Lent Term 2026 O. Randal-Williams

IB Topological Spaces // Example Sheet 1

1. Determine whether the following subsets $A \subseteq \mathbb{R}^2$ are open, closed, or neither:
   (a) $A = \{(x, y) : x < 0\} \cup \{(x, y) : x > 0 \text{ and } y > 1/x\}$,
   (b) $A = \{(x, \sin(1/x) : x > 0\} \cup \{(0, y) : y \in [-1, 1]\}$,
   (c) $A = \{(x, y) : x \in \mathbb{Q} \text{ and } x = y^n \text{ for some positive integer } n\}$.
2. Show that $[0, 1] \times (0, 1]$ and $(0, 1] \times (0, 1]$, with their natural metric topologies, are homeomorphic.
3. Recall that the *cofinite topology* on a set X is $\mathcal{T}_{\text{cofinite}} := \{X \setminus F : F \subseteq X \text{ is finite}\} \cup \{\emptyset\}$. Show that this is indeed a topology. Assuming that the set X is infinite, when does a sequence (x_n) in $(X, \mathcal{T}_{\text{cofinite}})$ converge, and what does it converge to?
4. Let X be a set and $\mathcal{S} \subseteq P(X)$ be a collection of subsets of X which cover X. Prove that there is a unique topology $\mathcal{T}_\mathcal{S}$ on X for which $\mathcal{S}$ is a subbasis. Show that continuity can be checked on subbases: If $(Y, \mathcal{T}_Y)$ is a topological space and $f : (Y, \mathcal{T}_Y) \to (X, \mathcal{T}_\mathcal{S})$ is a function, then f is continuous if and only if $f^{-1}(S) \in \mathcal{T}_Y$ for all $S \in \mathcal{S}$.
5. Show that the collection of closed subsets of a topological space X contains $\emptyset$ and X and is preserved by arbitrary intersections and by finite unions. Show that a function between topological spaces is continuous if and only if the preimage of every closed set is closed.
6. Find a closed set $C \subseteq \mathbb{R}$ so that $\overline{\mathring{C}} \neq C$ and an open set $U \subseteq \mathbb{R}$ so that $\mathring{\overline{U}} \neq U$.
7. Let X and Y be topological spaces with Y Hausdorff.
   (a) Show that $\Delta := \{(y, y) : y \in Y\}$ is a closed subset of $Y \times Y$ with the product topology.
   (b) Let $f, g : X \to Y$ be continuous. Show that $\{x \in X : f(x) = g(x)\}$ is a closed subset of X.
8. Recall the (right) order topology $\mathcal{T}_{\text{ord}}$ on $\mathbb{R}$, in which the open sets are (a, ∞) for $a \in [-\infty, \infty]$, giving a topological space $\mathbb{R}_{\text{ord}}$.
   (a) Are the functions $(x, y) \mapsto x + y, (x, y) \mapsto x \cdot y : \mathbb{R}_{\text{ord}} \times \mathbb{R}_{\text{ord}} \to \mathbb{R}_{\text{ord}}$ continuous?
   (b) Show that a continuous function $f : \mathbb{R}_{\text{ord}} \to \mathbb{R}$ must be constant. *[It may help to first show that it is weakly monotonic increasing.]*
9. The *lower limit topology* $\mathcal{T}_{\text{ll}}$ on $\mathbb{R}$ is the topology with basis the half-open intervals $[a, b)$ with $a < b$. Write $\mathbb{R}_{\text{ll}}$ for the topological space $(\mathbb{R}, \mathcal{T}_{\text{ll}})$.
   (a) Show that $\mathcal{T}_{\text{ll}}$ is finer than $\mathcal{T}_{\text{Euc}}$, i.e. $id : \mathbb{R}_{\text{ll}} \to \mathbb{R}$ is continuous.
   (b) Show that $\mathbb{R}_{\text{ll}}$ is Hausdorff.
   (c) When does a sequence (x_n) in $\mathbb{R}_{\text{ll}}$ converge, and what is its limit?
   (d) Describe in elementary terms what it means for a function $\mathbb{R}_{\text{ll}} \to \mathbb{R}$ to be continuous.
10. Let $X = [0, 1] \cup [2, 3]$ and define an equivalence relation $\sim$ on X by $x \sim y$ whenever $x = y$ or $\{x, y\} = \{1, 2\}$. Show that $X_{/\sim}$ is homeomorphic to $[0, 2]$ with the usual topology.
11. Define an equivalence relation $\sim$ on $\mathbb{R}^2$ by

$$(x_0, y_0) \sim (x_1, y_1) \iff x_0 + y_0^2 = x_1 + y_1^2.$$

    Show that the quotient space is homeomorphic to a familiar space.

12. Consider the topological space X obtained from the disjoint union $[0,1] \times \{0\} \sqcup (2,3] \times [0,1]$ as the quotient by the relation $(t,0) \sim (t+2,0)$ for $t \in (0,1]$. Show that there is a continuous bijection

$$\Phi : X \longrightarrow \{(x,y) \in [0,1]^2 : x > 0 \text{ or } y = 0\}$$

but that this is *not* a homeomorphism.

Optional Questions

13. Recall that if $\{X_\alpha\}_{\alpha \in I}$ is an arbitrary collection of sets then we may form the product $\prod_{\alpha \in I} X_\alpha$. Suppose each X_α has a given topology $\mathcal{T}_\alpha$.

    (a) Show that the collection $\mathcal{B}_{box} := \{\prod_{\alpha \in I} U_\alpha : U_\alpha \in \mathcal{T}_\alpha\}$ is a basis for a topology $\mathcal{T}_{\text{box}}$ on $\prod_{\alpha \in I} X_\alpha$, the *box topology*. Show that the projections $\pi_\alpha : (\prod_{\alpha \in I} X_\alpha, \mathcal{T}_{\text{box}}) \to (X_\alpha, \mathcal{T}_\alpha)$ are all continuous

    (b) Explain why there exists a smallest topology $\mathcal{T}_{\text{prod}}$ on $\prod_{\alpha \in I} X_\alpha$, the *product topology*, for which the projections $\pi_\alpha : \prod_{\alpha \in I} X_\alpha \to X_\alpha$ are all continuous. Describe a basis $\mathcal{B}_{\text{prod}}$ for this topology analogous to $\mathcal{B}_{box}$ above.

    (c) Consider the set $\mathbb{R}^{\mathbb{N}} = \prod_{n \in \mathbb{N}} \mathbb{R}$ of sequences of real numbers, and its subset Z of those sequences which are eventually zero. Show that the function $x \mapsto (x, x, x, \ldots) : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ is continuous when the target is given the product topology, but not when it is given the box topology. Determine the closure of Z in each of the box and product topologies.

Comments or corrections to `or257@cam.ac.uk`