E_∞-algebras and general linear groups

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Premise

All based on joint work with S. Galatius and A. Kupers,
E_∞-cells and general linear groups of infinite fields

We want to study the group homology of $GL_n(A)$ for various rings A, especially the behaviour with respect to varying n.

To do so, consider the totality

$$\mathbf{R}^+ = \coprod_{n \geq 0} BGL_n(A),$$

which is a unital E_∞-algebra in the category of $\mathbb{N}$-graded spaces.

We have tried to understand cellular E_∞-algebra structures on $\mathbf{R}^+$, and in doing so have been led to many results which can be stated without reference to E_∞-algebras.

I will first explain some of these results, and later give an idea of how they all fit together.

The Steinberg module

The Steinberg module

Let $\mathbb{F}$ be a field and V be an $\mathbb{F}$-vector space.

The "Tits building" is

$T(V)$ = the nerve of the poset of proper subspaces of V,

in which a (nondegenerate) p-simplex is a flag of subspaces

$$0 < V_0 < V_1 < \cdots < V_p < V.$$

It is acted upon by $GL(V)$.

Theorem (Solomon–Tits)
$T(V)$ is homotopy equivalent to a wedge of $(\dim(V) - 2)$-spheres.

The unique homology group

$$St(V) := \widetilde{H}_{\dim(V)-2}(T(V); \mathbb{Z})$$

is the "Steinberg module". As $GL(V)$ acts on $T(V)$, it acts on $St(V)$.

Bilinear forms on the Steinberg module

As $T(V)$ is a $(\dim(V) - 2)$-dimensional simplicial complex, $St(V)$ is a submodule of $\widetilde{C}_{\dim(V)-2}(T(V); \mathbb{Z})$.

This has a basis of complete flags in V: give it a bilinear form by declaring the complete flags to be orthonormal.

This restricts to a positive-definite symmetric bilinear form

$$\langle -, - \rangle : St(V) \otimes St(V) \longrightarrow \mathbb{Z},$$

e.g. if a is an "apartment" then $\langle a, a \rangle = \dim(V)!$.

Theorem (Galatius–Kupers–R-W)
On coinvariants this induces $[St(V) \otimes St(V)]_{GL(V)} \xrightarrow{\sim} \mathbb{Z}$.

Corollary (Galatius–Kupers–R-W)
For any connected commutative ring $\mathbb{k}$*, the* $\mathbb{k}[GL(V)]$*-module* $\mathbb{k} \otimes_{\mathbb{Z}} St(V)$ *is indecomposable.*

Bilinear forms on the Steinberg module

There are natural multiplication maps $St(V) \otimes St(W) \to St(V \oplus W)$, which give

$$\mathbb{Z}\{1\} \oplus \bigoplus_{n \geq 1} [St(\mathbb{F}^n) \otimes St(\mathbb{F}^n)]_{GL_n(\mathbb{F})}$$

the structure of a commutative ring.

Theorem (Galatius–Kupers–R-W)
The isomorphisms $[St(\mathbb{F}^n) \otimes St(\mathbb{F}^n)]_{GL_n(\mathbb{F})} \xrightarrow{\sim} \mathbb{Z}$ *assemble to a ring isomorphism*

$$\mathbb{Z}\{1\} \oplus \bigoplus_{n \geq 1} [St(\mathbb{F}^n) \otimes St(\mathbb{F}^n)]_{GL_n(\mathbb{F})} \cong \Gamma_{\mathbb{Z}}[x]$$

to the divided power algebra on x i.e. $\langle 1, \frac{x^1}{1!}, \frac{x^2}{2!}, \frac{x^3}{3!}, \ldots \rangle_{\mathbb{Z}} \subset \mathbb{Q}[x]$.

The proofs of these results use a presentation of $St(\mathbb{F}^n)$ due to Lee–Szczarba, and elementary but complicated manipulations of matrices.

Rognes’ connectivity conjecture

Rognes' connectivity conjecture

Let A be a connected commutative ring for which f.g. projective modules are free. (e.g. a field or local ring)

Rognes has defined a "rank filtration"

$$* \subset F_0\mathbf{K}(A) \subset F_1\mathbf{K}(A) \subset F_2\mathbf{K}(A) \subset \cdots \subset \mathbf{K}(A)$$

of the algebraic K-theory spectrum of A, and has identified the filtration quotients

$$\frac{F_n\mathbf{K}(A)}{F_{n-1}\mathbf{K}(A)} \simeq \mathbf{D}(A^n)_{hGL_n(A)}$$

as the homotopy orbits of certain $GL_n(A)$-spectra $\mathbf{D}(A^n)$, the nth *stable building* of A.

(Idea: the Tits building is the first space in this spectrum; the kth space is made from k-dimensional flags of submodules of A^n.)

Based on calculations for $n \leq 3$, Rognes conjectured that for A local or Euclidean the spectrum $\mathbf{D}(A^n)$ is $(2n-3)$-connected.

Rogness' connectivity conjecture

We do not know how to prove Rognes' conjecture, however for applications it seems to be enough to know that the homotopy orbit spectrum $\mathbf{D}(A^n)_{hGL_n(A)}$ is $(2n-3)$-connected.

Theorem (Galatius–Kupers–R-W)

(i) *If A is a connected semi-local ring with all residue fields infinite, then $\mathbf{D}(A^n)_{hGL_n(A)}$ is $(2n-3)$-connected.*

(ii) *If A is an infinite field then in addition*

$$H_{2n-2}(\mathbf{D}(A^n)_{hGL_n(A)}) = \begin{cases} \mathbb{Z} & \text{if } n = 1, \\ \mathbb{Z}/p & \text{if } n = p^k \text{ with } p \text{ prime}, \\ 0 & \text{otherwise}. \end{cases}$$

Rognes had also conjectured that $H_{2n-2}(\mathbf{D}(A^n))_{GL_n(A)}$ is torsion for $n > 1$, which is similar to (ii).

Homological stability

What is homological stability?

Have stabilisation maps

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} : GL_{n-1}(A) \longrightarrow GL_n(A)$$

and *homological stability* hopes these are homology isomorphisms in a range of degrees going to ∞ with n.

Equivalently, it hopes that

$$H_d(GL_n(A), GL_{n-1}(A)) = 0 \text{ for all } d \leq f(n)$$

for some divergent function f.

One can ask this question for homology with $\mathbb{k}$-coefficients: the function f may then depend on $\mathbb{k}$.

Stability with $\mathbb{Z}$-coefficients is known when A has finite “stable rank”, by work of Maazen and van der Kallen: then $f(n) = \frac{n - sr(A)}{2}$ will do.
e.g. A commutative of finite Krull dimension

The Nesterenko–Suslin theorem

Sometimes one has homological stability in a range of degrees much larger than the slope $\frac{1}{2}$ range of Maazen and van der Kallen.

Nesterenko–Suslin: If A is a semi-local ring with all residue fields infinite, then

$$H_d(GL_n(A), GL_{n-1}(A); \mathbb{Z}) = 0 \text{ for } d < n,$$

and $H_n(GL_n(A), GL_{n-1}(A); \mathbb{Z}) \cong K_n^M(A)$, nth Milnor K-theory.

Recall: Milnor K-theory $K_*^M(A)$ is the graded ring generated by $K_1^M(A) = A^\times$ and subject to the relations $a \cdot b = 0 \in K_2^M(A)$ whenever $a, b \in A^\times$ satisfy $a + b = 1$.

A degree above the Nesterenko–Suslin theorem

We study these relative homology groups one degree further up (and rationally). We first show that

$$\bigoplus_{n \geq 1} H_{n+1}(GL_n(A), GL_{n-1}(A); \mathbb{Q})$$

can be made into a $K^M_*(A) \otimes \mathbb{Q}$-module, then analyse how it may be generated efficiently.

Theorem (Galatius–Kupers–R-W)
If A is a connected semi-local ring with all residue fields infinite, then there is a map of graded $\mathbb{Q}$-vector spaces

$$\mathrm{Harr}_3(K^M_*(A) \otimes \mathbb{Q}) \longrightarrow \mathbb{Q} \otimes_{K^M_*(A) \otimes \mathbb{Q}} \bigoplus_{n \geq 1} H_{n+1}(GL_n(A), GL_{n-1}(A); \mathbb{Q})$$

which is an isomorphism in gradings ≥ 5.

Here Harr = Harrison homology = André–Quillen homology.
Third Harrison homology measures "relations between relations" in a presentation of the quadratic algebra $K^M_*(A) \otimes \mathbb{Q}$.

Improved homological stability

Under further assumptions on A, our methods (which I have not yet told you) instead give improved homological stability results:

Theorem (Galatius–Kupers–R-W)

(i) *If A is a connected semi-local ring with all residue fields infinite and such that $K_2(A) \otimes \mathbb{Q} = 0$ (e.g. $\bar{\mathbb{F}}_q$, $\mathbb{F}_q(t)$, number field, $\bar{\mathbb{Q}}$) then*

$$H_d(GL_n(A), GL_{n-1}(A); \mathbb{Q}) = 0 \text{ for } d < \tfrac{4n-1}{3}.$$

(ii) *If A is a connected semi-local ring with all residue fields infinite and p is a prime number such that $A^\times \otimes \mathbb{Z}/p = 0$ then*

$$H_d(GL_n(A), GL_{n-1}(A); \mathbb{Z}/p) = 0 \text{ for } d < \tfrac{5n}{4}.$$

(iii) *If $\mathbb{F}$ is an algebraically closed field then, for all primes p,*

$$H_d(GL_n(\mathbb{F}), GL_{n-1}(\mathbb{F}); \mathbb{Z}/p) = 0 \text{ for } d < \tfrac{5n}{3}.$$

Resolving some conjectures

The last part implies that if $\mathbb{F}$ is an algebraically closed field then

$$H_{n+1}(GL_n(\mathbb{F}), GL_{n-1}(\mathbb{F}); \mathbb{Z}/p) = 0$$

for all $n > 1$ and all primes p.

This resolves a conjecture of Mirzaii on divisibility of certain "higher pre-Bloch groups" $\mathfrak{p}_n(\mathbb{F})$, and a similar conjecture of Yagunov on a different notion of "higher pre-Bloch groups" $\wp_n(\mathbb{F})$ and $\wp_n(\mathbb{F})_{cl}$.

In a different direction, we can complete an approach of Mirzaii to proving Suslin's "injectivity conjecture":

Theorem (Galatius–Kupers–R-W)
If $\mathbb{F}$ is an infinite field and $\mathbb{k}$ is a field in which $(n-1)!$ is invertible then the stabilisation map

$$H_n(GL_{n-1}(\mathbb{F}); \mathbb{k}) \longrightarrow H_n(GL_n(\mathbb{F}); \mathbb{k})$$

is injective.

Homology of Steinberg modules

Finally, and returning to the beginning, we prove a vanishing theorem for the homology of the Steinberg module.

Theorem (Galatius–Kupers–R-W)
If A is a connected semi-local ring with infinite residue fields, then

$$H_d(GL_n(A); St(A^n)) = 0$$

for $d < \frac{1}{2}(n-1)$.

Analogous results in the case of fields have been obtained by Ash–Putman–Sam and Miller–Nagpal–Patzt.

What do these things have to do with each other?

Reformulation

These results arose in our analysis of

$$\mathbf{R}^{+} = \bigsqcup_{n \geq 0} BGL_n(A)$$

as a unital E_∞-algebra in the category of $\mathbb{N}$-graded spaces.

Rognes' conjecture and our description of bilinear forms on the Steinberg module essentially corresponds to computing the "E_2-homology" of $\mathbf{R}^{+}$ in a range of degrees.

This determines the "E_∞-homology" of $\mathbf{R}^{+}$ in this range of degrees, implying that there is a cell structure on $\mathbf{R}^{+}$ in the category of E_∞-algebras with highly constrained cells.

The applications to homological stability are calculations using this constrained cell structure and the description

$$H_d(GL_n(A), GL_{n-1}(A)) = H_{n,d}(\mathbf{R}^{+}/\sigma)$$

for $\sigma \in H_0(BGL_1(A))$. I will not describe these calculations today.

Homotopy theory of E_k-algebras

Graded objects

Let C denote sSet, sSet_*, Sp, or (because we are eventually interested in taking $\Bbbk$-homology) $\mathrm{sMod}_{\Bbbk}$.

Write $\otimes$ for the cartesian, smash, or tensor product.

We will consider $\mathbb{N}$-graded objects in C, meaning $\mathrm{C}^{\mathbb{N}} := \mathrm{Fun}(\mathbb{N}, \mathrm{C})$. This is given the Day convolution monoidal structure:

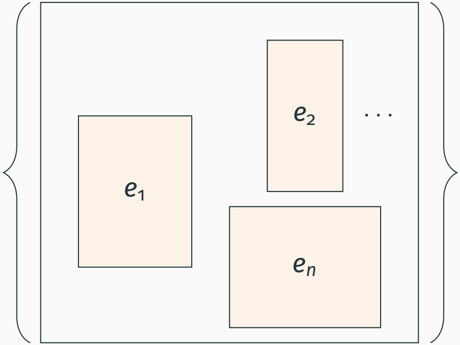
$$(X \otimes Y)(n) = \bigsqcup_{a+b=n} X(a) \otimes Y(b).$$

Define bigraded homology groups as $H_{n,d}(X; \Bbbk) := H_d(X(n); \Bbbk)$.

E_k-algebras

Let $\mathcal{C}_k$ denote the non-unital ($\mathcal{C}_k(0) = \varnothing$) little k-cubes operad.

$\mathcal{C}_2(n) =$



The categories $\mathsf{C}^{\mathbb{N}}$ are all tensored over Top: can make sense of the monad

$$E_k(X) := \bigsqcup_{n \geq 1} \mathcal{C}_k(n) \odot_{\mathfrak{S}_n} X^{\otimes n}$$

and so of E_k-algebras $\mathbf{X}$ in $\mathsf{C}^{\mathbb{N}}$. Call the category of these $\mathrm{Alg}_{E_k}(\mathsf{C}^{\mathbb{N}})$.

E_k-indecomposables

As we work with the *non-unital* little k-cubes operad, every pointed object can be considered as an E_k-algebra in a trivial way, giving an inclusion

$$inc : \mathsf{C}_*^{\mathbb{N}} \to \mathsf{Alg}_{E_k}(\mathsf{C}_*^{\mathbb{N}}).$$

This has a left adjoint $\mathbf{X} \mapsto Q^{E_k}(\mathbf{X})$, the E_k-*indecomposables* of $\mathbf{X}$.

e.g. $Q^{E_k}(\mathbf{E_k}(X)) = X$.

This functor is not homotopy invariant, so we should instead evaluate the *derived* E_k-*indecomposables*

$$Q_{\mathbb{L}}^{E_k}(\mathbf{X}) := Q^{E_k}(\text{cofibrant replacement of } \mathbf{X}),$$

a.k.a. topological Quillen homology (for the operad $\mathcal{C}_k$).

Write $H_{n,d}^{E_k}(\mathbf{X}) = H_{n,d}(Q_{\mathbb{L}}^{E_k}(\mathbf{X}))$, the "$E_k$-homology".

Computing derived E_k-indecomposables

$Q^{E_k}_{\mathbb{L}}(\mathbf{X})$ may also be computed by a k-fold bar construction.

Instances of this have been given by Getzler–Jones, Basterra–Mandell, Fresse, Francis.

Specifically, if $\mathbf{X}$ is an E_k-algebra with unitalisation $\mathbf{X}^+$, then there is an equivalence

$$\mathbb{1} \oplus \Sigma^k Q^{E_k}_{\mathbb{L}}(\mathbf{X}) \simeq B^{E_k}(\mathbf{X}^+)$$

with the k-fold bar construction.

Considering the k-fold bar construction as the bar construction of the $(k-1)$-fold bar construction gives a spectral sequence

$$E^2_{n,p,q} = Tor_p^{H_{*,*}(B^{E_{k-1}}(\mathbf{X}^+);\mathbb{k})}(\mathbb{k},\mathbb{k})_{n,q} \Rightarrow H_{n,p+q}(B^{E_k}(\mathbf{X}^+);\mathbb{k}).$$

This allows one, in principle, to calculate E_k-homology by taking iterated bar constructions.

The general linear groups

The general linear group E_∞-algebra

Let A be a connected commutative ring for which f.g. projective modules are free.

The symmetric monoidal category P_A of f.g. projective A-modules and their isomorphisms has classifying space

$$\mathbf{R}^+ = B\mathsf{P}_A \simeq \coprod_{n \geq 0} BGL_n(A)$$

and is equipped with an action of an E_∞-operad. We consider this as $\mathbb{N}$-graded via the rank functor $r : \mathsf{P}_A \to \mathbb{N}$.

Alternatively, can consider the terminal object $\mathbf{t} \in \mathsf{sSet}^{\mathsf{P}_A}$, which has a unique E_∞^+-algebra structure, cofibrantly replace it by $\mathbf{T}$ as an E_∞^+-algebra, then take the Kan extension along r:

$$\mathbf{R}^+ = r_*(\mathbf{T}) \in \mathsf{Alg}_{E_\infty^+}(\mathsf{sSet}^{\mathbb{N}}).$$

We indeed have $\mathbf{R}^+(n) \simeq \mathrm{colim}_{r/n}\, \mathbf{T} = \mathbf{T}(A^n)/GL_n(A) \simeq BGL_n(A)$.

The advantage of the second description is that many constructions commute with Kan extension: we can instead compute them for the simple object $\mathbf{T} \overset{\sim}{\to} \mathbf{t}$ (at the expense of working in the complicated category $\mathrm{sSet}^{\mathsf{P}_A}$).

In particular, $B^{E_k}(\mathbf{T})$ has the following description: evaluated at a projective module M it is the k-fold simplicial pointed set $\widetilde{D}^k(M)$ with $(p_1, p_2, \ldots, p_k)$-simplices given by

$$\frac{\{M_{i_1,i_2,\ldots,i_k} \leq M \text{ for } 1 \leq i_j \leq p_j\}}{\{\text{those for which } \bigoplus M_{i_1,i_2,\ldots,i_k} \to M \text{ is not an iso}\}},$$

face maps given by direct sum of submodules, and degeneracies given by inserting trivial modules.

Thus we have $(\Sigma^k Q^{E_k}_{\mathbb{L}}(\mathbf{R}))(n) \simeq \widetilde{D}^k(A^n)_{hGL_n(A)}$ for $n > 0$.

The k-fold Tits building

Rognes defines a k-fold analogue of the Tits building for M as the k-fold simplicial pointed set $D^k(M)$ with $(p_1, p_2, \ldots, p_k)$-simplices given by

$$\frac{\{\text{lattices } \varphi : [p_1] \times \cdots \times [p_k] \to \mathrm{Sub}(M)\}}{\{\text{non-full lattices}\}}$$

where a “lattice” is a functor to the poset of direct summands of M such that

$$\operatorname*{colim}_{[a_1 \leq b_1] \times \ldots \times [a_k \leq b_k] \setminus \{b\}} \varphi \longrightarrow \varphi(b)$$

is an isomorphism onto a direct summand, and a lattice is “full” if $\varphi(a_1, \ldots, a_k) = 0$ whenever some $a_i = 0$, and $\varphi(p_1, ..., p_k) = M$.

Rognes' stable building $\mathbf{D}(A^n)$ is the spectrum with kth space $D^k(A^n)$.

When $k = 1$ and A is a field we have $D^1(A^n) \simeq \Sigma^2 T(A^n)$, the double suspension of the Tits building. By the Solomon–Tits theorem, this is a wedge of n-spheres.

The key theorem

Theorem (Galatius–Kupers–R-W)
If A is a field then the natural map $D^2(M) \to D^1(M) \wedge D^1(M)$ is an isomorphism, so $D^2(A^n)$ is a wedge of $2n$-spheres.

Proof: A nontrivial (p, q)-simplex of $D^2(M)$ is a diagram of subspaces of M as shown, where: $V_{i,j}$ vanish if i or j is 0; $V_{p,q} = M$; and by the lattice condition the induced map

$V_{0,q}$	$\cdots$	$\cdots$	$V_{p,q}$
$\vdots$	$\vdots$		$\vdots$
$V_{0,1}$	$V_{1,1}$	$\cdots$	$\vdots$
$V_{0,0}$	$V_{1,0}$	$\cdots$	$V_{p,0}$

$$\operatorname{colim}(V_{i,j+1} \leftarrow V_{i,j} \to V_{i+1,j}) \to V_{i+1,j+1}$$

is a monomorphism. This means that $V_{i,j} = V_{i+1,j} \cap V_{i,j+1}$, so this bisimplex is uniquely determined by the pair of flags

$$0 = V_{0,q} \leq V_{1,q} \leq \cdots \leq V_{p,q} = M \qquad 0 = V_{p,0} \leq V_{p,1} \leq \cdots \leq V_{p,q} = M$$

via $V_{i,j} = V_{i,q} \cap V_{p,j}$. □

The same connectivity holds for A a connected semi-local ring with all residue fields infinite but the proof is much harder, involving the contractibility of a complex of "submodules in general position".

Rings with many units

There are maps $\widetilde{D}^k(M) \to D^k(M)$ given by sending a k-fold splitting to the associated k-fold flag. These are never isomorphisms, but we have the following:

Theorem (Galatius–Kupers–R-W)
If A is a ring with many units then the map on homotopy orbits

$$\widetilde{D}^k(M)_{hGL(M)} \longrightarrow D^k(M)_{hGL(M)}$$

is a homology equivalence.

A ring A has "many units" if for each $n \in \mathbb{N}$ there are elements a_1, a_2, $\ldots$, $a_n \in A$ all of whose partial sums are units (e.g. semi-local with infinite residue fields). This condition was discovered by Suslin and Nesterenko: it implies that the inclusions

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \longrightarrow \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

induce isomorphisms on group homology (which is what we need).

Proof of Rognes' conjecture

$D^2(A^n)$ is $(2n-1)$-connected by the Key Theorem, so $D^2(A^n)_{hGL_n(A)}$ is also $(2n-1)$-connected, and this is the same as $\widetilde{D}^2(A^n)_{hGL_n(A)}$.

Now $\{\widetilde{D}^k(A^n)_{hGL_n(A)}\}_{n\in\mathbb{N}}$ is obtained from $\{\widetilde{D}^2(A^n)_{hGL_n(A)}\}_{n\in\mathbb{N}}$ by taking a $(k-2)$-fold bar construction, so $\widetilde{D}^k(A^n)_{hGL_n(A)} \simeq D^k(A^n)_{hGL_n(A)}$ is $(2n-1+k-2)$-connected.

This is the kth space of the spectrum $\mathbf{D}(A^n)_{hGL_n(A)}$, which is therefore $(2n-3)$-connected. This proves part (i).

For part (ii), when A is a field we have

$$\widetilde{H}_{2n}(\widetilde{D}^2(A^n)_{hGL_n(A)}) = [H_{2n}(D^1(A^n)\wedge D^1(A^n))]_{GL_n(A)} = [St(A^n)\otimes St(A^n)]_{GL_n(A)}.$$

The first results I mentioned say that these coinvariants are $\mathbb{Z}$ and combine to form a divided power algebra $\Gamma_{\mathbb{Z}}[x]$. The bar spectral sequence shows that

$$\widetilde{H}_{2n+1}(\widetilde{D}^3(A^n)_{hGL_n(A)}) = Tor_1^{\Gamma_{\mathbb{Z}}[x]}(\mathbb{Z},\mathbb{Z})_n = \begin{cases} \mathbb{Z} & \text{if } n=1, \\ \mathbb{Z}/p & \text{if } n=p^k \text{ with } p \text{ prime}, \\ 0 & \text{otherwise.} \end{cases}$$

Literature

Based on work with S. Galatius and A. Kupers:

E_∞-cells and general linear groups of infinite fields.
arXiv:2005.05620.

Cellular E_k-algebras.
arXiv:1805.07184.

For further applications of these ideas see also:

E_2-cells and mapping class groups.
Publ. Math. Inst. Hautes Études Sci. 130 (2019), 1–61.

E_∞-cells and general linear groups of finite fields.
arXiv:1810.11931.