IB Topological Spaces // Example Sheet 3

1. Show that the cone $\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2-z^2=0\}$ is not a topological manifold.

2. Let $F:\mathbb{R}^{d+n}\to\mathbb{R}^n$ be a continuously differentiable function such that whenever $F(x)=0$ the derivative D_xF of F at x is surjective. Using the implicit function theorem from Part IB Analysis II show that $F^{-1}(0)$ is a topological manifold of dimension d.

3. Show that a connected-sum of two connected topological manifolds of dimension $d\geq 2$ is connected. What about $d=1$?

4. For a pair of self-homeomorphisms $\varphi_0,\varphi_1 : X \overset{\cong}{\to} X$, an *isotopy* between them is a continuous function $\Phi:[0,1]\times X\to X$ such that $\varphi_0=\Phi_{\{0\}\times X}$, $\varphi_1=\Phi_{\{1\}\times X}$, and each $\Phi_{\{t\}\times X}$ is a homeomorphism. The existence of an isotopy from φ_0 to φ_1 defines an equivalence relation on self-homeomorphisms. *[Check this if you like.]*

   (a) Show that a self-homeomorphism of $[0,1]$ is isotopic to either the identity or to $t\mapsto 1-t$.

   (b) Show that a self-homomorphism of S^1 is isotopic to either the identity or to a reflection.

   (c) Show that a self-homeomorphism φ of $D^n:=\{v\in\mathbb{R}^n : |v|\leq 1\}$ which satisfies $\varphi(v)=v$ whenever $|v|=1$ is isotopic to the identity. *[Try to isotope φ so that it is the identity on increasingly-large neighbourhoods of the boundary of the disc.]*

5. Let X be a connected topological manifold of dimension $d\geq 2$. If $x_1,\ldots,x_n$ is a collection of distinct points in X and $y_1,\ldots,y_n$ is another, show there is a self-homeomorphism φ of X such that $\varphi(x_i)=y_i$ for all $i=1,\ldots,n$. Show that the conclusion is not true if X has dimension 1.

6. Supply the details of the proof that the quotient $P/\sim$ of a polygon $P\subset\mathbb{R}^2$ given by identifying pairs of edges is a topological surface.

   By drawing pictures, explain how one may triangulate such a surface.

7. For triangulable surfaces S_1 and S_1, show that (at least for for well-chosen discs with which to form the connected-sum) $S_1\# S_2$ is triangulable and $\chi(S_1\# S_2)=\chi(S_1)+\chi(S_2)-2$.

8. (a) A *polygonal decomposition* of a compact topological surface S consists of a finite collection of *faces* $\mathcal{F}=\{P_\alpha : \alpha\in I\}$, which are polygons, together with continuous functions $f_\alpha : P_\alpha\to S$ which are homeomorphisms on their image, such that if two subspaces $f_\alpha(P_\alpha)$ intersect they do so either along an entire edge, or in a single vertex. The *edges* $\mathcal{E}$ are the subspaces $f_\alpha(e)$ where e is an edge of the polygon P_α, and the *vertices* $\mathcal{V}$ are the subspaces $f_\alpha(v)$ where v is a vertex of the polygon P_α. Comment on why S may be triangulated, and writing F,E,V for the cardinalities of $\mathcal{F},\mathcal{E},\mathcal{V}$ show that

   $$\chi(S)=V-E+F.$$

   (b) Consider a polygonal decomposition where all vertices have valence ≥ 3 and every face contains ≥ 3 vertices. Let F_n denote the number of faces bound by precisely n edges, and V_m the number of vertices where precisely m edges meet. Show that $\sum_n n\,F_n=2E=\sum_m m\,V_m$. If $V_3=0$, deduce that $E\geq 2V$, whilst if $F_3=0$ deduce that $E\geq 2F$. If the surface is a sphere, deduce that $V_3+F_3>0$.

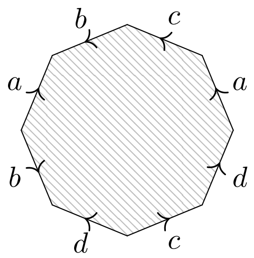
   (c) For a polygonal decomposition of a sphere, further show that

   $$\sum_n(6-n)F_n=12+2\sum_m(m-3)V_m.$$

   If each face has at least 3 edges and at least 3 edges meet at each vertex, deduce that $3F_3+2F_4+F_5\geq 12$.

(d) The surface of a football is decomposed into hexagons and pentagons, with precisely 3 faces meeting at each vertex. How many pentagons are there?

9. Express the following surface as a connected-sum of tori and projective planes:



[You do not need to do so explicitly!]

10. If $S = P/\sim$ is a compact topological surface obtained by gluing the edges of a $2n$-gon together in pairs, what are the maximum and minimum possible values of $\chi(S)$?

Optional Questions

11. Show that

$$\{(x, y, z) \in \mathbb{R}^3 : x^4 + y^4 + z^4 + 2(x^2y^2 + x^2z^2 + y^2z^2) + 8xyz - 10(x^2 + y^2 + z^2) + 20 = 0\}$$

is a compact topological surface. By plotting it on a computer, work out which surface it is in the classification.

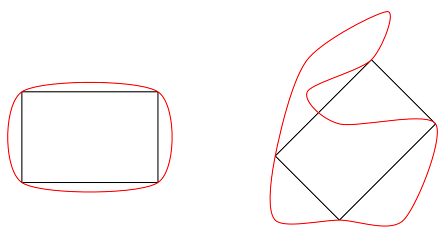
12. (a) Let X be the set of unordered pairs of points on a circle S^1. Explain why X is naturally a quotient space of the torus T^2. By considering T^2 as a quotient space of a square, or otherwise, show that X is homeomorphic to $\mathbb{RP}^2 \backslash \mathring{D}$, the complement of an open disc $\mathring{D} \subset \mathbb{RP}^2$ in $\mathbb{RP}^2$.

(b) Let $\phi : D := \{|z| \leq 1\} \to \mathbb{R}^2$ be a continuous injection of a closed disc D, with boundary $C = \phi(S^1) \subset \mathbb{R}^2$. Define a map $f : C \times C \to \mathbb{R}^3$ via

$$(u, v) \mapsto \left(\frac{u+v}{2}, |u - v|\right) \in \mathbb{R}^2 \times \mathbb{R}.$$

Show that f defines a map on X, which extends to a continuous map $\hat{\phi} : \mathbb{RP}^2 \to \mathbb{R}^3$.

(c) Using without proof that a compact non-orientable topological surface cannot be continuously embedded in $\mathbb{R}^3$, deduce that C bounds an *inscribed rectangle*, i.e. there are four pairwise-distinct points on the curve which form the vertices of a rectangle in $\mathbb{R}^2$ (the rectangle need not be wholly contained in $\phi(D)$, cf. figure below).



Comments or corrections to `or257@cam.ac.uk`