

TQFTs v. exotic spheres

Oscar Randal-Williams

University of Cambridge

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joint with Ben Gripaios

TQFTs do not detect the Milnor sphere

Theorem (Gripaios–R–W). Let Σ_{Milnor}^7 denote Milnor's exotic 7-sphere and $F : \text{Bord}_7^{\text{SO}} \rightarrow \text{Vect}_{\mathbb{k}}$ be a TQFT. Then

$$F(\Sigma_{\text{Milnor}}^7) = F(S^7) \in \mathbb{k}.$$

Recall that the **cobordism category** $\text{Bord}_d^{\text{SO}}$ has:

- *objects*: closed oriented smooth $(d - 1)$ -manifolds;
- *morphisms* $P \rightsquigarrow Q$: oriented smooth d -dimensional bordisms, up to diffeomorphism rel ∂ , composed by gluing;
- *symmetric monoidal structure*: $\sqcup$, with unit $\emptyset$.

A **TQFT** is a symmetric monoidal functor $F : \text{Bord}_d^{\text{SO}} \rightarrow \text{Vect}_{\mathbb{k}}$.

A closed oriented d -manifold M is a bordism $\emptyset \rightsquigarrow \emptyset$, so

$$F(M) \in \text{Hom}_{\mathbb{k}}(\mathbb{k}, \mathbb{k}) = \mathbb{k}.$$

TQFTs v. exotic smooth structures

Earlier work:

Reutter '23. A *semisimple* TQFT $\text{Bord}_4^{\text{SO}} \rightarrow \text{Vect}_{\mathbb{K}}$ cannot distinguish homeomorphic 4-manifolds.

Reutter–Schommer–Pries '25. Semisimple TQFTs can detect *some* exotic spheres (“Hitchin spheres”, those detected by the index of their Dirac operator).

Reutter–Schommer–Pries Question 1.3. Do semisimple TQFTs detect all exotic spheres in dimensions > 4 ?

No. Milnor’s Σ^7 is a counterexample, even omitting semisimplicity.

Reminder on homotopy spheres

$$\Theta_d := \{\text{oriented homotopy } d\text{-spheres } \Sigma^d\} / \cong .$$

Kervaire–Milnor '63. For $d \neq 4$ the group Θ_d is finite, and there is an exact sequence

$$0 \longrightarrow \mathrm{bP}_{d+1} \longrightarrow \Theta_d \longrightarrow \pi_d(\mathbb{S})/\mathrm{im}(J),$$

where bP_{d+1} = those Σ bounding a **parallelisable** $(d+1)$ -manifold.

bP_{4k} is cyclic of order related to numerators of Bernoulli numbers:

$$\mathrm{bP}_8 = \mathbb{Z}/28 \quad \mathrm{bP}_{12} = \mathbb{Z}/992 \quad \mathrm{bP}_{16} = \mathbb{Z}/8128 \quad \mathrm{bP}_{20} = \mathbb{Z}/261632 \cdots$$

Generated by the boundary of the E_8 -plumbing of copies of TS^{2k} , called the *Milnor sphere* $\Sigma_{\mathrm{Milnor}}^{4k-1}$.

Dimension 7. Here $\pi_7(\mathbb{S}) = \mathrm{im}(J)$, so $\Theta_7 = \mathrm{bP}_8 = \mathbb{Z}/28\{\Sigma_{\mathrm{Milnor}}^7\}$.

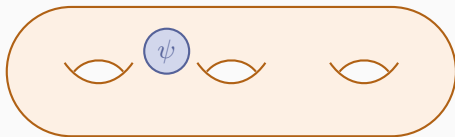
Clutching and implanting

Smale: For $d \geq 6$ every homotopy d -sphere arises as $D^d \cup_{\varphi} D^d$ for a $\varphi \in \text{Diff}^+(S^{d-1})$ [**Cerf:** unique up to isotopy].

Such a φ may be represented uniquely up to isotopy as $\psi' := \psi \cup \text{id}$ for a diffeomorphism $\psi \in \text{Diff}_{\partial}(D^{d-1})$.

Thus $\Sigma_{\psi} := D^d \cup_{\psi'} D^d$ gives an isomorphism $\pi_0 \text{Diff}_{\partial}(D^{d-1}) \cong \Theta_d$.

Implanting. As ψ is the identity near ∂ , it extends by the identity along any $D^{d-1} \subset N^{d-1}$: we have $\psi \mapsto \psi'' : \Theta_d \rightarrow \pi_0 \text{Diff}^+(N)$.



$$\psi'' = \psi \cup \text{id}$$

N

The proof

Strategy

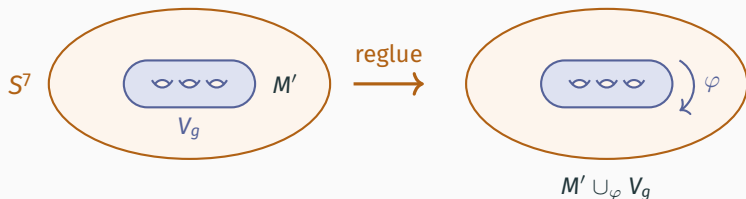
Heegaard splitting. $V_g := \natural^g S^3 \times D^4 \subset S^7$ has complement

$$M' := S^7 \setminus \text{int}(V_g) = \natural^g D^4 \times S^3 \cong V_g$$

glued along

$$W_g := \partial V_g = \#^g S^3 \times S^3.$$

Regluing by $\varphi \in \text{Diff}^+(W_g)$ gives a new manifold $M' \cup_{\varphi} V_g$



on which a TQFT F evaluates as

$$F(M' \cup_{\varphi} V_g) : \mathbb{k} \xrightarrow{F(V_g)} F(W_g) \xrightarrow{F(\varphi)} F(W_g) \xrightarrow{F(M')} \mathbb{k}.$$

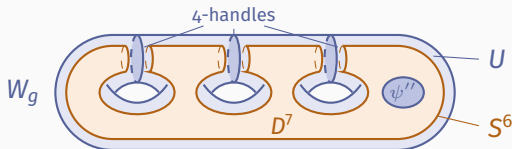
So want φ with (a) $M' \cup_{\varphi} V_g \cong \Sigma_{\text{Milnor}}^7$, and (b) $F(\varphi) = F(\text{Id}) = \text{Id}$.

Step (a)

For $\psi \in \text{Diff}_{\partial}(D^6)$ with $\psi'' \in \text{Diff}^+(W_g)$ its implantation, have:

Lemma. $M' \cup_{\psi''} V_g \cong \Sigma_{\psi}$.

Proof. Attach 4-handles along the g disjoint copies of $D^3 \times S^3$ in $W_g \setminus \text{int}(D^6)$ to get a cobordism $U : W_g \rightsquigarrow S^6$ with $V_g = U \cup_{\text{id}} D^7$.



Since ψ'' is supported in D^6 , it extends (by the identity) over U , restricting to ψ' on the outgoing S^6 . Thus

$$\begin{aligned} M' \cup_{\psi''} V_g &= M' \cup_{\psi''} (U \cup_{\text{id}} D^7) \\ &\cong (M' \cup_{\text{id}} U) \cup_{\psi'} D^7 = D^7 \cup_{\psi'} D^7 = \Sigma_{\psi}. \end{aligned}$$

□

Step (b)

We must show that the implanted diffeomorphism acts trivially:

$$F(\psi'') = \text{Id} \in \text{GL}(F(W_g)).$$

F is a TQFT so $F(W_g) \in \text{Vect}_{\mathbb{k}}$ is dualisable and hence finite dimensional, so have a finite dimensional representation

$$F : \pi_0 \text{Diff}^+(W_g) \longrightarrow \text{GL}(F(W_g)).$$

Strategy:

- (i) $\pi_0 \text{Diff}^+(W_g)$ is *finitely generated*,
- (ii) $[\psi'']$ lies in the *finite residual* of $\pi_0 \text{Diff}^+(W_g)$,
- (iii) every finitely generated subgroup of $\text{GL}(F(W_g))$ is *residually finite* (Mal'cev).

Background: residual finiteness

Definition. The **finite residual** of a group G is $\text{fr}(G) := \bigcap_{[G:H] < \infty} H$.

G is **residually finite** if $\text{fr}(G)$ is trivial, i.e. if every non-trivial element remains non-trivial in some finite quotient group.

Examples: finite groups; $\mathbb{Z}$; f.g. abelian, free, polycyclic and f.g. nilpotent groups; profinite groups; and subgroups and products of such.

Non-examples: infinite simple groups; $\mathbb{Q}$.

Naturality. For $\alpha : G \rightarrow Q$ have $\alpha(\text{fr}(G)) \subseteq \text{fr}(Q)$. In particular

$$Q \text{ residually finite} \implies \text{fr}(G) \subseteq \ker(\alpha).$$

G is **locally residually finite** if every finitely generated subgroup is residually finite (e.g. $\mathbb{Q}$)

Mal'cev '40: $\text{GL}_n(\mathbb{k})$ is locally residually finite.

The mapping class group of $W_g = \#^g S^n \times S^n$

Kreck '79: For $n \geq 3$ there are group extensions

$$1 \longrightarrow \Theta_{2n+1} \longrightarrow \pi_0 \text{Diff}^+(W_g) \longrightarrow \pi_0 \text{Diff}^+(W_g) / \Theta_{2n+1} \longrightarrow 1$$

and

$$1 \longrightarrow H_n(W_g; S\pi_n SO(n)) \longrightarrow \pi_0 \text{Diff}^+(W_g) / \Theta_{2n+1} \longrightarrow G_g \longrightarrow 1$$

where

- (i) $S\pi_n SO(n) := \text{im}(\pi_n SO(n) \rightarrow \pi_n SO)$,
- (ii) G_g consists of the automorphisms of the intersection form on $H_n(W_g; \mathbb{Z}) = \mathbb{Z}^{2g}$ preserving the quadratic refinement $q : H_n(W_g; \mathbb{Z}) \rightarrow \pi_{n-1} SO(n)$ which records normal bundles of embedded spheres.

$\Rightarrow \pi_0 \text{Diff}^+(W_g)$ is finitely generated (cf. Sullivan).

These extensions have been studied in some detail, and are more interesting than one might first guess.

The mapping class group of $W_g = \#^g S^n \times S^n$

Krannich–R-W ’20. $\pi_0 \text{Diff}^+(W_g)$ need not be residually finite.

In particular, it need not be an arithmetic group (cf. Sullivan).

The precise form we will use is:

Krannich–Kupers–Mezher ’25, systematising K–R–W ’20: Let $[\Sigma_\psi] \in \text{bP}_{4k}$ and $g \geq 2$. Then $[\psi''] \in \pi_0 \text{Diff}^+(W_g^{4k-2})$ lies in the finite residual of this group.

While it is not necessary for the argument, let me advertise the following recent theorem of the same spirit:

Mezher ’24: Let M be a smooth closed 2-connected manifold of dimension $d \geq 6$. Then

- (i) The finite residual of $\pi_0 \text{Diff}(M)$ is contained in the image of the implantation map $\Theta_{d+1} \rightarrow \pi_0 \text{Diff}(M)$.
- (ii) $\pi_0 \text{Homeo}(M)$ is an arithmetic group (so is residually finite).

Deligne '78: For $g \geq 2$ the central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{Sp}}_{2g}(\mathbb{Z}) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1$$

pulled back from the universal cover of $\mathrm{Sp}_{2g}(\mathbb{R})$ is not residually finite; its finite residual is $2\mathbb{Z}$.

After passing from $\mathrm{Sp}_{2g}(\mathbb{Z})$ to a certain finite-index subgroup $\mathrm{Sp}_{2g}^q(\mathbb{Z})$, Deligne's extension can be (divided by 2 and) mapped to

$$1 \longrightarrow \Theta_{4k-1} \longrightarrow \pi_0 \mathrm{Diff}^+(W_g^{4k-2}) \longrightarrow \pi_0 \mathrm{Diff}^+(W_g^{4k-2}) / \Theta_{4k-1} \longrightarrow 1$$

by a map sending $1 \in \mathbb{Z}$ to $\Sigma_{\mathrm{Milnor}}^{4k-1} \in \mathrm{bP}_{4k} \leq \Theta_{4k-1}$.

$$\Rightarrow \mathrm{bP}_{4k} \leq \mathrm{fr}(\pi_0 \mathrm{Diff}^+(W_g^{4k-2})).$$

This is basically a calculation.

Putting it together

Consider the representation induced by the TQFT,

$$F : \pi_0 \text{Diff}^+(W_g) \longrightarrow \text{GL}(F(W_g)).$$

1. $\pi_0 \text{Diff}^+(W_g)$ is finitely generated, so $\text{im}(F)$ is finitely generated.
2. $F(W_g)$ is dualisable, hence finite-dimensional, so $\text{im}(F)$ is a finitely generated *linear* group; by Mal'cev it is residually finite.
3. $[\psi'']$ lies in $\text{fr}(\pi_0 \text{Diff}^+(W_g))$, so

$$F([\psi'']) \in \text{fr}(\text{im}(F)) = 1, \quad \text{i.e.} \quad F(\psi'') = \text{Id}.$$

This is (b).

Combined with (a), namely $M' \cup_{\psi''} V_g \cong \Sigma_{\text{Milnor}}^7$, we have

$$F(\Sigma_{\text{Milnor}}^7) : \mathbb{k} \xrightarrow{F(V_g)} F(W_g) \xrightarrow{\text{Id}} F(W_g) \xrightarrow{F(M')} \mathbb{k},$$

which is $F(S^7)$. □

Extensions

Extensions I: dimensions, target categories

The argument is more general than the special case I presented.

(I) It applies to any $[\Sigma] \in \mathrm{bP}_{4k} \leq \Theta_{4k-1}$ for any $k \geq 2$, and the argument generalises to show that for a $(4k-1)$ -dimensional cobordism $M : P \rightsquigarrow Q$ we have

$$F(M \# \Sigma) = F(M) : F(P) \longrightarrow F(Q).$$

(II) The only property of the target category $\mathrm{Vect}_{\mathbb{k}}$ it used was: automorphism groups of dualisable objects are locally residually finite.

The following categories have this property:

- (i) modules over any commutative ring,
- (ii) graded modules over a graded commutative ring (in particular super vector spaces),
- (iii) the derived category of vector spaces over a field,
- (iv) quasicoherent sheaves on a scheme.

Extensions II: structures, extendedness

(III) The source can be replaced by $\text{Bord}_{4k-1}^\theta$ for any tangential structure θ , with the same statement.

For this one needs an analysis of the (stably-)framed mapping class group of W_g , which is a significant part of the paper.

(IV) The result concerns TQFTs in the least fancy sense. If one has a TQFT in a more fancy sense, for example extended downwards (to be a d -category) and upwards (to be an (∞, d) -category), then it in particular gives a plain TQFT and so the result applies to it (assuming the target category satisfies our hypothesis).

On the other hand, the TQFT $\text{Id} : \text{Bord}_7^{\text{SO}} \rightarrow \text{Bord}_7^{\text{SO}}$ does detect the Milnor sphere.

Question: Is there an “algebraic” C and a TQFT $F : \text{Bord}_7^{\text{SO}} \rightarrow C$ which detects the Milnor sphere?

Thank you!

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Stably-framed mapping class groups

Let $\Theta_{4k-1}^{\text{sfr}}$ denote the abelian group of homotopy $(4k-1)$ -spheres equipped with a stable framing.

The Kervaire–Milnor sequence with stable framings is

$$1 \longrightarrow \mathbb{Z} = \text{bP}_{4k}^{\text{sfr}} \longrightarrow \Theta_{4k-1}^{\text{sfr}} \longrightarrow \pi_{4k-1}(\mathbb{S}) \longrightarrow 1$$

and (for a certain stable framing ℓ_W on W_g) there is an extension

$$1 \longrightarrow \Theta_{4k-1}^{\text{sfr}} \longrightarrow \pi_0(\text{Diff}^{\text{sfr}}(W_g^{4k-2}, D^{4k-2}; \ell_W)) \longrightarrow \text{Sp}_{2g}^q(\mathbb{Z}) \longrightarrow 1$$

and even an extension

$$1 \rightarrow \text{bP}_{4k}^{\text{sfr}} \rightarrow \pi_0(\text{Diff}^{\text{sfr}}(W_g^{4k-2}, D^{4k-2}; \ell_W)) \rightarrow \pi_{4k-1}(\mathbb{S}) \times \text{Sp}_{2g}^q(\mathbb{Z}) \rightarrow 1$$

Theorem. The finite residual of $\pi_0(\text{Diff}^{\text{sfr}}(W_g^{4k-2}, D^{4k-2}; \ell_W))$ is $\text{bP}_{4k}^{\text{sfr}}$.

Framed mapping class groups

The framed mapping class group relates to the stably framed one:

$$\pi_0(\text{Diff}^{\text{fr}}(W_g^{4k-2}, D^{4k-2}; \ell_W)) \longrightarrow \pi_0(\text{Diff}^{\text{sfr}}(W_g^{4k-2}, D^{4k-2}; \ell_W))$$

has finite kernel and image of finite index.

Some elementary group theory shows this implies that the induced map

$$\text{fr}(\pi_0(\text{Diff}^{\text{fr}}(W_g^{4k-2}, D^{4k-2}; \ell_W))) \longrightarrow \text{fr}(\pi_0(\text{Diff}^{\text{sfr}}(W_g^{4k-2}, D^{4k-2}; \ell_W))) = \text{bP}_{4k}^{\text{sfr}}$$

is surjective.

Any $[\Sigma_\psi] \in \text{bP}_{4k}$ can be lifted to $\text{bP}_{4k}^{\text{sfr}}$ (choose a bounding stable framing) and can therefore be lifted to a

$$[\psi''] \in \text{fr}(\pi_0(\text{Diff}^{\text{fr}}(W_g^{4k-2}, D^{4k-2}; \ell_W))).$$

As “framings” are initial among tangential structures, this can be used to prove the Theorem for any tangential structure θ .

Thank you again!

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