# Metastability
## in the homology of mapping class groups

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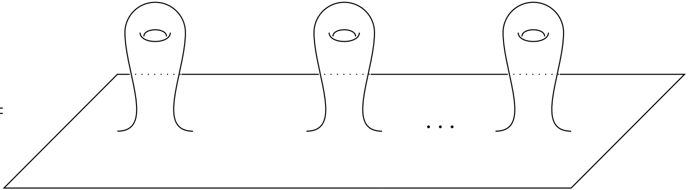
with S. Galatius and A. Kupers; available as arXiv:1805.07187.

The surface

$$\Sigma_{g,1} =$$



has a mapping class group

$$\Gamma_{g,1} = \pi_0(\mathrm{Diff}_\partial(\Sigma_{g,1})).$$

Putting such surfaces next to each other provides homomorphisms

$$\Gamma_{g,1} \times \Gamma_{h,1} \xrightarrow{\circ_{g,h}} \Gamma_{g+h,1}$$

which endows

$$\bigoplus_{g \geq 0} H_*(\Gamma_{g,1}; \mathbb{k})$$

with an associative unital multiplication $\cdot$.

Homological stability for the $\Gamma_{g,1}$ concerns the effect on homology of

$$\Gamma_{g-1,1} \xrightarrow{e \times Id} \Gamma_{1,1} \times \Gamma_{g-1,1} \xrightarrow{\circ_{1,g-1}} \Gamma_{g,1}.$$

This is precisely the map

$$\sigma \cdot - : H_d(\Gamma_{g-1,1}; \mathbb{k}) \longrightarrow H_d(\Gamma_{g,1}; \mathbb{k})$$

given by left multiplication by the generator $\sigma \in H_0(\Gamma_{1,1}; \mathbb{k})$.

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**Question**: For fixed $d$, is this map surjective / injective for $g \gg d$?

Relative homology measures the failure of homological stability.

Theorem (Boldsen, R-W; earlier results by Harer, Ivanov)

$H_d(\Gamma_{g,1}, \Gamma_{g-1,1}; \Bbbk) = 0$ *for* $d \leq \frac{2g-2}{3}$.

## Main Theorem (Galatius–Kupers–R-W)

*There are maps*

$$\varphi_* \colon H_{d-2}(\Gamma_{g-3,1}, \Gamma_{g-4,1}; \mathbb{k}) \longrightarrow H_d(\Gamma_{g,1}, \Gamma_{g-1,1}; \mathbb{k})$$

*which are epimorphisms for* $d \leq \frac{3g-1}{4}$ *and isomorphisms for* $d \leq \frac{3g-5}{4}$.

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There are elaborations for surfaces with additional boundaries and with marked points, and for homology with certain twisted coefficients.

## The idea

Stability concerns the structure of

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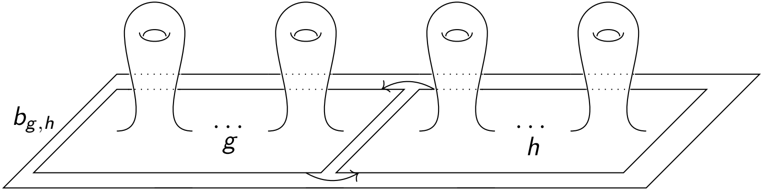
$\bigoplus_{g \geq 0} H_*(\Gamma_{g,1}; \Bbbk)$ **is a $\Bbbk$-algebra: should study algebra generators and relations (and relations between relations, and ...) instead!**

The homomorphism

$$\Gamma_{g,1} \times \Gamma_{h,1} \xrightarrow{\text{swap}} \Gamma_{h,1} \times \Gamma_{g,1} \xrightarrow{\circ_{h,g}} \Gamma_{g+h,1}$$

differs from $\circ_{g,h}$ by conjugation by the diffeomorphism




Conjugation acts as the identity on group homology, so the multiplication on $\bigoplus_{g \geq 0} H_*(\Gamma_{g,1}; \mathbb{k})$ is commutative.

In fact this structure makes

$$\coprod_{g \geq 0} \Gamma_{g,1}$$

into a braided monoidal groupoid, so makes

$$\mathbf{R}^+ := \coprod_{g \geq 0} B\Gamma_{g,1}$$

into a unital $E_2$-algebra. This gives

$$H_*(\mathbf{R}^+) = \bigoplus_{g \geq 0} H_*(\Gamma_{g,1}; \mathbb{k})$$

further structure: a Browder bracket

$$[-,-] : H_d(\Gamma_{g,1}; \mathbb{k}) \otimes H_{d'}(\Gamma_{g',1}; \mathbb{k}) \longrightarrow H_{d+d'+1}(\Gamma_{g+g',1}; \mathbb{k})$$

as well as Dyer–Lashof operations in $\mathbb{F}_p$-homology, and more.

Rather than trying to study "generators" and "relations" for

$$H_*(\mathbf{R}^+) = \bigoplus_{g \geq 0} H_*(\Gamma_{g,1}; \Bbbk)$$

as an algebraic object having this rich structure, we shall study the $E_2$-algebra

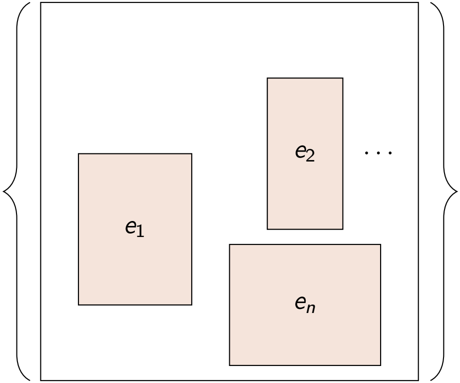
$$\mathbf{R}^+ = \bigsqcup_{g \geq 0} B\Gamma_{g,1}$$

and attempt to describe its $E_2$-algebra "generators" and "relations".

We can worry about extracting homological information out of this later.

The little 2-cubes operad $\mathcal{C}_2$ has

$\mathcal{C}_2(n) =$




Associated monad

$$X \mapsto E_2(X) = \coprod_{n \geq 1} \mathcal{C}_2(n) \times_{\Sigma_n} X^n$$

given by space of unordered little 2-cubes each labelled by $X$. Forgetting intermediate cubes gives a map

$$\alpha : E_2(E_2(X)) \longrightarrow E_2(X).$$

An non-unital *$E_2$-algebra* $\mathbf{X} = (X, \mu)$ is a space $X$ and a $\mu : E_2(X) \to X$ compatible with $\alpha$ in the evident way. Our space

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To record individual genera, consider $\mathbf{R}$ as a $\mathbb{N}$-*graded pointed space*: the functor

$$\mathbf{R} : \mathbb{N} \longrightarrow \mathsf{Top}_*$$

given by $\mathbf{R}(g) = (B\Gamma_{g,1})_+$.

$$\Rightarrow \mathbf{R} \in \mathsf{Alg}_{E_2}(\mathsf{Top}_*^{\mathbb{N}}).$$

For $X \in \mathsf{Top}_*^{\mathbb{N}}$ write

$$H_{g,d}(X) := \tilde{H}_d(X(g)).$$

**Attaching cells**: The graded sphere $S^{g,d-1} \in \mathrm{Top}_*^{\mathbb{N}}$ is given by

$$S^{g,d-1}(g) = \begin{cases} * & \text{if } h \neq g \\ S^{d-1} & \text{if } h = g, \end{cases}$$

and the graded disc $D^{g,d}$ is similar.

A map $f : S^{g,d-1} \to \mathbf{X}$ extends to an $E_2$-map $f' : \mathbf{E}_2(S^{g,d-1}) \to \mathbf{X}$ from the free $E_2$-algebra on $S^{g,d-1}$, and we can form the push-out

$$\begin{array}{ccc} \mathbf{E}_2(S^{g,d-1}) & \xrightarrow{\quad f' \quad} & \mathbf{X} \\ \Big\downarrow{\scriptstyle \mathbf{E}_2(inc)} & & \Big\downarrow \\ \mathbf{E}_2(D^{g,d}) & \longrightarrow & \mathbf{X} \cup_f^{E_2} \mathbf{D}^{g,d} \end{array}$$

in $\mathrm{Alg}_{E_2}(\mathrm{Top}_*^{\mathbb{N}})$. This is attaching a $(g,d)$-dimensional $E_2$-cell to $\mathbf{X}$.

A *cellular* $E_2$*-algebra* is one constructed from $*$ by attaching cells in this way.

**Detecting cells**: For $\mathbf{X} \in \mathrm{Alg}_{E_2}(\mathrm{Top}_*^{\mathbb{N}})$ define

$$E_2(X) = \bigvee_{n \geq 1} \mathcal{C}_2(n)_+ \wedge_{\Sigma_n} X^{\wedge n} \underset{c}{\overset{\mu_X}{\rightrightarrows}} X \longrightarrow Q^{E_2}(\mathbf{X})$$

where $c$ collapses all factors with $n > 1$ to the basepoint, and applies $\mathcal{C}_2(1)_+ \to S^0$. This is the $E_2$-*indecomposables* of $\mathbf{X}$.

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**Observe**: $Q^{E_2} : \mathsf{Alg}_{E_2}(\mathsf{Top}_*^{\mathbb{N}}) \to \mathsf{Top}_*^{\mathbb{N}}$ preserves colimits.

$$\Rightarrow Q^{E_2}(\mathbf{X} \cup_f^{E_2} \mathbf{D}^{g,d}) \cong Q^{E_2}(\mathbf{X}) \cup_{Q^{E_2}(f)} D^{g,d},$$

so $Q^{E_2}(\mathbf{X})$ has one ordinary $(g,d)$-cell for each $E_2$-$(g,d)$-cell of $\mathbf{X}$.

$E_2$-**homology**: $Q^{E_2}$ is not homotopy invariant and must be derived: we can let

$$Q^{E_2}_{\mathbb{L}}(\mathbf{X}) = Q^{E_2}(c\mathbf{X}) = \left\{ \substack{\text{a graded cell complex with one} \\ (g,d)\text{-cell for each } E_2\text{-}(g,d)\text{-cell of } c\mathbf{X}} \right\}$$

for a cellular approximation $c\mathbf{X} \overset{\sim}{\to} \mathbf{X}$.

Write

$$H^{E_2}_{g,d}(\mathbf{X}; \mathbb{k}) := H_{g,d}(Q^{E_2}_{\mathbb{L}}(\mathbf{X}); \mathbb{k}).$$

If $\mathbb{k}$ is a field, the discussion so far shows

$$\dim_{\mathbb{k}} H^{E_2}_{g,d}(\mathbf{X}; \mathbb{k}) \leq \substack{\text{number of } E_2\text{-}(g,d)\text{-cells in any} \\ E_2\text{-cellular approximation of } \mathbf{X}.}$$

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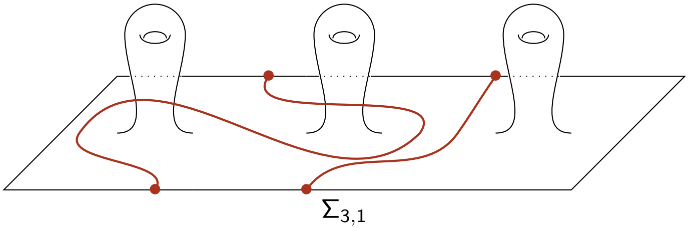
## Theorem

*If we take $\mathbb{k}$-linear singular simplices this is sharp: a $\mathbf{X} \in \mathrm{Alg}_{E_2}(\mathrm{sMod}^{\mathbb{N}}_{\mathbb{k}})$ has a cellular approximation $c\mathbf{X} \xrightarrow{\sim} \mathbf{X}$ with $\dim_{\mathbb{k}} H^{E_2}_{g,d}(\mathbf{X};\mathbb{k})$-many $E_2$-$(g,d)$-cells.*
*Furthermore $c\mathbf{X}$ can be taken to be "CW", not just "cellular".*

There is a model for $Q^{E_2}_{\mathbb{L}}(\mathbf{X})$ in terms of a two-fold bar construction; instances have been given by Getzler–Jones, Basterra–Mandell, Fresse, Francis. For the $E_2$-algebra $\mathbf{R} = \coprod_{g\geq 1} B\Gamma_{g,1}$ this leads us to study the simplicial complex whose $p$-simplices are $(p+1)$ arcs on the surface $\Sigma_{g,1}$, which cut it into $(p+2)$ components each of which have non-zero genus.



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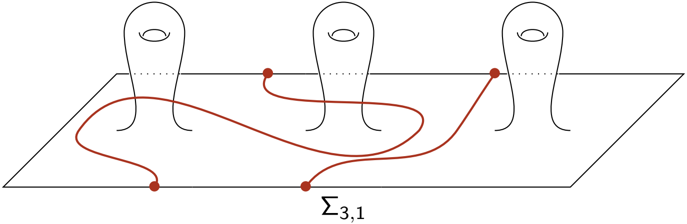


We show that this simplicial complex is $(g-3)$-connected, so

**Theorem (Galatius–Kupers–R-W)**

$H^{E_2}_{g,d}(\mathbf{R}) = 0$ *for* $d < g - 1$.

Thus there is an $E_2$-cellular approximation $c\mathbf{R} \xrightarrow{\sim} \mathbf{R}$ only having $(g, d)$-cells for $d \geq g - 1$.

F. Cohen has calculated the homology of free unital $E_2$-(and more generally $E_k$-)algebras. Working for simplicity over $\mathbb{Q}$, one has

$$\begin{aligned} H_{*,*}(\mathbf{E}_2^+(X);\mathbb{Q}) &= \text{free Gerstenhaber algebra on } H_{*,*}(X;\mathbb{Q}) \\ &= {}^{\text{free graded commutative algebra on}}_{\text{the free graded Lie algebra on}}\ H_{*,*}(X;\mathbb{Q}) \end{aligned}$$

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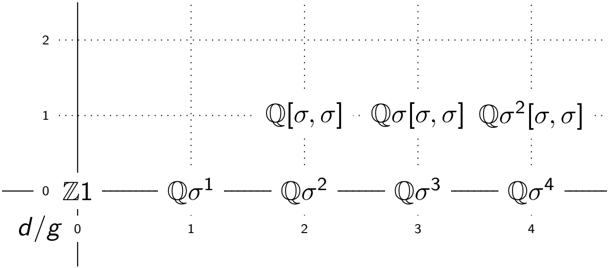
For example

$$H_{*,*}(\mathbf{E}_2^+(S^{1,0}_\sigma);\mathbb{Q}) = \mathbb{Q}[\sigma,[\sigma,\sigma]]/([\sigma,\sigma]^2)$$



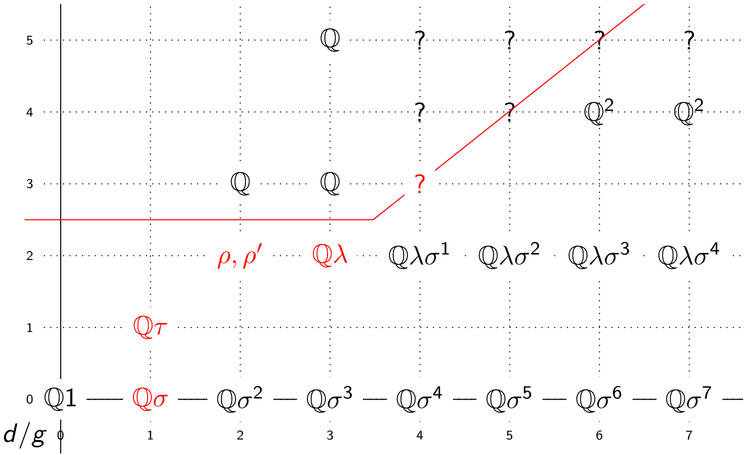
Low-dimensional homology of $\Gamma_{g,1}$ has been studied in detail by many mathematicians:

Abhau, Benson, Bödigheimer, Boes, F. Cohen, Ehrenfried, Godin, Harer, Hermann, Korkmaz, Looijenga, Meyer, Morita, Mumford, Pitsch, Sakasai, Stipsicz, Tommasi, Wang, ...

What is known in homological degrees $\leq 3$ is:

| $d/g$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 5 | | | | $\mathbb{Q}$ | ? | ? | ? | ? |
| 4 | | | | | ? | ? | $\mathbb{Q}^2$ | $\mathbb{Q}^2$ |
| 3 | | | $\mathbb{Q}$ | $\mathbb{Q}$ | | | | |
| 2 | | | | $\mathbb{Q}\lambda$ | $\mathbb{Q}\lambda\sigma$ | $\mathbb{Q}\lambda\sigma^2$ | $\mathbb{Q}\lambda\sigma^3$ | $\mathbb{Q}\lambda\sigma^4$ |
| 1 | | $\mathbb{Q}\tau$ | | | | | | |
| 0 | $\mathbb{Q}1$ — | $\mathbb{Q}\sigma$ — | $\mathbb{Q}\sigma^2$ — | $\mathbb{Q}\sigma^3$ — | $\mathbb{Q}\sigma^4$ — | $\mathbb{Q}\sigma^5$ — | $\mathbb{Q}\sigma^6$ — | $\mathbb{Q}\sigma^7$ — |

This allows us to construct an explicit $E_2$-cell structure in homological degrees $\leq 2$ and $d < g - 1$ as:




The cells $\rho$ and $\rho'$ are attached along $\partial(\rho) = [\sigma, \sigma]$ and $\partial(\rho') = \sigma \cdot \tau$. The lowest slope $\frac{d}{g}$ in which there may be an additional $E_2$-cell is $\frac{3}{4}$.

**Homological stability**: Construct the $\mathbf{R}^+$-module cofibre sequence

$$S^{1,0} \otimes \mathbf{R}^+ \xrightarrow{\sigma\cdot -} \mathbf{R}^+ \longrightarrow \mathbf{R}^+/\sigma.$$

This has $H_{g,d}(\mathbf{R}^+/\sigma) = H_d(\Gamma_{g,1}, \Gamma_{g-1,1}; \mathbb{Q})$, so homological stability means finding a vanishing line for this.

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Filtering $\mathbf{R}^+$ by its $E_2$-skeleta gives a spectral sequence going from

$$E^1_{g,p,q} = H_{g,p+q,q}(\mathbf{E}_2^+(S^{1,0,0}_\sigma \oplus S^{3,2,2}_\lambda \oplus S^{2,2,2}_\rho \oplus \cdots)/\sigma)$$

to $H_{g,p+q}(\mathbf{R}^+/\sigma)$, where the generators $\cdots$ all have slope $\geq \frac{3}{4}$. Cohen's calculation identifies the $E^1$-page, and the $d^1$-differential satisfies $d^1(\rho) = [\sigma, \sigma]$. It is then an elementary piece of homological algebra to show that $E^2_{g,p,q} = 0$ for $\frac{p+q}{g} < \frac{2}{3}$.

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This recovers the known homological stability range, with slope $\frac{2}{3}$. Analysing the argument, all that is used particular to mapping class group—in addition to the vanishing line for $E_2$-cells—is that

$$H_1(\Gamma_{1,1}) \longrightarrow H_1(\Gamma_{2,1})$$

is onto (which follows from the fact that the $\Gamma_{g,1}$ are generated by non-separating Dehn twists). This argument extends to $\mathbb{Z}$-coefficients.

**Secondary homological stability**: Construct the $\mathbf{R}^+$-module cofibre sequence

$$S^{3,2} \otimes \mathbf{R}^+/\sigma \xrightarrow{\lambda\cdot -} \mathbf{R}^+/\sigma \longrightarrow \mathbf{R}^+/(\sigma, \lambda).$$

This gives $(\lambda \cdot -)_* : H_{d-2}(\Gamma_{g-3,1}, \Gamma_{g-4,1}; \mathbb{Q}) \to H_d(\Gamma_{g,1}, \Gamma_{g-1,1}; \mathbb{Q})$ so secondary stability (with $\mathbb{Q}$-coefficients) means finding a slope $\frac{3}{4}$ vanishing line for $H_{g,d}(\mathbf{R}^+/(\sigma, \lambda))$.

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Proceed exactly as before: there is a spectral sequence going from

$$E^1_{g,p,q} = H_{g,p+q,q}(\mathbf{E}_2^+(S^{1,0,0}_\sigma \oplus S^{3,2,2}_\lambda \oplus S^{2,2,2}_\rho \oplus \cdots)/(\sigma, \lambda))$$

to $H_{g,p+q}(\mathbf{R}^+/(\sigma, \lambda))$, where the generators $\cdots$ all have slope $\geq \frac{3}{4}$. Still have $d^1(\rho) = [\sigma, \sigma]$, and again it is an elementary piece of homological algebra to show that $E^2_{g,p,q} = 0$ for $\frac{p+q}{g} < \frac{3}{4}$.

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This argument **does not** extend to $\mathbb{Z}$-coefficients.

**$\mathbb{Z}$-coefficients (outline).** We show that

$$H_2(\Gamma_{3,1};\mathbb{Z}) \longrightarrow H_2(\Gamma_{3,1},\Gamma_{2,1};\mathbb{Z}) \stackrel{\partial}{\longrightarrow} H_1(\Gamma_{2,1};\mathbb{Z})$$

is $\mathbb{Z}\{\lambda\} \stackrel{10}{\to} \mathbb{Z}\{\mu\} \stackrel{\partial}{\to} \mathbb{Z}/10\{\sigma \cdot \tau\} \to 0$, so the map

$$\lambda \cdot - : H_0(\Gamma_{0,1},\Gamma_{-1,0};\mathbb{Z}) = \mathbb{Z}\{1\} \longrightarrow H_2(\Gamma_{3,1},\Gamma_{2,1};\mathbb{Z}) = \mathbb{Z}\{\mu\}$$

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Instead, take $\mu \in H_{3,2}(\mathbf{R}^+/\sigma)$, use that $\mathbf{R}^+/\sigma$ is a $\mathbf{R}^+$-module to represent it by a $\mathbf{R}^+$-module map

$$\mu : S^{3,2} \otimes \mathbf{R}^+ \longrightarrow \mathbf{R}^+/\sigma,$$

check that the $\mathbf{R}^+$-module map

$$S^{3,2} \otimes S^{1,0} \otimes \mathbf{R}^+ \xrightarrow{S^{3,2}\otimes\sigma} S^{3,2} \otimes \mathbf{R}^+ \xrightarrow{\mu} \mathbf{R}^+/\sigma,$$

which is an element of $H_{4,2}(\mathbf{R}^+/\sigma) = 0$, vanishes, and hence extend $\mu$ to a map

$$\varphi : S^{3,2} \otimes \mathbf{R}^+/\sigma \longrightarrow \mathbf{R}^+/\sigma.$$

As always in obstruction theory, there is a choice of extensions

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**Strategy**: We know how to construct $\mathbf{R}^+$ as a CW-$E_2$-algebra having no $(g, d)$-cells with $d < g - 1$; this comes with a skeletal filtration, inducing a filtration on $\mathbf{R}^+/\sigma$.

We show that $\varphi$ can be given the structure of a filtered map; this is quite subtle: need to show that all choices of $\varphi$'s come from filtered maps. This gives a filtration on the cofibre $\mathbf{C}_\varphi$, so a spectral sequence going from

$$H_{*,*,*}\left((S^{0,0,0}_{\mathbb{F}_\ell} \oplus S^{3,3,3}_{\mathbb{F}_\ell}\rho_4) \otimes \overline{\mathbf{E}_2(S^{1,0,0}_{\mathbb{F}_\ell}\sigma \oplus S^{1,1,1}_{\mathbb{F}_\ell}\tau \oplus S^{2,2,2}_{\mathbb{F}_\ell}\rho_1 \oplus S^{2,2,2}_{\mathbb{F}_\ell}\rho_2 \oplus S^{3,2,2}_{\mathbb{F}_\ell}\rho_3 \oplus \bigoplus_{\alpha \in I} S^{g_\alpha, d_\alpha, d_\alpha}_{\mathbb{F}_\ell})/\sigma}\right)$$

to $H_{*,*}(\mathbf{C}_\varphi; \mathbb{F}_\ell)$ (here $\frac{d_\alpha}{g_\alpha} \geq \frac{3}{4}$). Then we use Cohen's calculations of the $\mathbb{F}_\ell$-homology of free $E_2$-algebras, and compute the effect of the $d^1$-differential: we find that $E^2_{g,p,q} = 0$ for $\frac{p+q}{g} < \frac{3}{4}$.