Lent Term 2023 O. Randal-Williams

Part III Characteristic classes and K-theory // Example Sheet 2

Hand in work to questions marked $*$ to my pigeon hole at CMS by 09:00 on Wednesday 22nd February if you would like it marked.

1. A *division algebra* structure on a finite-dimensional real vector space V is a map $\cdot : V \otimes V \to V$ such that $- \cdot x : V \to V$ is an isomorphism for all $x \neq 0$. Given such a structure, show that the tangent bundle of the real projective space $\mathbb{P}(V)$ is trivial as follows:

   (i) Choose a basis $e_1, \ldots, e_n$ of V, and define isomorphisms $v_i : V \to V$ intrinsically by $v_i(x \cdot e_1) = x \cdot e_i$. Show that $v_1(x) = x$ and that the $v_i(x)$ are linearly independent.

   (ii) For each $\ell \in \mathbb{P}(V)$ use $v_2, \ldots, v_n$ to define linear maps $\bar{v}_2, \ldots, \bar{v}_n : \ell \to \ell^\perp$ which are linearly independent. (Form $\perp$ with respect to an auxiliary inner product on V.) Show these define a trivialisation of $T\mathbb{P}(V)$.

   Deduce that if $(V, \cdot)$ is a division algebra then $\dim V$ is a power of 2. (The division algebras you know, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$, indeed have this property.)

2. $*$ If a compact n-manifold embeds into $\mathbb{R}^{n+1}$, show that all its Stiefel–Whitney classes are zero. Show that this need not be the case if it immerses into $\mathbb{R}^{n+1}$.

3. Let $\alpha(n)$ denote the number of 1's when n is written in binary. By computing Stiefel–Whitney classes, show that for each n there is an n-dimensional manifold [*try products of real projective spaces*] which does not immerse into $\mathbb{R}^{2n-\alpha(n)-1}$.

   [*In 1985 R. L. Cohen proved that every compact n-manifold immerses into* $\mathbb{R}^{2n-\alpha(n)}$.]

4. If $f : M^d \to \mathbb{R}^n$ is an embedding of a compact manifold, with $(n-d)$-dimensional normal bundle $\nu_f \to M$, then show that $e(\nu_f) = 0 \in H^{n-d}(M; \mathbb{F}_2)$. You will need to use a tubular neighbourhood, excision, and the commutativity of a diagram

$$\begin{array}{ccc} H^i(X,A) \otimes H^j(X,A) & \xrightarrow{\smile} & H^{i+j}(X,A) \\ \downarrow & \nearrow_{\smile} & \\ H^i(X) \otimes H^j(X,A) & & \end{array}$$

   which you should construct.

   Hence show that $\mathbb{RP}^{2^k}$ does not embed in $\mathbb{R}^{2^{k+1}-1}$.

   [*In 1944 H. Whitney proved that every compact n-manifold embeds into* $\mathbb{R}^{2n}$.]

5. If $\pi : E \to X$ is a d-dimensional complex vector bundle over a finite CW-complex of dimension n, show that if $n < 2d$ then it has a nowhere vanishing section. [*Go by induction over cells.*]

   Similarly, if $\pi_i : E_i \to X$, $i = 1, 2$, are d-dimensional complex vector bundles over a finite CW-complex of dimension n and $E_1 \oplus \underline{\mathbb{C}^1} \cong E_2 \oplus \underline{\mathbb{C}^1}$, show that if $n + 1 < 2(d + 1)$ then $E_1 \cong E_2$. [*Translate from isomorphisms to vector bundles over $X \times [0, 1]$.*]

6. $*$ Using Q5 and the clutching description of vector bundles, compute $K^0(S^2)$.

7. $*$ Compute $K^*(S^1 \times S^1)$ and $K^*(\mathbb{RP}^2)$ as abelian groups, and hence compute $K^*(S)$ for every compact closed surface S.

8. Compute the graded ring structure on $K^*(S^1 \times S^1)$.

9. Show that for any finitely-generated abelian group A there exists a finite cell complex X with $\tilde{K}^0(X) \cong A$.

Additional Questions

10. (i) Let $(\gamma_{\mathbb{F}}^{n,N})^{\perp}$ denote the orthogonal complement of $\gamma_{\mathbb{F}}^{n,N} \leq Gr_n(\mathbb{F}^N) \times \mathbb{F}^N$. Explain how to identify $\mathbb{S}(\gamma_{\mathbb{F}}^{n,N})$ with $\mathbb{S}((\gamma_{\mathbb{F}}^{n-1,N})^{\perp})$.

    (ii) Writing $c_i = c_i(\gamma_{\mathbb{C}}^{2,4})$ and $c_i' = c_i((\gamma_{\mathbb{F}}^{2,4})^{\perp})$, show that there is a well-defined map

$$\mathbb{Z}[c_1, c_2, c_1', c_2']/(c_1 + c_1', c_2 + c_1c_1' + c_2', c_1c_2' + c_2c_1', c_2c_2') \longrightarrow H^*(Gr_2(\mathbb{C}^4); \mathbb{Z}).$$

    Using the previous part, show this map is an isomorphism.

11. If M is a compact n-manifold, it has a fundamental class $[M] \in H_n(M; \mathbb{F}_2)$. If $\sum d_i = n$ then we can form $\langle \prod_i w_{d_i}(TM), [M] \rangle \in \mathbb{F}_2$.

    If there is a compact $(n+1)$-manifold W with boundary M, show that $TW|_M = TM \oplus \underline{\mathbb{R}}_M$, and hence show that $\langle \prod_i w_{d_i}(TM), [M] \rangle = 0$ for all sequences $\{d_i\}$. Deduce that $\mathbb{RP}^{2n}$ is not the boundary of any compact manifold with boundary.

Comments or corrections to `or257@cam.ac.uk`