Michaelmas Term 2025 O. Randal-Williams

Part III Algebraic Topology // Example Sheet 2

1. Construct a natural map $H^n(X) \to \mathrm{Hom}_{\mathbb{Z}}(H_n(X), \mathbb{Z})$, and similarly for relative (co)homology, and prove that these maps commute with the ∂-maps in the long exact sequence for a pair. Show that your map is a surjection, but that it is not always an isomorphism.

2.* A 2×2 integer matrix A induces a continuous map $f_A : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ by $[\binom{x}{y}] \mapsto [A\binom{x}{y}]$. Show that the induced map on $H_1(-;\mathbb{Z})$ is given by the matrix A. Show that the induced map on $H_2(-;\mathbb{Z})$ is given by multiplication by $\det(A)$. *[Hint: for the latter it will be convenient to consider the local homology at the point* $[\binom{0}{0}] \in \mathbb{R}^2/\mathbb{Z}^2$*.]*

3. If $f : X \to X$ is a homeomorphism, let T_f be the quotient space of $X \times [0,1]$ by $(x,0) \sim (f(x),1)$. By considering the open cover of T_f given by the complement of $X \times \{\frac{1}{3}\}$ and the complement of $X \times \{\frac{2}{3}\}$, construct a long exact sequence of the form

$$\cdots \longrightarrow H_{n+1}(T_f) \longrightarrow H_n(X) \xrightarrow{1-f_*} H_n(X) \longrightarrow H_n(T_f) \longrightarrow \cdots .$$

Calculate $H_*(T_f)$ when (a) $f : S^n \to S^n$ is the antipodal map, (b) $f : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ is induced by the matrix $\left(\begin{smallmatrix} 3 & 4 \\ 2 & 3 \end{smallmatrix}\right)$.

4. Say a map $f : X \to Y$ between cell complexes is *cellular* if $f(X^n) \subset Y^n$ for every n. Show how to associate to such an f a chain map $f_\bullet^{cell} : C_\bullet^{cell}(X) \to C_\bullet^{cell}(Y)$ and show that the induced map $f_*^{cell} : H_*^{cell}(X) \to H_*^{cell}(Y)$ agrees with $f_* : H_*(X) \to H_*(Y)$ under a suitable identification of the homology groups.

5.* If $f : S^n \to X$ is a map, let $X \cup_f D^{n+1}$ be the space obtained by gluing D^{n+1} to X along the map f.

(i) If $f \simeq f' : S^n \to X$, show that $X \cup_f D^{n+1} \simeq X \cup_{f'} D^{n+1}$.

(ii) Let $Y = S^n \cup_f D^{n+1}$ be constructed using a map $f : S^n \to S^n$ of degree $m > 1$. Show that the natural quotient map $Y \to Y/S^n \cong S^{n+1}$ is trivial on homology $H_{*>0}$, but is non-trivial on cohomology $H^{*>0}$. What happens if we instead consider the inclusion $S^n \hookrightarrow Y$?

6.

(i) Let X be a cell complex and $A \subset X$ be a subcomplex. Prove that the pair (X, A) is good.

(ii) Let X be a cell complex and $K \subset X$ a compact subspace. Prove that K intersects only finitely many open cells in X. Hence show that any element of $H_i(X)$ lies in the image of $H_i(X^m) \to H_i(X)$ for some $m \gg 0$.

7. If X and Y are finite cell complexes with cells $\{e_\alpha\}_{\alpha \in I}$ and $\{f_\beta\}_{\beta \in J}$, construct a cell structure on $X \times Y$ with cells $\{e_\alpha \times f_\beta\}_{(\alpha,\beta) \in I \times J}$. Hence show that there is an isomorphism of chain complexes $C_\bullet^{cell}(X \times Y) \cong C_\bullet^{cell}(X) \otimes C_\bullet^{cell}(Y)$, where the latter has differential $d(e_\alpha \otimes f_\beta) = d(e_\alpha) \otimes f_\beta + (-1)^{dim(e_\alpha)} e_\alpha \otimes d(f_\beta)$. *[Hint: to understand this sign, it may help to think about the cellular chain complex of* $D^p \times D^q$*.]*

Use this to calculate $H_*(\mathbb{RP}^2 \times \mathbb{RP}^2)$.

8. Show that for $m, n \in \mathbb{N}$ and any space X there are short exact sequences of chain complexes

$$0 \longrightarrow C^\bullet(X) \longrightarrow C^\bullet(X) \longrightarrow C^\bullet(X; \mathbb{Z}/m) \longrightarrow 0$$

$$0 \longrightarrow C^\bullet(X;\mathbb{Z}/n) \longrightarrow C^\bullet(X;\mathbb{Z}/n\cdot m) \longrightarrow C^\bullet(X;\mathbb{Z}/m) \longrightarrow 0$$

and hence describe "Bockstein operations"

$$\tilde{\beta} : H^i(X;\mathbb{Z}/m) \longrightarrow H^{i+1}(X) \qquad \text{and} \qquad \beta : H^i(X;\mathbb{Z}/m) \longrightarrow H^{i+1}(X;\mathbb{Z}/n).$$

How are these two operations related? Compute the effect of β and $\tilde{\beta}$ for $m = 2$, $n = 2^r$, and $X = \mathbb{RP}^k$.

When $n = m$ show that $\beta(x \smile y) = \beta(x) \smile y + (-1)^{|x|} x \smile \beta(y)$.

9. A map $\pi : E \to B$ is called a *covering map* if there is an open cover $\{U_\alpha\}$ of B such that $\pi^{-1}(U_\alpha)$ is a disjoint union $\coprod V_{\alpha,\beta}$ with each $\pi|_{V_{\alpha,\beta}} : V_{\alpha,\beta} \to U_\alpha$ a homeomorphism.

(i) If $\pi : E \to B$ is a covering map with finite fibres of cardinality N, show how to construct a map $\pi^! : H_*(B) \to H_*(E)$ such that $\pi_* \circ \pi^!$ is multiplication by N.

(ii) In the same situation, if B is a finite cell complex show that $\chi(E) = N \cdot \chi(B)$.

(iii) Show there is a covering map $\Sigma_g \to \Sigma_h$ if and only if $g = kh - k + 1$ for some $k \in \mathbb{N}$.

10. For $A, B \subset X$ open sets, explain how to construct a *relative cup product*

$$\smile : H^i(X,A) \times H^j(X,B) \longrightarrow H^{i+j}(X, A \cup B)$$

[*Hint: it may be helpful to consider the cochain complex $C^*_{A+B}(X)$ of cochains vanishing on simplices lying wholly in A or B, and use the Small Simplices Theorem.*] Using this, show that if X has a cover by n contractible open sets, then the *cup-length*

$$\max\left\{k \mid \exists\, a_1, \ldots, a_k \ \in H^{*>0}(X),\ a_1 \smile \ldots \smile a_k \neq 0\right\}$$

is strictly smaller than n. What does this say about the ring $H^*(\Sigma X)$, where Σ is the suspension operation?

11. [*Optional: this is more tricky than it looks.*]

(i) Let $e : [0,1]^k \to S^n$ be a map which is a homeomorphism onto its image $D \subset S^n$. By considering the open sets

$$A = S^n \setminus e([0,1]^{k-1} \times [0,1/2]) \qquad B = S^n \setminus e([0,1]^{k-1} \times [1/2,1])$$

in S^n, show by induction on k that $\widetilde{H}_i(S^n \setminus D) = 0$.

(ii) If $e : S^k \to S^n$ is a map which is a homeomorphism onto its image $S \subset S^n$, compute $\widetilde{H}_i(S^n \setminus S)$. Think about what this means in the case $(n,k) = (2,1)$.

Comments or corrections to `or257@cam.ac.uk`