Lent Term 2026 O. Randal-Williams

IB Topological Spaces // Example Sheet 2

1. Which of the following subsets of $\mathbb{R}^2$ are connected? Which are path-connected?
   (a) $\{(x, y) \in \mathbb{R}^2 : |(x, y) - (-1, 0)| \leq 1 \text{ or } |(x, y) - (1, 0)| < 1\}$.
   (b) $\{(x, y) \in \mathbb{R}^2 : x = 0 \text{ or } y = qx \text{ for } q \in \mathbb{Q}\}$.
   (c) $\{(x, y) \in \mathbb{R}^2 : x = 0 \text{ or } y = qx \text{ for } q \in \mathbb{Q}\} \setminus \{(0, 0)\}$.

2. (a) If $f : [0, 1] \to [0, 1]$ is continuous show there is an $x \in [0, 1]$ such that $f(x) = x$.
   (b) If $f : S^1 \to \mathbb{R}$ is continuous, show there is a point $x \in S^1$ such that $f(x) = f(-x)$.

3. Show that there is no continuous injective map $f : \mathbb{R}^2 \to \mathbb{R}$.

4. Recall from Q9 on Example Sheet 1 the topological space $\mathbb{R}_{\text{ll}}$ given by the real numbers with the lower limit topology (having basic open sets $[a, b)$ for $a < b$).
   (a) Describe the connected components of $\mathbb{R}_{\text{ll}}$.
   (b) Show that every continuous function $f : \mathbb{R} \to \mathbb{R}_{\text{ll}}$ is constant.

5. Show that a connected component of a topological space is closed. Need it be open?

6. If $A \subset \mathbb{R}^2$ is a countable subset, show that $\mathbb{R}^2 \setminus A$ is path-connected.

7. Is there is Hausdorff topology on $[0, 1]$ which is coarser than the usual (Euclidean) topology?

8. If X is a set, when is $(X, \mathcal{T}_{\text{cofinite}})$ compact? When is $(X, \mathcal{T}_{\text{cocountable}})$ compact?

9. Say a topological space is *normal* if whenever C_1 and C_2 are disjoint closed subsets, there are disjoint open sets U_1 and U_2 such that $C_i \subseteq U_i$. Show that a compact Hausdorff space is normal.

10. Let $K_1 \supseteq K_2 \supseteq K_3 \supseteq \ldots$ be a decreasing sequence of non-empty compact subsets of a Hausdorff topological space X. Let $K := \bigcap_{n=1}^{\infty} K_n$.
    (a) Show that K is non-empty.
    (b) If the K_n are all connected, show that K is connected. *[Hint: use the previous question.]*
    (c) Give an example with $X = \mathbb{R}^2$ to show that the conclusion of part (b) need not be true if the K_n are assumed to be "closed" instead of "compact".

11. Let X be any topological space and Y be a compact topological. Show that the image of a closed set under the projection map $\pi_X : X \times Y \to Y$ is closed.

    Now let X be any space and Y be a compact Hausdorff topological space. Show that a function $f : X \to Y$ is continuous if and only if its graph $\Gamma_f := \{(x, y) \in X \times Y : y = f(x)\}$ is closed in $X \times Y$.

12. The *one-point compactification* of a topological space $(X, \mathcal{T})$ consists of the set $X^+ := X \cup \{\infty\}$ together with the collection of subsets

$$\mathcal{T}^+ := \mathcal{T} \cup \{(X \setminus K) \cup \{\infty\} : K \text{ a closed and compact subspace of } (X, \mathcal{T})\}.$$

    Show that $(X^+, \mathcal{T}^+)$ is a topological space, and that it is compact. Describe the one-point compactifications of $\mathbb{R}$ and $\mathbb{R}^2$ in terms of more familiar topological spaces.

Optional Questions

13. Define a relation $\asymp$ on a topological space X by $x \not\asymp y$ whenever there is a decomposition $X = U \sqcup V$ into disjoint open subsets with $x \in U$ and $y \in V$. (So $x \asymp y$ means that for any decomposition of X into disjoint open sets, x and y lie in the same part.)

    (a) Show $\asymp$ an equivalence relation.

    (b) Show that if x and y lie in the same connected component of X then $x \asymp y$.

    (c) Show that the equivalence classes for $\asymp$ are not the same as connected components in general. *[They do coincide if X is compact Hausdorff. You have all the tools necessary to prove this, but it is quite involved.]*

14. Let X and Y be topological spaces, and $F(X,Y)$ denote the set of continuous functions from X to Y. For a compact set $K \subseteq X$ and an open set $U \subseteq Y$, let

$$V(K,U) := \{f : X \to Y : f(K) \subseteq U\}.$$

    The *compact-open topology* on $F(X,Y)$ is the topology with subbasis the sets $V(K,U)$.

    (a) If $\phi : Z \times X \to Y$ is a continuous function, show that $z \mapsto \phi(z,-) : Z \to F(X,Y)$ is continuous.

    (b) If $\Phi : Z \to F(X,Y)$ is continuous and X is compact, show that $(z,x) \mapsto \Phi(z)(x) : Z \times X \to Y$ is continuous.

    (c) If $V \subseteq X \times Y$ is open, show that the set

$$\Gamma(V) := \{f \in F(X,Y) : \text{the graph of } f \text{ lies inside } V\}$$

    is open. These sets form a subbasis for the *graph topology*. Show that if X is compact then the graph and compact-open topologies coincide.

    (d) If the topology on Y is induced by a metric d on Y, show that a sequence (f_n) in $F(X,Y)$ converges to $f_\infty \in F(X,Y)$, in the sense of topological spaces, if and only it converges uniformly to f_∞ in the sense of IB Analysis II.

    Describe another topology on $F(X,Y)$ in which a sequence (f_n) converges if and only if it converges pointwise in the sense of IB Analysis II.

15. Show that $\mathbb{R}_{ll}$ is normal, and that $\mathbb{R}_{ll} \times \mathbb{R}_{ll}$ is not normal.

Comments or corrections to `or257@cam.ac.uk`