

Part III Algebraic Topology // Example Sheet 1**1.**

- (i) Prove that homotopy equivalence is an equivalence relation on topological spaces.
- (ii) Which of the following are homotopy equivalent to S^1 ? (a) the annulus $\{1 < |z| < r\}$, (b) a bagel, (c) a genus two surface with a disc sewn across one of the holes, (d) the complement of a point in the real projective plane $\mathbb{RP}^2$.

2. Compute $H^0(X)$ for a topological space X . Give an example of a space X for which $H_0(X)$ and $H^0(X)$ are not isomorphic.

3. What can you say about the abelian group A and/or the homomorphism α given exact sequences of the form

- (i) $0 \rightarrow \mathbb{Z}/2 \rightarrow A \rightarrow \mathbb{Z} \rightarrow 0$;
- (ii) $0 \rightarrow A \rightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0$;
- (iii) $0 \rightarrow \mathbb{Z}/4 \xrightarrow{\alpha} A \oplus \mathbb{Z}/2 \rightarrow \mathbb{Z}/4 \rightarrow 0$?

4.

- (i) The *suspension* ΣX of a space X is the quotient of $X \times [0, 1]$ by the map which collapses each end of the cylinder to a point: $X \times \{0\} \sim p_0$ and $X \times \{1\} \sim p_1$. Observe $\Sigma S^n \cong S^{n+1}$. Hence or otherwise prove there are maps $f : S^n \rightarrow S^n$ of any degree, for any $n > 0$.
- (ii) Suppose A is a closed manifold. Is ΣA necessarily homeomorphic to a closed manifold? Justify your answer.

5. If $f : S^n \rightarrow S^n$ has no fixed points, show that it is homotopic to the antipodal map. Hence show that if a group G acts freely on S^{2n} then $|G| \leq 2$.

6.

- (i) Finish the proof (begun in lectures) of the theorem that a short exact sequence of chain complexes has an associated long exact sequence on homology, by showing that the sequence obtained really is exact.
- (ii) Finish the proof (begun in lectures) of the 5-lemma.

7. If $X \subset \mathbb{R}^n$ is convex, show (*without using homotopy invariance!*) that $H_i(X) = 0$ for $i > 0$.

8.*

- (i) Let Σ_g denote the closed orientable surface of genus g , and $\Sigma_{g,1} := \Sigma_g - \text{int}(D^2)$ denote the space obtained by removing the interior of a closed disc in Σ_g ; its boundary $\partial \Sigma_{g,1}$ is a circle. Show that the inclusion $i : \partial \Sigma_{g,1} \rightarrow \Sigma_{g,1}$ induces the zero map on $H_1(-; \mathbb{Z})$.
- (ii) Hence compute the homology groups of Σ_g .

- (iii) Compute $H_*(\Sigma_2, A)$ where A is a simple closed curve which: (a) separates Σ_2 into two genus one pieces with one boundary component each; (b) is a non-separating simple closed curve cutting along which gives a genus one surface with two holes, and (c) bounds an embedded disc.

9.* Using Mayer–Vietoris, compute the cohomology groups of complex projective space $\mathbb{CP}^k$. For each n , construct a closed connected 4-dimensional manifold X_n with $H^1(X_n) = 0$ and $H^2(X_n) \cong \mathbb{Z}^n$. [*Hint: look up the “connect sum”.*]

10.

- (i) Define relative cohomology $H^*(X, A)$ in such a way that there is a long exact sequence

$$\cdots \rightarrow H^i(X, A) \rightarrow H^i(X) \rightarrow H^i(A) \rightarrow H^{i+1}(X, A) \rightarrow \cdots$$

- (ii) Compute the relative cohomology $H^*(D, \{p_1, \dots, p_k\})$ of the closed disc in $\mathbb{C}$ relative to k points.

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