

Part III Algebraic Topology // Example Sheet 3**1.***

- (i) Compute the cohomology ring of the closed oriented surface Σ_g of genus g .
- (ii) As $H^2(\Sigma_g) \cong \mathbb{Z}$ for every $g \geq 0$, define the degree of a map between oriented surfaces to be the induced map on H^2 . For which g is there a map $\Sigma_g \rightarrow \Sigma_1$ of positive degree? For which g is there a map $\Sigma_1 \rightarrow \Sigma_g$ of positive degree?

2. If $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ has components the elementary symmetric functions

$$(z_1, \dots, z_n) \mapsto (\sigma_i(z)) \quad \sigma_1 = \sum_j z_j \quad \sigma_2 = \sum_{i < j} z_i z_j \quad \cdots \quad \sigma_n = \prod_j z_j$$

then prove that f extends to a map $\psi : S^{2n} \rightarrow S^{2n}$ of degree $n!$.

Hence construct a map $\phi : (\mathbb{CP}^1)^n \rightarrow \mathbb{CP}^n$ of degree $n!$, and compute the effect of the map $\phi^* : H^2(\mathbb{CP}^n) \rightarrow H^2((\mathbb{CP}^1)^n)$. Deduce that there is a $x \in H^2(\mathbb{CP}^n)$ such that x^n is a generator of the abelian group $H^{2n}(\mathbb{CP}^n)$, and hence that $H^*(\mathbb{CP}^n) \cong \mathbb{Z}[x]/(x^{n+1})$ as a ring.

[Hint: relate $\mathbb{CP}^k$ to the space of degree k homogeneous polynomials in two variables.]

3. By considering a map to the wedge of two copies of $\mathbb{CP}^2$, or otherwise, compute $H^*(\mathbb{CP}^2 \# \mathbb{CP}^2)$ as a ring. Deduce that $\mathbb{CP}^2 \# \mathbb{CP}^2$ is not homotopy equivalent to $\mathbb{CP}^1 \times \mathbb{CP}^1$, even though they have the same (co)homology groups.

4. If X is a finite cell complex, show that $C_{\bullet}^{cell}(X)$ is (unnaturally) isomorphic to a direct sum of chain complexes of the form $0 \rightarrow B_n(X) \xrightarrow{A_n} Z_n(X) \rightarrow 0$, and hence show that

$$H^n(X) \cong \frac{H_n(X)}{\text{Tors}(H_n(X))} \oplus \text{Tors}(H_{n-1}(X)),$$

where $\text{Tors}(A) \leq A$ denotes the subgroup of elements of finite order.

5. Let $E \rightarrow X$ be a vector bundle with inner product $\langle \cdot, \cdot \rangle$. Let $F \subset E$ be a subbundle. Prove that the orthogonal complement bundle $F^\perp$ is locally trivial.

6. Show that a complex vector bundle has a canonical orientation.

7. If $\pi : E \rightarrow X$ is a d -dimensional real vector bundle which is not necessarily R -orientable, show that we still have $H^i(E, E^\#; R) = 0$ for $i < d$. If X is path-connected show that restriction to the fibre at $x \in X$ still gives an injective map $H^d(E, E^\#; R) \rightarrow H^d(E_x, E_x^\#; R) \cong R$.

Give an example to show that $H^{i+d}(E, E^\#; R)$ need not be isomorphic to $H^i(X; R)$ in general.

8.

- (i) Show that any map $f : \mathbb{RP}^n \rightarrow \mathbb{RP}^m$ induces a trivial map on reduced cohomology if $n > m$. What about if $n < m$?
- (ii) Show that $\mathbb{RP}^3$ is not homotopy equivalent to $\mathbb{RP}^2 \vee S^3$ although they have additively isomorphic (co)homology.

9.

- (i) If $f : S^n \rightarrow S^n$ satisfies $f(-x) = -f(x)$, show that it induces a map $\bar{f} : \mathbb{RP}^n \rightarrow \mathbb{RP}^n$. By considering the Gysin sequence show that f has odd degree.
- (ii) Show that any $g : S^n \rightarrow \mathbb{R}^n$ satisfies $g(x) = g(-x)$ for some $x \in S^n$.

10.*

- (i) Let $L = \gamma_{1,n+1}^{\mathbb{C}} \rightarrow \mathbb{CP}^n$ be the canonical 1-dimensional complex bundle. By considering $\pi_1^* L \otimes_{\mathbb{C}} \pi_2^* L \rightarrow \mathbb{CP}^n \times \mathbb{CP}^n$, with the $\pi_i : \mathbb{CP}^n \times \mathbb{CP}^n \rightarrow \mathbb{CP}^n$ being projections to the factors, prove that the Euler class of $L \otimes_{\mathbb{C}} L$ is equal to twice the Euler class of L .
- (ii) Show that the unit circle bundle in $L \otimes_{\mathbb{C}} L$ is homeomorphic to $\mathbb{RP}^{2n+1}$. Hence, compute the cohomology of $\mathbb{RP}^{2n+1}$ from knowledge of the cohomology of $\mathbb{CP}^n$.

11. Let $V_k(\mathbb{C}^n) \subset (\mathbb{C}^n)^k$ be the subspace of k -tuples of orthonormal vectors in $\mathbb{C}^n$ (a *Stiefel manifold*). Show there is a vector bundle $E_k \rightarrow V_k(\mathbb{C}^n)$ with fibre over $(v_1, \dots, v_k)$ given by the vector space $\text{span}(v_1, \dots, v_k) \leq \mathbb{C}^n$.

Show that the forgetful map $(v_1, \dots, v_k) \mapsto (v_1, \dots, v_{k-1}) : V_k(\mathbb{C}^n) \rightarrow V_{k-1}(\mathbb{C}^n)$ exhibits $V_k(\mathbb{C}^n)$ as the sphere bundle of a certain vector bundle over $V_{k-1}(\mathbb{C}^n)$. Hence compute $H^*(V_k(\mathbb{C}^n); \mathbb{Z})$ as a ring.

Deduce that the unitary group $U(n)$ has the same cohomology ring as $S^1 \times S^3 \times S^5 \times \dots \times S^{2n-1}$, and hence that

$$\sum_{j \geq 0} \text{rk } H^j(U(n); \mathbb{Z}) t^j = \prod_{i=1}^n (1 + t^{2i-1}).$$

12. [Optional: can only do (ii) if you have seen covering spaces.]

- (i) Show that a 1-dimensional vector bundle $\pi : E \rightarrow B$ is trivial if and only if it has a nowhere zero section.
- (ii) Show that isomorphism classes of 1-dimensional vector bundles over X are in bijection with isomorphism classes of 2-to-1 covering spaces of X . Hence shows that any 1-dimensional real vector bundle over S^n with $n > 1$ is trivial, and more generally that isomorphism classes of 1-dimensional real vector bundles over a finite cell complex X are naturally in 1-1 correspondence with elements of $H^1(X; \mathbb{Z}/2)$.

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