 2 or $\equiv 1 \pmod 4$. We now take the norm to obtain

$$N = x^2 + y^2 = N(z) = N(\alpha_1)N(\alpha_2)\cdots N(\alpha_q).$$

Now each $N(\alpha_i)$ is either p^2 with $p \equiv 3 \pmod 4$, or is just p for $p = 2$ or $p \equiv 1 \pmod 4$. So if p^m is the largest power of p divides n, we find that n must be even if $p \equiv 3 \pmod 4$.

Conversely, let $n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ be a product of distinct primes. Now for each p_i, either $p_i \equiv 3 \pmod 4$, and n_i is even, in which case

$$p_i^{n_i} = (p_i^2)^{n_i/2} = N(p_i^{n_i/2});$$

or $p_i = 2$ or $p_i \equiv 1 \pmod 4$, in which case, the above proof shows that $p_i = N(\alpha_i)$ for some α_i, so $p_i^{n_i} = N(\alpha_i^{n_i})$.

Since the norm is multiplicative, we can write n as the norm of some $z \in \mathbb{Z}[i]$. So

$$n = N(z) = N(x + iy) = x^2 + y^2,$$

as required. □

Example. Consider $65 = 5 \times 13$. Since $5, 13 \equiv 1 \pmod 4$, it is a sum of squares. Moreover, the proof tells us how to find 65 as the sum of squares. We have to factor 5 and 13 in $\mathbb{Z}[i]$. We have

$$\begin{aligned} 5 &= (2 + i)(2 - i) \\ 13 &= (2 + 3i)(2 - 3i). \end{aligned}$$

So we know

$$65 = N(2 + i)N(2 + 3i) = N((2 + i)(2 + 3i)) = N(1 + 8i) = 1^2 + 8^2.$$

But there is a choice here. We had to pick which factor is α and which is $\bar{\alpha}$. So we can also write

$$65 = N((2 + i)(2 - 3i)) = N(7 - 4i) = 7^2 + 4^2.$$

So not only are we able to write them as sum of squares, but this also gives us many ways of writing 65 as a sum of squares.

2.7 Algebraic integers

Definition (Algebraic integer)**.** An $\alpha \in \mathbb{C}$ is called an algebraic integer if it is a root of a monic polynomial in $\mathbb{Z}[X]$, ie. there is a monic $f \in \mathbb{Z}[X]$ such that $f(\alpha) = 0$.

We can immediately check that this is a sensible definition — not all complex numbers are algebraic integers, since there are only countably many polynomials with integer coefficients, hence only countably many algebraic integers, but there are uncountably many complex numbers.

Notation. For an algebraic integer α, we write $\mathbb{Z}[\alpha] \leq \mathbb{C}$ for the smallest subring containing α.

We can also construct $\mathbb{Z}[\alpha]$ by taking it as the image of the map $\phi : \mathbb{Z}[X] \to \mathbb{C}$ given by $g \mapsto g(\alpha)$. So we can also write

$$\mathbb{Z}[\alpha] \cong \mathbb{Z}[X]/I, \quad I = \ker \phi.$$

Note that I is non-zero by definition of an algebraic integer.

Proposition. Let $\alpha \in \mathbb{C}$ be an algebraic integer. Then the ideal

$$I = \ker(\phi : \mathbb{Z}[X] \to \mathbb{C}, f \mapsto f(\alpha))$$

is principal, and is equal to (f_α) for some irreducible monic f_α.

This is a non-trivial theorem, since $\mathbb{Z}[X]$ is not a principal ideal domain so there is no immediate guarantee that I is generated by one polynomial.

Definition (Minimal polynomial)**.** Let $\alpha \in \mathbb{C}$ be an algebraic integer. Then the *minimal polynomial* of α is the irreducible monic polynomial f_α such that $I = \ker(\phi) = (f_\alpha)$.

Proof. By definition, there is a monic $f \in \mathbb{Z}[X]$ such that $f(a) = 0$. So $f \in I$. So $I \neq 0$. Now let $f_\alpha \in I$ be such a polynomial of minimal degree. We may suppose that f_α is primitive. We want to show that $I = (f_\alpha)$, and that f_α is irreducible.

Let $h \in I$. In $\mathbb{Q}[X]$ we have the Euclidean algorithm, so we can write

$$h = f_\alpha q + r,$$

with $r = 0$ or $\deg r < \deg f_\alpha$. This was done over $\mathbb{Q}[X]$, not $\mathbb{Z}[X]$. We now clear denominators: multiplying by some non-zero $a \in \mathbb{Z}$ we get

$$ah = f_\alpha(aq) + (ar),$$

where now $(aq), (ar) \in \mathbb{Z}[X]$. We now evaluate these polynomials at α. Then we have

$$ah(\alpha) = f_\alpha(\alpha)aq(\alpha) + ar(\alpha).$$

We know $f_\alpha(\alpha) = h(\alpha) = 0$, since f_α and h are both in I. So $ar(\alpha) = 0$. So $(ar) \in I$. As $f_\alpha \in I$ has minimal degree, we cannot have $\deg(r) = \deg(ar) < \deg(f_a)$. So we must have $r = 0$.

Hence we know

$$ah = f_\alpha \cdot (aq)$$

is a factorization in $\mathbb{Z}[X]$. This is almost right, but we want to factor h, not ah. Again, taking contents of everything, we get

$$ac(h) = c(ah) = c(f_\alpha(aq)) = c(aq),$$

as f_α is primitive. In particular, $a \mid c(aq)$. This, by definition of content, means that (aq) can be written as $a\bar{q}$, where $\bar{q} \in \mathbb{Z}[X]$. Cancelling a, we get $q = \bar{q} \in \mathbb{Z}[X]$. So we know

$$h = f_\alpha q \in (f_\alpha).$$

So we know $I = (f_\alpha)$.

To show that f_α is irreducible, note that

$$\frac{\mathbb{Z}[X]}{(f_\alpha)} \cong \frac{\mathbb{Z}[X]}{\ker \phi} \cong \mathrm{im}(\phi) = \mathbb{Z}[\alpha] \leq \mathbb{C}.$$

Since $\mathbb{C}$ is an integral domain, so is $\mathrm{im}(\phi)$. So we know $\mathbb{Z}[X]/(f_\alpha)$ is an integral domain. So (f_α) is prime. So f_α is prime, hence it is irreducible. □

Example.

(i) We know $\alpha = i$ is an algebraic integer with $f_\alpha = X^2 + 1$.

(ii) Also, $\alpha = \sqrt{2}$ is an algebraic integer with $f_\alpha = X^2 - 2$.

(iii) More interestingly, $\alpha = \frac{1}{2}(1 + \sqrt{-3})$ is an algebraic integer with $f_\alpha = X^2 - X - 1$.

(iv) The monic polynomial $X^5 - X + d \in \mathbb{Z}[X]$ with $d \in \mathbb{Z}_{\geq 0}$ has precisely one real root α, which is an algebraic integer. It is a theorem, which will be proved in II Galois Theory, that this α cannot be constructed from integers via the operations $+, -, \times, \div, \sqrt[n]{\cdot}$. It is also a theorem in that course that degree 5 polynomials are the smallest degree for which this can happen.

Lemma. Let $\alpha \in \mathbb{Q}$ be an algebraic integer. Then $\alpha \in \mathbb{Z}$.

Proof. Let $f_\alpha \in \mathbb{Z}[X]$ be the minimal polynomial, which is irreducible. In $\mathbb{Q}[X]$, the polynomial $X - \alpha$ must divide f_α. However, by Gauss' lemma, we know $f \in \mathbb{Q}[X]$ is irreducible. So we must have $f_\alpha = X - \alpha \in \mathbb{Z}[X]$. So α is an integer. □

It turns out the collection of all algebraic integers form a subring of $\mathbb{C}$. This is not at all obvious — given $f, g \in \mathbb{Z}[X]$ monic such that $f(\alpha) = g(\alpha) = 0$, there is no easy way to find a new monic h such that $h(\alpha + \beta) = 0$. We will prove this later on in the course.

2.8 Hilbert's basis theorem

We now revisit the idea of Noetherian rings, something we have briefly mentioned when proving that PIDs are UFDs.

Definition (Noetherian ring)**.** A ring is *Noetherian* if for any chain of ideals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots,$$

then there is some N such that $I_N = I_{N+1} = I_{N+2} = \cdots$.

This condition is known as the *ascending chain condition* (ACC).

In this language, we have shown that PIDs are Noetherian. The following is the correct general context for this.

Definition (Finitely generated ideal)**.** An ideal I is *finitely generated* if it can be written as $I = (r_1, \cdots, r_n)$ for some $r_1, \cdots, r_n \in R$.

Proposition. A ring is Noetherian if and only if every ideal is finitely generated.

Proof. Suppose every ideal of R is finitely generated. Given the chain $I_1 \subseteq I_2 \subseteq \cdots$, consider the ideal

$$I = I_1 \cup I_2 \cup I_3 \cup \cdots.$$

This is an ideal, as you will check yourself in Example Sheet 2. We know I is finitely generated, say $I = (r_1, \cdots, r_n)$, with $r_i \in I_{k_i}$. Let

$$K = \max_{i=1,\cdots,n} \{k_i\}.$$

Then $r_1, \cdots, r_n \in I_K$. So $I_K = I$. So $I_K = I_{K+1} = I_{K+2} = \cdots$.

To prove the other direction, suppose there is an ideal $I \lhd R$ that is not finitely generated. We pick $r_1 \in I$. Since I is not finitely generated, we know $(r_1) \neq I$. So we can find some $r_2 \in I \setminus (r_1)$. Again $(r_1, r_2) \neq I$. So we can find $r_3 \in I \setminus (r_1, r_2)$. We continue on, and then can find an infinite strictly ascending chain

$$(r_1) \subseteq (r_1, r_2) \subseteq (r_1, r_2, r_3) \subseteq \cdots.$$

So R is not Noetherian. □

Theorem (Hilbert basis theorem)**.** Let R be a Noetherian ring. Then so is $R[X]$.

Proof. Let $J \lhd R[X]$ be an ideal.

Let $f_1 \in J$ be a polynomial of minimal degree. If $J \neq (f_1)$ then let $f_2 \in J \setminus (f_1)$ be a polynomial of minimal degree. If $J \neq (f_1, f_2)$ let $f_3 \in J \setminus (f_1, f_2)$ be a polynomial of minimal degree. Continuing in this way, if $J = (f_1, \ldots, f_i)$ then we are done, so suppose not.

Let $a_i \in R$ be the non-zero coefficient of the largest power of X in f_i, and consider the ideals $(a_1) \subseteq (a_1, a_2) \subseteq \cdots \subseteq (a_1, a_2, \ldots, a_i) \subseteq \cdots$ of R. As R is Noetherian these stabilise, so $(a_1, a_2, \ldots) = (a_1, a_2, \ldots, a_m)$ for some m.

Now $a_{m+1} \in (a_1, a_2, \ldots, a_m)$, so

$$a_{m+1} = \sum_{i=1}^{m} a_i \cdot b_i.$$

Thus the polynomial

$$g = \sum_{i=1}^{m} b_i f_i X^{\deg f_{m+1} - \deg f_i}$$

has the same degree and top coefficient as f_{m+1}. (Note that $\deg f_{m+1} \geq \deg f_i$ for $i \leq m$.) Thus $f_{m+1} - g$ has degree strictly smaller than f_{m+1}. But $g \in (f_1, \ldots, f_m)$ and $f_{m+1} \notin (f_1, \ldots, f_m)$, so

$$f_{m+1} - g \notin (f_1, \ldots, f_m),$$

which contradicts the fact that we chose f_{m+1} to have minimal degree among polynomials in J but not in $(f_1, \ldots, f_m)$.

Hence the process we started with must terminate at some point, so J is finitely-generated. □

Corollary. $\mathbb{Z}[X_1, X_2, \ldots, X_n]$ is Noetherian, and for F a field $F[X_1, X_2, \ldots, X_n]$ is Noetherian. □

Proposition. Let R be a Noetherian ring and I be an ideal of R. Then R/I is Noetherian.

Proof. Let $J \lhd R/I$ be an ideal. We want to show that J is finitely generated. By the ideal correspondence, it corresponds to some ideal $J' \lhd R$ containing I. This is an ideal of R, and is hence finitely generated since R is Noetherian. So $J' = (r_1, \cdots, r_n)$ for some $r_1, \cdots, r_n \in R$. Then J may be generated by $r_1 + I, \cdots, r_n + I$. □

A finitely-generated ring is a quotient of some $\mathbb{Z}[X_1, X_2, \ldots, X_n]$, giving:

Corollary. Any finitely-generated ring is Noetherian. □

3 Modules

Finally, we are going to look at modules. Recall that to define a vector space, we first pick some base field $\mathbb{F}$. We then defined a vector space to be an abelian group V with an action of $\mathbb{F}$ on V (ie. scalar multiplication) that is compatible with the multiplicative and additive structure of $\mathbb{F}$.

In the definition, we did not at all mention division in $\mathbb{F}$. So in fact we can make the same definition, but allow $\mathbb{F}$ to be a ring instead of a field. We call these *modules*. Unfortunately, most results we prove about vector spaces *do* use the fact that $\mathbb{F}$ is a field. So many linear algebra results do not apply to modules, and modules have much richer structures.

3.1 Definitions and examples

Definition (Module)**.** Let R be a commutative ring. We say a quadruple $(M, +, 0_M, \cdot)$ is an *R-module* if

(i) $(M, +, 0_M)$ is an abelian group

(ii) The operation $\cdot : R \times M \to M$ satisfies

(a) $(r_1 + r_2) \cdot m = (r_1 \cdot m) + (r_2 \cdot m)$;
(b) $r \cdot (m_1 + m_2) = (r \cdot m_1) + (r \cdot m_2)$;
(c) $r_1 \cdot (r_2 \cdot m) = (r_1 \cdot r_2) \cdot m$; and
(d) $1_R \cdot M = m$.

Note that there are two different additions going on — addition in the ring and addition in the module, and similarly two notions of multiplication. However, it is easy to distinguish them since they operate on different things. If needed, we can make them explicit by writing, say, $+_R$ and $+_M$.

We can think of modules as rings acting on abelian groups, just as groups can act on sets. Hence we might say "R acts on M" to mean M is an R-module.

Example. Let $\mathbb{F}$ be a field. An $\mathbb{F}$-module is precisely the same as a vector space over $\mathbb{F}$ (the axioms are the same).

Example. For any ring R, we have the R-module $R^n = R \times R \times \cdots \times R$ via

$$r \cdot (r_1, \cdots, r_n) = (rr_1, \cdots, rr_n),$$

using the ring multiplication. This is the same as the definition of the vector space $\mathbb{F}^n$ for fields $\mathbb{F}$.

Example. Let $I \lhd R$ be an ideal. Then it is a R-module via

$$r \cdot_M a = r \cdot_R a, \quad r_1 +_M r_2 = r_1 +_R r_2.$$

Also, R/I is an R-module via

$$r \cdot_M (a + I) = (r \cdot_R a) + I,$$

Example. A $\mathbb{Z}$-module is precisely the same as an abelian group. For A an abelian group, we have

$$\begin{aligned} \mathbb{Z} \times A &\longrightarrow A \\ (n, a) &\longmapsto \underbrace{a + \cdots + a}_{n \text{ times}}, \end{aligned}$$

where we adopt the notation

$$\underbrace{a + \cdots + a}_{-n \text{ times}} = \underbrace{(-a) + \cdots + (-a)}_{n \text{ times}},$$

and adding something to itself 0 times is just 0.

This definition is forced upon us, since by the axioms of a module, we must have $(1, a) \mapsto a$. Then we must send, say, $(2, a) = (1 + 1, a) \mapsto a + a$.

Example. Let $\mathbb{F}$ be a field, V a vector space over $\mathbb{F}$, and $\alpha : V \to V$ be a linear map. Then V is an $\mathbb{F}[X]$-module via

$$\begin{aligned} F[X] \times V &\to V \\ (f, v) &\mapsto f(\alpha)(v). \end{aligned}$$

This *is* a module. Picking a different α's will give a different $\mathbb{F}[X]$-module structures.

Example. Let $\phi : R \to S$ be a homomorphism of rings. Then any S-module M may be considered as an R-module via

$$\begin{aligned} R \times M &\longrightarrow M \\ (r, m) &\longmapsto \phi(r) \cdot_M m. \end{aligned}$$

Definition (Submodule)**.** Let M be an R-module. A subset $N \subseteq M$ is an *R-submodule* if it is a subgroup of $(M, +, 0_M)$, and if $n \in N$ and $r \in R$, then $rn \in N$. We write $N \leq M$.

Example. R itself is an R-module. A subset of R is a submodule if and only if it is an ideal.

Example. A subset of an $\mathbb{F}$-module V, where $\mathbb{F}$ is a field, is a $\mathbb{F}$-submodule if and only if it is a vector subspace of V.

Definition (Quotient module)**.** Let $N \leq M$ be an R-submodule. The *quotient module* M/N is the set of N-cosets in $(M, +, 0_M)$, with the R action given by

$$r \cdot (m + N) = (r \cdot m) + N.$$

This is well-defined and is indeed a module.

Note that modules are different from rings and groups. In groups, we had subgroups, and we had a special kind of subgroups called normal subgroups. We are only allowed to quotient by normal subgroups. In rings, we have subrings and ideals—which are not a special kind of subrings— and we only quotient by ideals. In modules we only have submodules, and we can quotient by arbitrary submodules.

Definition (R-module homomorphism and isomorphism)**.** A function $f : M \to N$ between R-modules is a *R-module homomorphism* if it is a homomorphism of abelian groups, and satisfies

$$f(r \cdot m) = r \cdot f(m)$$

for all $r \in R$ and $m \in M$.

An *isomorphism* is a bijective homomorphism, and two R-modules are isomorphic if there is an isomorphism between them.

Note that on the left, the multiplication is the action in M, while on the right, it is the action in N.

Example. If $\mathbb{F}$ is a field and V, W are $\mathbb{F}$-modules (i.e. vector spaces over $\mathbb{F}$), then an $\mathbb{F}$-module homomorphism is precisely an $\mathbb{F}$-linear map.

Theorem (First isomorphism theorem)**.** Let $f : M \to N$ be an R-module homomorphism. Then

$$\ker f = \{m \in M : f(m) = 0\}$$

is an R-submodule of M. Similarly,

$$\operatorname{im} f = \{f(m) : m \in M\}$$

is an R-submodule of N. Furthermore there is an R-module isomorphism

$$\frac{M}{\ker f} \cong \operatorname{im} f.$$

We will not prove this again. The proof is exactly the same.

Theorem (Second isomorphism theorem)**.** Let $A, B \leq M$. Then

$$A + B = \{m \in M : m = a + b \text{ for some } a \in A, b \in B\}$$

is a submodule of M and $A \cap B$ is a submodule of M. Furthermore there is an R-module isomorphism

$$\frac{A + B}{A} \cong \frac{B}{A \cap B}.$$

Theorem (Third isomorphism theorem)**.** Let $N \leq L \leq M$. Furthermore there is an R-module isomorphism

$$\frac{M}{L} \cong \left(\frac{M}{N}\right) / \left(\frac{L}{N}\right).$$

As usual, we have a correspondence

$$\{\text{submodules of } M/N\} \longleftrightarrow \{\text{submodules of } M \text{ which contain } N\}.$$

It is an exercise to see what these mean in the cases where R is a field, and modules are vector spaces.

We now have a new concept that was not present in rings and groups.

Definition (Annihilator)**.** Let M be a R-module, and $m \in M$. The *annihilator* of m is

$$\operatorname{Ann}(m) = \{r \in R : r \cdot m = 0\}.$$

For any set $S \subseteq M$, we define

$$\operatorname{Ann}(S) = \{r \in R : r \cdot m = 0 \text{ for all } m \in S\} = \bigcap_{m \in S} \operatorname{Ann}(m).$$

In particular, for the module M itself, we have

$$\operatorname{Ann}(M) = \{r \in R : r \cdot m = 0 \text{ for all } m \in M\} = \bigcap_{m \in M} \operatorname{Ann}(m).$$

Note that the annihilator is a subset of R. Moreover it is an ideal — if $r \cdot m = 0$ and $s \cdot m = 0$, then $(r + s) \cdot m = r \cdot m + s \cdot m = 0$. So $r + s \in \operatorname{Ann}(m)$. Moreover, if $r \cdot m = 0$, then also $(sr) \cdot m = s \cdot (r \cdot m) = 0$. So $sr \in \operatorname{Ann}(m)$.

Definition (Submodule generated by an element)**.** Let M be an R-module, and $m \in M$. The *submodule generated by* m is

$$Rm = \{r \cdot m \in M : r \in R\}.$$

Consider the R-module homomorphism

$$\begin{aligned} \phi : R &\longrightarrow M \\ r &\longmapsto rm. \end{aligned}$$

This is clearly a homomorphism. We have $\operatorname{Ann}(m) = \ker(\phi)$ and $Rm = \operatorname{im}(\phi)$, so by the first isomorphism theorem

$$Rm \cong R/\operatorname{Ann}(m).$$

As we mentioned, rings acting on modules is analogous to groups acting on sets: we think of this as the analogue of the orbit stabilizer theorem.

In general, we can generate a submodule with several elements.

Definition (Finitely-generated module)**.** An R-module M is *finitely-generated* if there is a finite list of elements $m_1, \cdots, m_k$ such that

$$M = Rm_1 + Rm_2 + \cdots + Rm_k = \{r_1m_1 + r_2m_2 + \cdots + r_km_k : r_i \in R\}.$$

This is analogous to the notion of a vector space being finite-dimensional. However, in the generality of modules it behaves somewhat differently. While this definition is rather concrete, it is often not the most helpful characterization of finitely-generated modules. Instead, we use the following lemma:

Lemma. An R-module M is finitely-generated if and only if there is a surjective R-module homomorphism $f : R^k \twoheadrightarrow M$ for some finite k.

Proof. If $M = Rm_1 + Rm_2 + \cdots + Rm_k$ then we define $f : R^k \to M$ by

$$(r_1, \cdots, r_k) \mapsto r_1m_1 + \cdots + r_km_k.$$

It is clear that this is an R-module homomorphism. It is by definition surjective, so we are done.

Conversely, given a surjection $f : R^k \twoheadrightarrow M$, we let

$$m_i = f(0, 0, \cdots, 0, 1, 0, \cdots, 0),$$

where the 1 appears in the ith position. We now claim that

$$M = Rm_1 + Rm_2 + \cdots + Rm_k.$$

So let $m \in M$. As f is surjective, we know

$$m = f(r_1, r_2, \cdots, r_k)$$

for some r_i. We then have

$$\begin{aligned} f(r_1, r_2, \cdots, r_k) &= f((r_1, 0, \cdots, 0) + (0, r_2, 0, \cdots, 0) + \cdots + (0, 0, \cdots, 0, r_k)) \\ &= f(r_1, 0, \cdots, 0) + f(0, r_2, 0, \cdots, 0) + f(0, 0, \cdots, 0, r_k) \\ &= r_1 f(1, 0, \cdots, 0) + r_2 f(0, 1, 0, \cdots, 0) + r_k f(0, 0, \cdots, 0, 1) \\ &= r_1m_1 + r_2m_2 + \cdots + r_km_k. \end{aligned}$$

So the m_i generate M. □

This view is a convenient way of thinking about finitely-generated modules. For example, we can immediately prove the following corollary:

Corollary. Let M be finitely-generated and $N \leq M$. Then M/N is also finitely-generated.

Proof. Since m is finitely-generated, we have some surjection $f : R^k \twoheadrightarrow M$. Moreover, we have the surjective quotient map $q : M \twoheadrightarrow M/N$. Then we get the following composition

$$R^k \xrightarrow{f} M \xrightarrow{q} M/N,$$

which is a surjection, since it is a composition of surjections. So M/N is finitely-generated. □

Example. A submodule of a finitely-generated module need not be finitely-generated.

We let $R = \mathbb{C}[X_1, X_2, \cdots]$. We consider the R-module $M = R$, which is finitely-generated (by the single element 1). A submodule of the ring is the same as an ideal. Moreover, an ideal is finitely-generated as an ideal if and only if it is finitely-generated as a module. We pick the submodule

$$I = (X_1, X_2, \cdots),$$

which we have already seen to be not finitely-generated.

Example. For a complex number α, the ring $\mathbb{Z}[\alpha]$ (ie. the smallest subring of $\mathbb{C}$ containing α) is a finitely-generated as a $\mathbb{Z}$-module if and only if α is an algebraic integer.

This is in the last Example Sheet. It allows us to prove that algebraic integers are closed under addition and multiplication, since it is easier to argue about whether $\mathbb{Z}[\alpha]$ is finitely-generated.

3.2 Direct sums and free modules

Definition (Direct sum of modules)**.** Let $M_1, M_2, \cdots, M_k$ be *R-modules*. The *direct sum* is the R-module

$$M_1 \oplus M_2 \oplus \cdots \oplus M_k,$$

which is the set $M_1 \times M_2 \times \cdots \times M_k$, with addition given by

$$(m_1, \cdots, m_k) + (m_1', \cdots, m_k') = (m_1 + m_1', \cdots, m_k + m_k'),$$

and the R action is given by

$$r \cdot (m_1, \cdots, m_k) = (rm_1, \cdots, rm_k).$$

We've been using one example of the direct sum already, namely

$$R^n = \underbrace{R \oplus R \oplus \cdots \oplus R}_{n \text{ times}}.$$

Definition (Linear independence)**.** Let $m_1, \cdots, m_k \in M$. Then $\{m_1, \cdots, m_k\}$ is *linearly independent* if

$$\sum_{i=1}^{k} r_i m_i = 0$$

implies $r_1 = r_2 = \cdots = r_k = 0$.

Most modules will not have a basis in the sense we are used to. The next best thing would be the following:

Definition (Freely generate)**.** A subset $S \subseteq M$ *generates* M *freely* if

(i) S generates M

(ii) Any set function $\psi : S \to N$ to an R-module N extends to an R-module map $\theta : M \to N$.

Note that if θ_1, θ_2 are two such extensions, we can consider $\theta_1 - \theta_2 : S \to M$. Then $\theta_1 - \theta_2$ sends everything in S to 0. So $S \subseteq \ker(\theta_1 - \theta_2) \leq M$. So the submodule generated by S lies in $\ker(\theta_1 - \theta_2)$ too. But this is by definition M. So $M \leq \ker(\theta_1 - \theta_2) \leq M$, ie. equality holds. So $\theta_1 - \theta_2 = 0$. So $\theta_1 = \theta_2$. So any such extension is unique.

Thus, what this definition tells us is that giving a map from M to N is exactly the same thing as giving a function from S to N.

Definition (Free module and basis)**.** An R-module is *free* if it is freely generated by some subset $S \subseteq M$, and S is called a *basis*.

We will soon prove that if R is a field, then every module is free. However, if R is not a field, then there are non-free modules.

Example. The $\mathbb{Z}$-module $\mathbb{Z}/2$ is not free. Suppose $\mathbb{Z}/2$ were generated by some $S \subseteq \mathbb{Z}/2$. Then this can only possibly be $S = \{1\}$. Then this implies there is a homomorphism $\theta : \mathbb{Z}/2 \to \mathbb{Z}$ sending 1 to 1. But it does not send $0 = 1 + 1$ to $1 + 1$, since homomorphisms send 0 to 0. So $\mathbb{Z}/2$ is not free.

Proposition. For a subset $S = \{m_1, \cdots, m_k\} \subseteq M$, the following are equivalent:

(i) S generates M freely.

(ii) S generates M and the set S is independent.

(iii) Every element of M is *uniquely* expressible as

$$r_1 m_1 + r_2 m_2 + \cdots + r_k m_k$$

for some $r_i \in R$.

Proof. The fact that (ii) and (iii) are equivalent is something we would expect from what we know from linear algebra, and in fact the proof is the same. So we only show that (i) and (ii) are equivalent.

Let S generate M freely. If S is not independent, then we can write

$$r_1 m_1 + \cdots + r_k m_k = 0,$$

with $r_i \in M$ and, say, r_1 non-zero. We define the set function $\psi : S \to R$ by sending $m_1 \mapsto 1_R$ and $m_i \mapsto 0$ for all $i \neq 1$. As S generates M freely, this extends to an R-module homomorphism $\theta : M \to R$.

By definition of a homomorphism, we can compute

$$\begin{aligned} 0 &= \theta(0) \\ &= \theta(r_1 m_1 + r_2 m_2 + \cdots + r_k m_k) \\ &= r_1 \theta(m_1) + r_2 \theta(m_2) + \cdots + r_k \theta(m_k) \\ &= r_1. \end{aligned}$$

This is a contradiction. So S must be independent.

To prove the other direction, suppose every element can be uniquely written as $r_1m_1 + \cdots + r_km_k$. Given any set function $\psi : S \to N$, we define $\theta : M \to N$ by

$$\theta(r_1m_1 + \cdots + r_km_k) = r_1\psi(m_1) + \cdots + r_k\psi(m_k).$$

This is well-defined by uniqueness, and is clearly a homomorphism. So it follows that S generates M □

Example. The set $\{2, 3\} \in \mathbb{Z}$ generates $\mathbb{Z}$. However, they do not generate $\mathbb{Z}$ freely, since

$$3 \cdot 2 + (-2) \cdot 3 = 0.$$

Recall from linear algebra that if a set S spans a *vector space* V, and it is not independent, then we can just pick some useless elements and throw them away in order to get a basis. However, this is no longer the case in modules: neither 2 nor 3 generate the $\mathbb{Z}$-module $\mathbb{Z}$.

Definition (Relations)**.** If M is a finitely-generated R-module, we have shown that there is a surjective R-module $\varphi : R^k \to M$. We call $\ker(\phi)$ the *relation module* for those generators.

Definition (Finitely presented module)**.** A finitely-generated module is *finitely-presented* if we have a surjective homomorphism $\phi : R^k \to M$ and $\ker \phi$ is finitely-generated.

Being finitely-presented means I can tell you everything about the module with finitely much paper. More precisely, if $\{m_1, \cdots, m_k\}$ generate M and $\{n_1, n_2, \cdots, n_k\}$ generate $\ker(\phi)$, then each

$$n_i = (r_{i1}, \cdots, r_{ik}) \in R^k$$

corresponds to the relation

$$r_{i1}m_1 + r_{i2}m_2 + \cdots + r_{ik}m_k = 0$$

in M. So M is the module generated by writing down R-linear combinations of $m_1, \cdots, m_k$, and say two elements are the same if they differ by these relations. Since there are only finitely many generators and finitely many such relations, we can specify the module with finitely much information.

Proposition (Invariance of dimension/rank)**.** Let R be a non-zero ring. If $R^n \cong R^m$ as an R-module, then $n = m$.

Proof. We know this is true if R is a field, so we reduce to that case.

We start with a general construction. If $I \lhd R$ be an ideal, and M be an R-module, we define

$$IM = \{am \in M : a \in I, m \in M\} \leq M.$$

So we can form the quotient module M/IM, which is an R-module again.

Now if $b \in I$, then its action on M/IM is

$$b(m + IM) = bm + IM = 0 + IM.$$

So we can make M/IM into an R/I module by

$$(r + I) \cdot (m + IM) = r \cdot m + IM.$$

Now let us go back to the case at hand., and suppose that $R^n \cong R^m$. Choose $I \lhd R$ a maximal ideal.[1] By the above construction we obtain an isomorphism $(R/I)^n \cong (R/I)^m$ of R/I-modules. But as I is maximal, R/I is a field, so $n = m$ by invariance of dimension for vector spaces. □

[1]There is one. An ideal of R is proper if and only if it does not contain 1_R, so an increasing union of proper ideals is proper; it then follows from Zorn's lemma that there is a maximal proper ideal. In fact the existence of maximal ideals is equivalent to the axiom of choice, and so to Zorn's lemma.

3.3 Matrices over Euclidean domains

Until further notice, we will assume R is a Euclidean domain, and we write $\phi : R \setminus \{0\} \to \mathbb{Z}_{\geq 0}$ for its Euclidean function. We know that in such a Euclidean domain, the greatest common divisor $\gcd(a, b)$ exists for all $a, b \in R$, and there are $x, y \in R$ such that $ax + by = \gcd(a, b)$.

Definition (Elementary row operations)**.** *Elementary row operations* on an $m \times n$ matrix A with entries in R are operations of the form

(ER1) Add $c \in R$ times the ith row to the jth row. This may be done by multiplying by the following matrix on the left:

$$\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & c & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix},$$

where c appears in the ith column of the jth row.

(ER2) Swap the ith and jth rows. This can be done by left multiplication by the permutation matrix

$$\begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 0 & & & 1 & & \\ & & & & 1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 1 & & \\ & & & 1 & & & 0 & & \\ & & & & & & & 1 & \\ & & & & & & & & \ddots \\ & & & & & & & & & 1 \end{pmatrix}.$$

The off-diagonal entries are in positions (i, j) and (j, i).

(ER3) We multiply the ith row by a *unit* $c \in R$. We do this via the following matrix:

$$\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & c & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

Notice all these matrices are invertible. We also have *elementary column operations* (EC1)–(EC3) defined in a similar fashion, corresponding to right multiplication by these matrices.

Definition (Equivalent matrices)**.** Two matrices are *equivalent* if we can get from one to the other via a sequence of such elementary row and column operations.

Note that if A and B are equivalent, then we can write

$$B = QAT^{-1}$$

for some invertible matrices Q and T^{-1}.

Our aim is to find, for each matrix, a matrix equivalent to it that is a simple as possible. Recall from IB Linear Algebra that if R is a field then we can put any matrix into the form

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

via elementary row and column operations. This is no longer true when working with rings. For example, over $\mathbb{Z}$, we cannot put the matrix

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

into that form, since no operation can turn the 2 into a 1. What we get is the following result:

Theorem (Smith normal form)**.** An $m \times n$ matrix over a Euclidean domain R is equivalent to a diagonal matrix

$$\begin{pmatrix} d_1 & & & & & & \\ & d_2 & & & & & \\ & & \ddots & & & & \\ & & & d_r & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix},$$

with the d_i all non-zero and

$$d_1 \mid d_2 \mid d_3 \mid \cdots \mid d_r.$$

Note that the divisibility criterion is similar to the classification of finitely-generated abelian groups. In fact, we will derive that as a consequence of the Smith normal form.

Definition (Invariant factors)**.** The d_k obtained in the Smith normal form are called the *invariant factors* of A.

Proof. Throughout the process, we will keep calling our matrix A, even though it keeps changing in each step, so that we don't have to invent hundreds of names for these matrices.

If $A = 0$, then done! So suppose $A \neq 0$. So some entry is not zero, say, $A_{ij} \neq 0$. Swapping the ith and first row, then jth and first column, we arrange that $A_{11} \neq 0$. We now try to reduce A_{11} as much as possible. We have the following two possible moves:

(i) If there is an A_{1j} not divisible by A_{11}, then we can use the Euclidean algorithm to write

$$A_{1j} = qA_{11} + r.$$

By assumption, $r \neq 0$. So $\phi(r) < \phi(A_{11})$ (where ϕ is the Euclidean function).

So we do a column operation to subtract q copies of the first column from the jth column. Then in position $(1, j)$, we now have r. We swap the first and jth column such that r is in position $(1, 1)$, and we have strictly reduced the value of ϕ at the first entry.

(ii) If there is an A_{i1} not divisible by A_{11}, we do the same thing, and this again reduces $\phi(A_{11})$.

We keep performing these until no move is possible. Since the value of $\phi(A_{11})$ strictly decreases every move, we must finish after finitely many applications. Then we know that we must have A_{11} dividing all A_{1j} and A_{i1}. Now we can just subtract appropriate multiples of the first column from others so that $A_{1j} = 0$ for $j \neq 1$. We do the same thing with rows so that the first row is cleared. Then we have a matrix of the form

$$A = \begin{pmatrix} d & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & C & \\ 0 & & & \end{pmatrix}.$$

We would like to say "do the same thing with C", but then this would get us a regular diagonal matrix, not necessarily in Smith normal form. So we need some preparation.

(iii) Suppose there is an entry of C not divisible by d, say A_{ij} with $i, j > 1$.

$$A = \begin{pmatrix} d & 0 & \cdots & 0 & \cdots & 0 \\ 0 & & & & & \\ \vdots & & & & & \\ 0 & & & A_{ij} & & \\ \vdots & & & & & \\ 0 & & & & & \end{pmatrix}$$

We suppose

$$A_{ij} = qd + r,$$

with $r \neq 0$ and $\phi(r) < \phi(d)$. We add column 1 to column j, and subtract q times row 1 from row i. Now we get r in the (i, j)th entry, and we want to send it back to the $(1, 1)$ position. We swap row i with row 1, swap column j with row 1, so that r is in the $(1, 1)$th entry, and $\phi(r) < \phi(d)$.

Now we have messed up the first row and column. So we go back and do (i) and (ii) again until the first row and columns are cleared. Then we get

$$A = \begin{pmatrix} d' & 0 & \cdots & 0 \\ 0 & & & \\ 0 & & C' & \\ 0 & & & \end{pmatrix},$$

where

$$\phi(d') \leq \phi(r) < \phi(d).$$

As this strictly decreases the value of $\phi(A_{11})$, we can only repeat this finitely many times. When we stop, we will end up with a matrix

$$A = \begin{pmatrix} d & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & C & \\ 0 & & & \end{pmatrix},$$

and d divides *every* entry of C. Now we apply the entire process to C. When we do this process, notice all allowed operations don't change the fact that d divides every entry of C.

So applying this recursively, we obtain a diagonal matrix with the claimed divisibility property. □

Note that if we didn't have to care about the divisibility property, we can just do (i) and (ii), and we can get a diagonal matrix. The trick to get to the Smith normal form is (iii).

Recall that the d_i are called the invariant factors. So it would be nice if we can prove that the d_i are indeed invariant. It is not clear from the algorithm that we will always end up with the same d_i. Indeed, we can multiply a whole row by -1 and get different invariant factors. However, it turns out that these are unique up to multiplication by units.

To study the uniqueness of the invariant factors of a matrix A, we relate them to other invariants, which involves *minors*.

Definition (Minor)**.** A $k \times k$ *minor* of a matrix A is the determinant of a $k \times k$ sub-matrix of A (ie. a matrix formed by removing all but k rows and all but k columns).

Any given matrix has many minors, since we get to decide which rows and columns we can throw away. The idea is to consider the ideal generated by all the minors of matrix.

Definition (Fitting ideal)**.** For a matrix A, the *kth Fitting ideal* $\mathrm{Fit}_k(A) \lhd R$ is the ideal generated by the set of all $k \times k$ minors of A.

A key property is that equivalent matrices have the same Fitting ideal, even if they might have very different minors.

Lemma. Let A and B be equivalent matrices. Then for all k we have

$$\mathrm{Fit}_k(A) = \mathrm{Fit}_k(B).$$

Proof. It suffices to show that changing A by a row or by a column operation does not change the Fitting ideal. Since taking the transpose does not change the determinant, ie. $\mathrm{Fit}_k(A) = \mathrm{Fit}_k(A^T)$, it suffices to consider just row operations.

The most difficult one is taking linear combinations. Let B be the result of adding c times the ith row to the jth row, and fix C a $k \times k$ minor of A. Suppose that the corresponding minor of B is C'. We then want to show that $\det C' \in \mathrm{Fit}_k(A)$.

If the jth row is outside of C, then the minor $\det C$ is unchanged. If both the ith and jth rows are in C, then C' is obtained from C by a row operation, which does not change the determinant.

Suppose the jth row is in C and the ith row is not. Suppose the ith row is $f_1, \cdots, f_k$. Then C is changed to C', differing only in the jth row, which is:

$$(C_{j1} + cf_1, C_{j_2} + cf_2, \cdots, C_{jk} + cf_k).$$

We compute $\det C'$ by expanding along this row, giving

$$\det C' = \det C + c \det D,$$

where D is the matrix obtained by replacing the jth row of C with $(f_1, \cdots, f_k)$. Now $\det C$ is definitely a minor of A, and $\det D$ is still a minor of A, just another one. Since ideals are closed under addition and multiplications, it follows that $\det(C') \in \mathrm{Fit}_k(A)$.

The other operations are much simpler. They just follow by standard properties of the effect of swapping rows or multiplying rows on determinants.

So after any row operation, the resultant submatrix C' satisfies $\det(C') \in \mathrm{Fit}_k(A)$. Since this is true for all minors, we must have $\mathrm{Fit}_k(B) \subseteq \mathrm{Fit}_k(A)$. But row operations are invertible. So we must have $\mathrm{Fit}_k(A) \subseteq \mathrm{Fit}_k(B)$ as well. So they must be equal. □

Corollary. If A has Smith normal form

$$B = \begin{pmatrix} d_1 & & & & & & \\ & d_2 & & & & & \\ & & \ddots & & & & \\ & & & d_r & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix},$$

then

$$\mathrm{Fit}_k(A) = (d_1 d_2 \cdots d_k).$$

In particular d_k is unique up to associates.

Proof. We have $\mathrm{Fit}_k(B) = (d_1 d_2 \cdots d_k)$ as the only possible contributing minors are from the diagonal submatrices, and the minor from the top left square submatrix divides all other diagonal ones. By the previous Lemma this is also $\mathrm{Fit}_k(A)$. The final claim follows since we can find d_k by dividing a generator of $\mathrm{Fit}_k(A)$ by a generator of $\mathrm{Fit}_{k-1}(A)$. □

Example. Consider the matrix $\left(\begin{smallmatrix} 2 & -1 \\ 1 & 2 \end{smallmatrix}\right)$ over $\mathbb{Z}$. We can perform the moves

$$\left(\begin{smallmatrix} 2 & -1 \\ 1 & 2 \end{smallmatrix}\right) \overset{EC1}{\rightsquigarrow} \left(\begin{smallmatrix} 1 & -1 \\ 3 & 2 \end{smallmatrix}\right) \overset{EC1}{\rightsquigarrow} \left(\begin{smallmatrix} 1 & 0 \\ 3 & 5 \end{smallmatrix}\right) \overset{ER1}{\rightsquigarrow} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 5 \end{smallmatrix}\right)$$

to put in into Smith normal form.

But if $\left(\begin{smallmatrix} d_1 & 0 \\ 0 & d_2 \end{smallmatrix}\right)$ is its Smith normal form then we also know that

$$(d_1) = (2, -2, 1, 2) = (1)$$

so $d_1 = \pm 1$, and

$$(d_1 d_2) = (\det\left(\begin{smallmatrix} 2 & -1 \\ 1 & 2 \end{smallmatrix}\right)) = (5)$$

so $d_1 d_2 = \pm 5$, and hence up to units $d_1 = 1$ and $d_2 = 5$.

We are now going to use Smith normal forms to do things. We will need some preparation, in the form of the following lemma:

Lemma. Let R be a principal ideal domain. Then any submodule of R^m is generated by at most m elements.

Proof. Let $N \leq R^m$ be a submodule. Consider the ideal

$$I = \{r \in R : (r, r_2, \cdots, r_m) \in N \text{ for some } r_2, \cdots, r_m \in R\}.$$

It is clear this is an ideal. Since R is a principal ideal domain, we must have $I = (a)$ for some $a \in R$. We now choose an element

$$n = (a, a_2, \cdots, a_m) \in N.$$

Now for any $(r_1, r_2, \cdots, r_m) \in N$ we know that $r_1 \in I$, so $a \mid r_1$. Thus we can write $r_1 = ra$. Then we can form

$$(r_1, r_2, \cdots, r_m) - r(a, a_2, \cdots, a_m) = (0, r_2 - ra_2, \cdots, r_m - ra_m) \in N.$$

This lies in $N' = N \cap (\{0\} \times R^{m-1}) \leq R^{m-1}$. Thus everything in N can be written as a multiple of n plus something in N'. But by induction, since $N' \leq R^{m-1}$, we know N' is generated by at most $m - 1$ elements. So there are $n_2, \cdots, n_m \in N'$ generating N'. So $n, n_2, \cdots, n_m$ generate N. □

Theorem. Let R be a Euclidean domain, and let $N \leq R^m$ be a submodule. Then there exists a basis $v_1, \cdots, v_m$ of R^m such that N is generated by $d_1 v_1, d_2 v_2, \cdots, d_r v_r$ for some $0 \leq r \leq m$ and some $d_i \in R$ such that $d_1 \mid d_2 \mid \cdots \mid d_r$.

Proof. By the previous lemma, N is generated by some elements $x_1, \cdots, x_n$ with $n \leq m$. Each x_i is an element of R^m. So we can think of it as a column vector of length m, and we can form a $m \times n$ matrix

$$A = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ x_1 & x_2 & \cdots & x_n \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}.$$

We can put this in Smith normal form. Since there are no more columns than there are rows, this is of the form

$$\begin{pmatrix} d_1 & & & & & & \\ & d_2 & & & & & \\ & & \ddots & & & & \\ & & & d_r & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \\ & & & & & & 0 \\ & & & & & & \vdots \\ & & & & & & 0 \end{pmatrix}$$

Recall that we got to the Smith normal form by row and column operations. Performing row operations is just changing the basis of R^m, while each column operation changes the generators of N.

So what this tells us is that there is a new basis $v_1, \cdots, v_m$ of R^m such that N is generated by $d_1 v_1, \cdots, d_r v_r$. By definition of Smith normal form, the divisibility condition holds. □

Corollary. Let R be a Euclidean domain. A submodule of R^m is free of rank at most m. In other words, the submodule of a free module is free, and of a smaller (or equal) rank.

Proof. Continuing with the notation from the argument above, we claim that the set $d_1 v_1, \ldots, d_r v_r$ *freely* generate N. This is because a linear dependence between them would give a linear dependence between the $v_1, \ldots, v_m$. □

Note that this is not true for all rings. For example, $(2, X) \lhd \mathbb{Z}[X]$ is a submodule of $\mathbb{Z}[X]$, but because it is not a principal ideal it cannot be isomorphic to $\mathbb{Z}[X]$.

Theorem (Classification of finitely-generated modules over a Euclidean domain)**.** Let R be a Euclidean domain, and M be a finitely-generated R-module. Then

$$M \cong \frac{R}{(d_1)} \oplus \frac{R}{(d_1)} \oplus \cdots \oplus \frac{R}{(d_r)} \oplus R \oplus R \oplus \cdots \oplus R$$

for some $d_i \neq 0$, and $d_1 \mid d_2 \mid \cdots \mid d_r$.

Proof. Since M is finitely-generated, there is a surjection $\phi : R^m \to M$. So by the first isomorphism theorem we have

$$M \cong \frac{R^m}{\ker \phi}.$$

Since $\ker \phi$ is a submodule of R^m, by the previous theorem, there is a basis $v_1, \cdots, v_m$ of R^m such that $\ker \phi$ is generated by $d_1 v_1, \cdots, d_r, v_r$ for $0 \leq r \leq m$ and $d_1 \mid d_2 \mid \cdots \mid d_r$. So

$$M \cong \frac{R^m}{((d_1, 0, \cdots, 0), (0, d_2, 0, \cdots, 0), \cdots, (0, \cdots, 0, d_r, 0, \cdots, 0))}.$$

This is

$$\frac{R}{(d_1)} \oplus \frac{R}{(d_2)} \oplus \cdots \oplus \frac{R}{(d_r)} \oplus R \oplus \cdots \oplus R,$$

with $m - r$ copies of R. □

This is particularly useful in the case where $R = \mathbb{Z}$, where R-modules are abelian groups.

Example. Let A be the abelian group generated by a, b, c with relations

$$\begin{aligned} 2a + 3b + \ c &= 0,\\ a + 2b \quad\ \ &= 0,\\ 5a + 6b + 7c &= 0. \end{aligned}$$

In other words, we have

$$A = \frac{\mathbb{Z}^3}{((2, 3, 1), (1, 2, 0), (5, 6, 7))}.$$

We would like to get a better description of A. (As things stand it is not even obvious if this module is the zero module or not.)

To work out a good description, we consider the matrix

$$X = \begin{pmatrix} 2 & 1 & 5 \\ 3 & 2 & 6 \\ 1 & 0 & 7 \end{pmatrix}.$$

To figure out its Smith normal form, we find the fitting ideals. We have

$$\mathrm{Fit}_1(X) = (1, \text{other stuff}) = (1).$$

So $d_1 = 1$.

We have to work out the second fitting ideal. In principle, we have to check all the minors, but we immediately notice

$$\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 1.$$

So $\mathrm{Fit}_2(X) = (1, \text{other stuff}) = (1)$, and so $d_2 = 1$. Finally, we find

$$\mathrm{Fit}_3(X) = \left(\begin{vmatrix} 2 & 1 & 5 \\ 3 & 2 & 6 \\ 1 & 0 & 7 \end{vmatrix}\right) = (3),$$

so $d_3 = 3$. Thus

$$A \cong \frac{\mathbb{Z}}{(1)} \oplus \frac{\mathbb{Z}}{(1)} \oplus \frac{\mathbb{Z}}{(3)} \cong \frac{\mathbb{Z}}{(3)} \cong C_3.$$

We re-state the previous theorem in the specific case where R is $\mathbb{Z}$, since this is particularly useful.

Corollary (Classification of finitely-generated abelian groups)**.** Any finitely-generated abelian group is isomorphic to

$$C_{d_1} \times \cdots \times C_{d_r} \times C_\infty \times \cdots \times C_\infty,$$

where $C_\infty \cong \mathbb{Z}$ is the infinite cyclic group, with $d_1 \mid d_2 \mid \cdots \mid d_r$.

Proof. Let $R = \mathbb{Z}$, and apply the classification of finitely-generated R-modules. □

Note that if the group is finite, then there cannot be any C_∞ factors. So it is just a product of finite cyclic groups.

Corollary. If A is a finite abelian group, then

$$A \cong C_{d_1} \times \cdots C_{d_r},$$

with $d_1 \mid d_2 \mid \cdots \mid d_r$.

This is the result we stated near the beginning of the course.

Recall that we were also to decompose a finite abelian group into products of the form C_{p^k}, where p is a prime, and we said it was just the Chinese remainder theorem. This is again in general true, but we, again, need the Chinese remainder theorem.

Lemma (Chinese remainder theorem)**.** Let R be a Euclidean domain, and $a, b \in R$ be such that $\gcd(a, b) = 1$. Then

$$\frac{R}{(ab)} \cong \frac{R}{(a)} \oplus \frac{R}{(b)}$$

as R-modules.

The proof is just that of the Chinese remainder theorem written in ring language.

Proof. Consider the R-module homomorphism

$$\begin{aligned} \phi : \frac{R}{(a)} \oplus \frac{R}{(b)} &\longrightarrow \frac{R}{(ab)} \\ (r_1 + (a), r_2 + (b)) &\longmapsto br_1 + ar_2 + (ab). \end{aligned}$$

To show this is well-defined, suppose

$$(r_1 + (a), r_2 + (b)) = (r_1' + (a), r_2' + (b)).$$

Then

$$\begin{aligned} r_1 &= r_1' + xa \\ r_2 &= r_2' + yb. \end{aligned}$$

So

$$br_1 + ar_2 + (ab) = br_1' + xab + ar_2' + yab + (ab) = br_1' + ar_2' + (ab).$$

So this is indeed well-defined. It is clear that this is a module map.

We now have to show it is surjective and injective. So far, we have not used the hypothesis, that $\gcd(a, b) = 1$. As $\gcd(a, b) = 1$, by the Euclidean algorithm we can write

$$1 = ax + by$$

for some $x, y \in R$. So we have

$$\phi(y + (a), x + (b)) = by + ax + (ab) = 1 + (ab).$$

So $1 \in \operatorname{im} \phi$. Since this is an R-module map, we get

$$\phi(r(y + (a), x + (b))) = r \cdot (1 + (ab)) = r + (ab).$$

The key fact is that $R/(ab)$ as an R-module is generated by 1. Thus ϕ is surjective.

Finally, we have to show it is injective, i.e. that the kernel is trivial. Suppose

$$\phi(r_1 + (a), r_2 + (b)) = 0 + (ab).$$

Then

$$br_1 + ar_2 \in (ab).$$

So we can write

$$br_1 + ar_2 = abx$$

for some $x \in R$. Since $a \mid ar_2$ and $a \mid abx$, we know $a \mid br_1$. Since a and b are coprime, unique factorization implies $a \mid r_1$. Similarly, we know $b \mid r_2$.

$$(r_1 + (a), r_2 + (b)) = (0 + (a), 0 + (b)).$$

So the kernel is trivial. □

Theorem (Primary decomposition theorem)**.** Let R be a Euclidean domain, and M be a finitely-generated R-module. Then

$$M \cong N_1 \oplus N_2 \oplus \cdots \oplus N_t,$$

where each N_i is either R or is $R/(p^n)$ for some prime $p \in R$ and some $n \geq 1$.

Proof. We already know that

$$M \cong \frac{R}{(d_1)} \oplus \cdots \oplus \frac{R}{(d_r)} \oplus R \oplus \cdots \oplus R,$$

so it suffices to show that each $R/(d_1)$ can be written in that form. We let

$$d = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$$

with p_i distinct primes. So each $p_i^{n_i}$ is coprime to each other. So by the previous Lemma iterated a few times, we have

$$\frac{R}{(d_1)} \cong \frac{R}{(p_1^{n_1})} \oplus \cdots \oplus \frac{R}{(p_k^{n_k})}.$$

□

3.4 Modules over $\mathbb{F}[X]$ and normal forms for matrices

For a field $\mathbb{F}$, the polynomial ring $\mathbb{F}[X]$ is a Euclidean domain, so the results of the last few sections apply to it. If V is a vector space on $\mathbb{F}$, and $\alpha : V \to V$ is a linear map, then we can make V into an $\mathbb{F}[X]$-module via

$$\begin{aligned} \mathbb{F}[X] \times V &\longrightarrow V \\ (f, v) &\longmapsto (f(\alpha))(v). \end{aligned}$$

Let us write V_α for this $\mathbb{F}[X]$-module.

Lemma. If V is a finite-dimensional vector space, then V_α is a finitely-generated $\mathbb{F}[X]$-module.

Proof. If $v_1, \cdots, v_n$ generate V as an $\mathbb{F}$-module, i.e. span V as a vector space over $\mathbb{F}$, then they also generate V_α as an $\mathbb{F}[X]$-module since $\mathbb{F} \leq \mathbb{F}[X]$. □

Example. Suppose $V_\alpha \cong \mathbb{F}[X]/(X^r)$ as $\mathbb{F}[X]$-modules. Then in particular they are isomorphic as $\mathbb{F}$-modules (since being a map of $\mathbb{F}$-modules has fewer requirements than being a map of $\mathbb{F}[X]$-modules).

Under this bijection, the elements $1, X, X^2, \cdots, X^{r-1} \in \mathbb{F}[X]/(X^r)$ form a vector space basis for V_α. Viewing $\mathbb{F}[X]/(X^r)$ as an $\mathbb{F}$-vector space, the action of X has the matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

We also know that in V_α, the action of X is by definition the linear map α. So with respect to this basis α is given by the same matrix.

Example. Suppose

$$V_\alpha \cong \frac{\mathbb{F}[X]}{((X-\lambda)^r)}$$

for some $\lambda \in \mathbb{F}$. Consider the new linear map

$$\beta = \alpha - \lambda \cdot \mathrm{id} : V \longrightarrow V.$$

Then $V_\beta \cong \mathbb{F}[Y]/(Y^r)$, for $Y = X - \lambda$. So there is a basis for V so that β is represented by

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

In this basis α is represented by

$$\begin{pmatrix} \lambda & 0 & \cdots & 0 & 0 \\ 1 & \lambda & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \lambda \end{pmatrix}$$

So it is a Jordan block (except the Jordan blocks are the other way round, with zeroes below the diagonal).

Example. Suppose $V_\alpha \cong \mathbb{F}[X]/(f)$ for some polynomial f, for

$$f = a_0 + a_1 X + \cdots + a_{r-1} X^{r-1} + X^r.$$

This has a basis $1, X, X^2, \cdots, X^{r-1}$ as well, in which α is represented by the matrix

$$c(f) = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{r-1} \end{pmatrix}.$$

We call this the *companion matrix* for the monic polynomial f.

These are different things that can possibly happen. Since we have already classified all finitely-generated $\mathbb{F}[X]$ modules, this allows us to put matrices in a rather nice form.

Theorem (Rational canonical form)**.** Let $\alpha : V \to V$ be a linear endomorphism of a finite-dimensional vector space over $\mathbb{F}$, and V_α be the associated $\mathbb{F}[X]$-module. Then

$$V_\alpha \cong \frac{\mathbb{F}[X]}{(f_1)} \oplus \frac{\mathbb{F}[X]}{(f_2)} \oplus \cdots \oplus \frac{\mathbb{F}[X]}{(f_s)},$$

with $f_1 \mid f_2 \mid \cdots \mid f_s$. Thus there is a basis for V in which the matrix for α is the block diagonal

$$\begin{pmatrix} c(f_1) & 0 & \cdots & 0 \\ 0 & c(f_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c(f_s) \end{pmatrix}$$

Proof. We know that V_α is a finitely-generated $\mathbb{F}[X]$-module. By the structure theorem for these, we know

$$V_\alpha \cong \frac{\mathbb{F}[X]}{(f_1)} \oplus \frac{\mathbb{F}[X]}{(f_2)} \oplus \cdots \oplus \frac{\mathbb{F}[X]}{(f_s)} \oplus 0.$$

(There can be no copies of $\mathbb{F}[X]$, since $V_\alpha = V$ is finite-dimensional over $\mathbb{F}$ but $\mathbb{F}[X]$ is not.) The divisibility criterion also follows from the structure theorem. Then the form of the matrix is immediate. □

This is really a canonical form. The Jordan normal form is not canonical, since we can move the blocks around. The structure theorem determines the factors f_i up to units, and once we require it is monic, there is no choice left.

In terms of matrices, this says that if α is represented by a matrix $A \in M_{n,n}(F)$ in some basis, then A is conjugate to a matrix of the form above.

From the rational canonical form, we can immediately read off the minimal polynomial as f_s. This is since if we view V_α as the decomposition above, we find that $f_s(\alpha)$ kills everything in $\frac{\mathbb{F}[X]}{(f_s)}$. It also kills the other factors since $f_i \mid f_s$ for all i. So $f_s(\alpha) = 0$. We also know that no smaller polynomial kills V, since it does not kill $\frac{\mathbb{F}[X]}{(f_s)}$.

Similarly, we find that the characteristic polynomial of α is $f_1 f_2 \cdots f_s$.

Recall we had a different way of decomposing a module over a Euclidean domain, namely the primary decomposition: this will give us the Jordan normal form. Before we can use that, we need to know what the primes are. This is why we need to work over $\mathbb{C}$.

Lemma. The prime elements of $\mathbb{C}[X]$ are the $X - \lambda$ for $\lambda \in \mathbb{C}$ (up to multiplication by units).

Proof. Let $f \in \mathbb{C}[X]$. If f is constant, then it is either a unit or 0. Otherwise, by the fundamental theorem of algebra, it has a root λ. So it is divisible by $X - \lambda$. So if f is irreducible, it must have degree 1. And clearly everything of degree 1 is prime. □

Applying the primary decomposition theorem to $\mathbb{C}[X]$-modules gives us the Jordan normal form.

Theorem (Jordan normal form)**.** Let $\alpha : V \to V$ be an endomorphism of a vector space V over $\mathbb{C}$, and V_α be the associated $\mathbb{C}[X]$ module. Then

$$V_\alpha \cong \frac{\mathbb{C}[X]}{((X-\lambda_1)^{a_1})} \oplus \frac{\mathbb{C}[X]}{((X-\lambda_2)^{a_2})} \oplus \cdots \oplus \frac{\mathbb{C}[X]}{((X-\lambda_t)^{a_t})},$$

where $\lambda_i \in \mathbb{C}$ do *not* have to be distinct. Thus there is a basis of V in which α has matrix

$$\begin{pmatrix} J_{a_1}(\lambda_1) & & & 0 \\ & J_{a_2}(\lambda_2) & & \\ & & \ddots & \\ 0 & & & J_{a_t}(\lambda_t) \end{pmatrix},$$

where

$$J_m(\lambda) = \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 1 & \lambda & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & \lambda \end{pmatrix}$$

is an $m \times m$ matrix.

Proof. Apply the prime decomposition theorem to V_α. Then all primes are of the form $X - \lambda$. We then use our second example at the beginning of the chapter to get the form of the matrix. □

The blocks $J_m(\lambda)$ are called the *Jordan λ-blocks*. It turns out that the Jordan blocks are unique up to reordering, but it does not immediately follow from what we have said so far, and we will not prove it. It is done in the IB Linear Algebra course.

We can also read off the minimal and characteristic polynomials of α. The minimal polynomial is

$$\prod_\lambda (X-\lambda)^{a_\lambda},$$

where a_λ is the size of the largest λ-block. The characteristic polynomial of α is

$$\prod_\lambda (X-\lambda)^{b_\lambda},$$

where b_λ is the sum of the sizes of the λ-blocks. Alternatively, it is

$$\prod_{i=1}^{t} (X-\lambda_i)^{a_i}.$$

From the Jordan normal form, we can also read off the size of the λ-space of α, as the number of λ-blocks.