SEMINAR ON COBORDISM CATEGORIES AND THE MADSEN–WEISS THEOREM

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The goal of this seminar is to understand a proof of a major recent theorem of Madsen and Weiss [MW07]. This theorem concerns the cohomology of "the space $\mathcal{M}_g$ of all of orientable genus g surfaces", in the limit as $g \to \infty$. With rational coefficients the answer takes a particularly simple form:

$$\lim_{g\to\infty} H^*(\mathcal{M}_g; \mathbb{Q}) = \mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \ldots]$$

where $\kappa_i \in H^{2i}(\mathcal{M}_g; \mathbb{Q})$ are specific cohomology classes, the Miller–Morita–Mumford classes. This result had been conjectured in 1983 by Mumford, from the point of view of algebraic geometry, where the space $\mathcal{M}_g$ also arises. The proof however is entirely topological.

A second proof of the theorem was given by Galatius, Madsen, Tillmann, and Weiss [GTMW09], based on an earlier relationship discovered by Tillmann [Til97] between the spaces $\mathcal{M}_g$ and the 2-dimensional cobordism category (the category whose objects are compact 1-manifolds, and whose morphisms are compact surfaces with incoming and outgoing boundary, where the composition law is by gluing surfaces together). A third proof was given by Galatius and myself [GRW10], also based on cobordism categories but involving several technical simplifications. An exposition of this proof (though not quite identical) has been written by Hatcher [Hat11], and is very accessible.

In this seminar we will go through this last proof, using [GRW10, Hat11] as our main references. As our focus will be on oriented surfaces, we will be able to forget about "θ-structures" as described in [GRW10], and just work with oriented manifolds everywhere. This leads to many simplifications, and [Hat11] is written with many of these simplifications implemented: although most references below are to [GRW10] I suggest also reading [Hat11] in parallel.

We will meet 13:30 - 15:00 in MR13 (in the basement of Pavilion E) on the following days; talks are nominally 1 hour, followed by half an hour of discussion and clarification. Please contact me on `or257@cam.ac.uk` or come and see me in E2.06 if you are unsure about any aspect of your talk.

Talk 0. (Introduction, ORW, October 12) In this introductory talk, I will carefully explain the statement of the Madsen–Weiss theorem that we shall aim for. This means that I will explain the definitions of the spaces $\mathcal{M}_{g,1} = B\mathrm{Diff}_\partial(\Sigma_{g,1})$ and $\Omega^\infty \mathbf{MTSO(2)}$ involved.

Talk 1. (Spaces of manifolds, NP, November 6) This talk will introduce the space $\psi_d(U)$ of d-dimensional submanifolds of an open set $U \subset \mathbb{R}^n$, following Section 2. You should define the topology on this space, but you should not try to prove all of its properties; however, it is important to state Theorems 2.7, 2.9, and 2.11, as these are what one uses in practice. You should avoid saying anything about θ-structures, and instead simply define an analogous space $\psi_{d,+}(U)$ of oriented manifolds. You should define what a smooth map into $\psi_{d,+}(U)$ is, and state the smooth approximation result (Lemma 2.18).

The conclusion of this talk should be proving Theorem 3.22, identifying the homotopy type of $\psi_{d,+}(\mathbb{R}^n)$.

Talk 2. (The cobordism category, KH, November 9) This talk will start by defining the embedded oriented d-dimensional cobordism categories $\mathcal{C}_{d,+}(\mathbb{R}^n)$ and $\mathcal{C}_{d,+} = \mathrm{colim}_{n\to\infty} \mathcal{C}_{d,+}(\mathbb{R}^n)$, following [GRW10, §3.2], and should then mention the relation between the spaces $\mathcal{C}_{d,+}(P,Q)$ of morphisms in this category and the spaces $B\mathrm{Diff}_\partial(W)$ introduced in Talk 0.

You should then describe the notion of a simplicial set and simplicial space, explain how the nerve of a topological category (such as $\mathcal{C}_{d,+}$) gives a simplicial space, define the geometric realisation of a simplicial space and establish its basic properties. A reference for this and many further (semi-)simplicial things, adapted to what we need, is [ER16], but see also [Seg68] (though note that what Segal calls "semi-simplicial" is what is called "simplicial" these days).

Talk 3. (The "Main Theorem" of GMTW, DC, November 13) You should explain the proofs of Theorem 3.9 and Theorem 3.10 in detail, arriving at an equivalence $B\mathcal{C}_{d,+}(\mathbb{R}^n) \simeq \psi_{d,+}(n,1)$. You should then *state* Theorem 3.13, and show how to put several things together to prove the "Main Theorem" of [GTMW09],

$$\Omega B\mathcal{C}_{d,+} \simeq \Omega^\infty \mathbf{MTSO(d)}.$$

Then, in preparation for the next talk, you should state Lemma 3.14 and say something about its proof, e.g. following [Seg74, Proposition 1.5] (or [ER16, Theorem 2.15], but discuss it with me).

Talk 4. (Proof of Theorem 3.13, NP, November 16) Explain the proof of the missing ingredient, Theorem 3.13. This involves Proposition 3.6 and Corollary 3.11, expressing $\pi_0(\psi_{d,+}(n,k))$ in terms of oriented cobordism so that we know it is a group, but the main geometric content is Proposition 3.21, which should be explained in detail, with many good pictures.

This finishes the proof of the "Main Theorem" of [GTMW09]. From now on we specialise to $d = 2$. The goal from now on is to show that a certain subcategory $\mathcal{D} \subset \mathcal{C}_{2,+}$ having a single object (so $\mathcal{D}$ is a monoid) has the same classifying space as the whole category $\mathcal{C}_2^+$. The group-completion theorem then relates the homology of $\Omega B\mathcal{D} \simeq \Omega B\mathcal{C}_{2,+} \simeq \Omega^\infty \mathbf{MTSO(2)}$ with a stabilisation of the homology of $\mathcal{D}$, and the Madsen–Weiss theorem will be immediate from this.

Talk 5. (The group-completion theorem, BM, November 20) You should present a proof of the group-completion theorem, in particular the low-tech proof given in Appendix D of [Hat11], which uses nothing beyond Hatcher's book.

Talk 6. (The controlled cobordism category, KL, November 23) You should start by defining all of the objects that we will need from the beginning of Section 4 ($\psi_{2,+}(n,1)^\bullet$, $\mathcal{C}_{2,+}^\bullet$, $\psi_{2,+}^{nc}(n,1)^\bullet$, $\psi_{2,+}^{nc}(n,1)^\bullet_{\mathbf{Conn}}$) and then quickly explain the equivalence $B\mathcal{C}_{2,+}^\bullet \simeq \psi_{2,+}^{nc}(\infty,1)^\bullet_{\mathbf{Conn}}$ of Theorem 4.5 by discussing how the steps in Talk 3 go through. You should then give the proofs of Lemmas 4.6 and 4.7, and adapt the proof of Proposition 4.26 to show that $\mathcal{C}_{2,+}^\bullet$ has the same classifying space as the full subcategory $\mathcal{D}$ on any of its objects. Hence explain how $\Omega B\mathcal{D} \simeq \Omega B\mathcal{C}_{2,+} \simeq \Omega^\infty \mathbf{MTSO(2)}$ will follow if the inclusion

$$\psi_{2,+}^{nc}(\infty,1)^\bullet_{\mathbf{Conn}} \longrightarrow \psi_{2,+}^{nc}(\infty,1)^\bullet$$

is a homotopy equivalence.

Talk 7. (Parameterised surgery, ORW, November 27) I will explain why the last map is a homotopy equivalence, following Section 4.2.

Talk 8. (Putting it all together and calculations, NP, November 30) You should start by putting everything we have done together to show that

$$\operatorname*{colim}_{g\to\infty} H_*(\mathcal{M}_{g,1};\mathbb{Z}) \cong H_*(\Omega_0^\infty \mathbf{MTSO(2)};\mathbb{Z}),$$

(to verify the hypothesis of the group-completion theorem, adapt the discussion on p. 1295). You should then follow Appendix C of [Hat11] and show that the cohomology ring $H^*(\Omega_0^\infty \mathbf{MTSO(2)};\mathbb{Q})$ as the polynomial algebra on certain classes κ_i of degree $2i$. You should also mention that the spaces $\mathcal{M}_{g,1}$ have *homological stability* (see [Wah13]) and perhaps also calculate and interpret $H_1(\Omega_0^\infty \mathbf{MTSO(2)};\mathbb{Z}) = 0$ (I can explain ways to do this).

References

[ER16] Johannes Ebert and Oscar Randal-Williams, *Semi-simplicial spaces*, arXiv:1705.03774, 2016.

[GRW10] Søren Galatius and Oscar Randal-Williams, *Monoids of moduli spaces of manifolds*, Geom. Topol. **14** (2010), no. 3, 1243–1302.

[GTMW09] Søren Galatius, Ulrike Tillmann, Ib Madsen, and Michael Weiss, *The homotopy type of the cobordism category*, Acta Math. **202** (2009), no. 2, 195–239.

[Hat11] Allen Hatcher, *A short exposition of the Madsen-Weiss theorem*, https://www.math.cornell.edu/~hatcher/Papers/MW.pdf, 2011.

[MW07] Ib Madsen and Michael Weiss, *The stable moduli space of Riemann surfaces: Mumford's conjecture*, Ann. of Math. (2) **165** (2007), no. 3, 843–941.

[Seg68] Graeme Segal, *Classifying spaces and spectral sequences*, Inst. Hautes Études Sci. Publ. Math. (1968), no. 34, 105–112.

[Seg74] ———, *Categories and cohomology theories*, Topology **13** (1974), 293–312.

[Til97] Ulrike Tillmann, *On the homotopy of the stable mapping class group*, Invent. Math. **130** (1997), no. 2, 257–275.

[Wah13] Nathalie Wahl, *Homological stability for mapping class groups of surfaces*, Handbook of moduli. Vol. III, Adv. Lect. Math. (ALM), vol. 26, Int. Press, Somerville, MA, 2013, pp. 547–583.

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