Homeomorphisms of Euclidean space

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Automorphisms of Euclidean space

Homeomorphisms

$$\begin{aligned} Top(d) &= Homeo(\mathbb{R}^d) \\ &= \left\{ f : \mathbb{R}^d \to \mathbb{R}^d \,\middle|\, f \text{ is a homeomorphism} \right\} \end{aligned}$$

(compact-open topology, i.e. uniform convergence on compact sets)

Compactly-supported diffeomorphisms

$$Diff_c(\mathbb{R}^d) = \left\{ f : \mathbb{R}^d \to \mathbb{R}^d \,\middle|\, \begin{array}{c} f \text{ is a diffeomorphism which agrees} \\ \text{with the identity outside a compact set} \end{array} \right\}$$

(Whitney C^∞-topology)

Cousins $Homeo_c(\mathbb{R}^d) \simeq *$ $Diff(\mathbb{R}^d) \simeq GL_d(\mathbb{R}) \simeq O(d)$

Morlet '69. $\frac{Homeo_c(\mathbb{R}^d)}{Diff_c(\mathbb{R}^d)} \simeq \Omega^d_0 \frac{Homeo(\mathbb{R}^d)}{Diff(\mathbb{R}^d)}$ (for $d \neq 4$)

$$\Rightarrow Diff_c(\mathbb{R}^d) \simeq \Omega^{d+1} \frac{Top(d)}{O(d)}$$

Automorphisms of Euclidean space

$$\mathit{Diff}(S^d) \simeq O(d+1) \times \mathit{Diff}_c(\mathbb{R}^d)$$

$\Rightarrow \mathit{Diff}_c(\mathbb{R}^d)$ measures how the full group of smooth symmetries of the d-sphere differs from the obvious linear symmetries.

That $\mathit{Diff}_c(\mathbb{R}^d)$ can fail to be contractible, or that $O(d) \to \mathit{Top}(d)$ can fail to be a homotopy equivalence, is outside of our intuition:

$$\mathit{Diff}_c(\mathbb{R}^1) \simeq *$$
$$\mathit{Diff}_c(\mathbb{R}^2) \simeq * \quad \text{[Smale '59]}$$
$$\mathit{Diff}_c(\mathbb{R}^3) \simeq * \quad \text{[Hatcher '83, Bamler–Kleiner '19]}$$

Watanabe '09, '18. $\mathit{Diff}_c(\mathbb{R}^d) \not\simeq *$ for all $d \geq 4$.

What is it then?

Goal

I wish to

(i) explain what is currently known about this question, then

(ii) formulate a proposal for something like a full answer, in homotopy-theoretic terms.

It will be important to formulate the proposal using $Top(d)$ rather than $Diff_c(\mathbb{R}^d)$, so I will state results in terms of $Top(d)$.

I will in fact state results in terms of the rational homotopy groups of the clasifying space $BTop(d)$:

$$\pi_i(BTop(d))_{\mathbb{Q}} = \pi_{i-1}(Top(d)) \otimes \mathbb{Q}.$$

Warm-up: $O(d)$

$$\frac{O(d+1)}{O(d)} = S^d$$

$\Rightarrow$ exact sequence of homotopy groups

$$\to \pi_i(S^d) \to \pi_i(BO(d)) \to \pi_i(BO(d+1)) \to \pi_{i-1}(S^d) \to \pi_{i-1}(BO(d)) \to$$

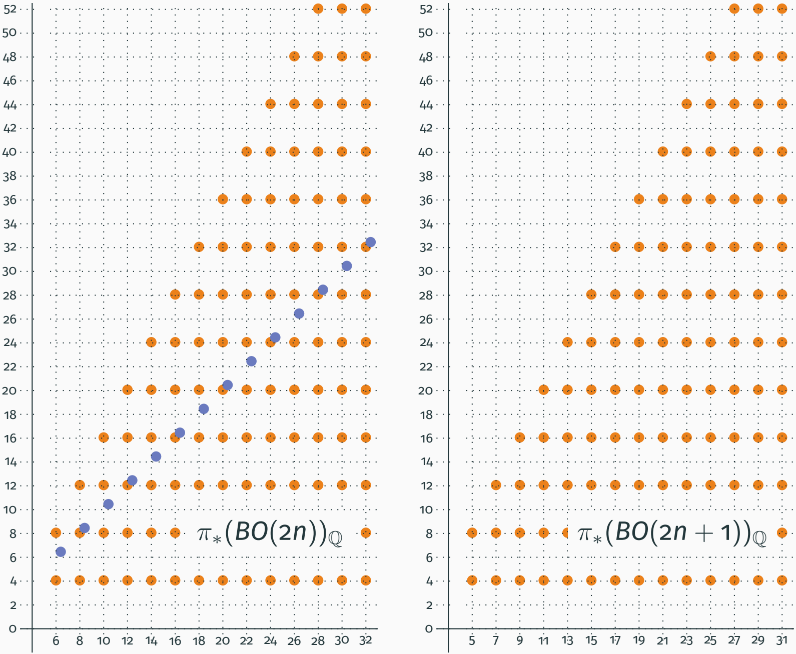
(1) It is a fundamental (and easy) result in algebraic topology that $\pi_i(S^d) = 0$ for $i < d$.

$$\Rightarrow \pi_i(BO(d)) = \pi_i(BO(d+1)) \text{ for } i < d.$$

(2) The calculation of $\pi_*(S^d)_{\mathbb{Q}}$ is also fundamental (but less easy).

$$\Rightarrow \pi_*(BO(d))_{\mathbb{Q}} = \bigoplus_{i=1}^{\lfloor (d-1)/2 \rfloor} \mathbb{Q}[4i] \oplus \begin{cases} \mathbb{Q}[d] & d \text{ even} \\ 0 & d \text{ odd.} \end{cases}$$

$\mathbb{Q}[4i]$ = "topological Pontrjagin classes" $\qquad$ $\mathbb{Q}[d]$ = "Euler class"



$\pi_*(BO(2n))_{\mathbb{Q}}$
$\pi_*(BO(2n+1))_{\mathbb{Q}}$
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6 8 10 12 14 16 18 20 22 24 26 28 30 32
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Three short stories

Classical approach to diffeomorphism groups is via

homotopy theory + surgery theory + pseudoistopy theory

Intrinsic limiting factor, the “pseudoisotopy stable range”:

Igusa ’88. For d-manifolds this is at least $\min(\frac{d-7}{2}, \frac{d-4}{3}) \sim \frac{d}{3}$.

Applied to $Diff_c(\mathbb{R}^d)$, and translated via Morlet:

Farrell–Hsiang ’78. We have

$$\pi_*(BTop(d))_\mathbb{Q} = \bigoplus_{i\geq 1} \mathbb{Q}[4i] \oplus \begin{cases} \mathbb{Q}[d] & d \text{ even} \\ \mathbb{Q}[d+5] \oplus \mathbb{Q}[d+9] \oplus \cdots & d \text{ odd} \end{cases}$$

in degrees $* \lesssim \frac{4}{3}d$.

$$\mathbb{Q}[4i] = \text{“topological Pontrjagin classes”}$$

$$\mathbb{Q}[d] = K_0(\mathbb{Z})_\mathbb{Q} = \text{“Euler class”}$$

$$\mathbb{Q}[d+4j+1] = K_{4j+1}(\mathbb{Z})_\mathbb{Q}$$

Configuration space integrals: Kontsevich, Watanabe

Kontsevich '92.

"Configuration space integrals" $\rightsquigarrow$ classes in $H^*(BDiff_c^{fr}(\mathbb{R}^d);\mathbb{Q})$

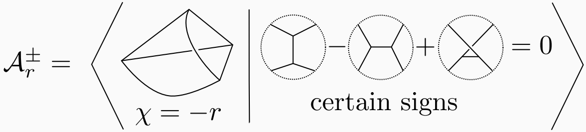
Organised in term of graphs: "graph complex".

Watanabe analysed this construction in detail in the extremal degrees corresponding to trivalent graphs. Translated via Morlet:

Watanabe '09, '18. For $d \geq 5$ there is a surjection

$$\pi_{(r+1)\cdot(d-3)+4}(BTop(d))_{\mathbb{Q}} \longrightarrow \mathcal{A}_r^{(-1)^d}$$

onto a certain vector space of trivalent graphs



onto a certain vector space of dimension

Topological Pontrjagin classes: Weiss

$$\pi_*(BO(d))_{\mathbb{Q}} = \bigoplus_{i=1}^{\lfloor (d-1)/2 \rfloor} \mathbb{Q}[4i] \oplus \begin{cases} \mathbb{Q}[d] & d \text{ even} \\ 0 & d \text{ odd} \end{cases}$$

$$\pi_*(BTop(d))_{\mathbb{Q}} = \bigoplus_{i \geq 1} \mathbb{Q}[4i] \oplus \begin{cases} \mathbb{Q}[d] & d \text{ even} \\ \mathbb{Q}[d+5] \oplus \mathbb{Q}[d+9] \oplus \cdots & d \text{ odd} \end{cases}, * \lesssim \tfrac{4}{3}d$$

Weiss '22. For all $d \gg 0$ and certain $i > \lfloor (d-1)/2 \rfloor$ there are

$$0 \neq w_{d,i} \in \pi_{4i}(BTop(d))_{\mathbb{Q}},$$

"belonging" to the summand $\mathbb{Q}[4i]$.

(This is **far** outside of the Igusa stable range $* \lesssim \frac{4}{3}d$.)

A pattern

A pattern: *Top*(2*n*)

Inspired by ideas in Weiss' argument, Alexander Kupers and I began a programme to determine

$$\pi_*(BTop(2n))_{\mathbb{Q}} \text{ or equivalently } \pi_*(BDiff_c(\mathbb{R}^{2n}))_{\mathbb{Q}}$$

as completely as possible.

Theorem (Kupers–R-W '20). For $d = 2n \geq 6$ we have

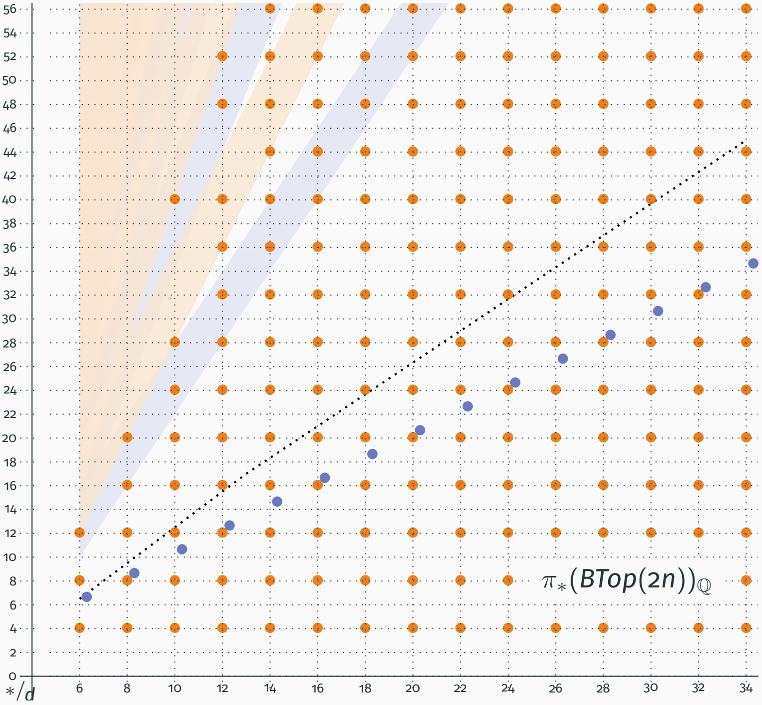
$$\pi_*(BTop(2n))_{\mathbb{Q}} = \bigoplus_{i \geq 1} \mathbb{Q}[4i] \oplus \mathbb{Q}[2n]$$

modulo classes in the bands of degrees

$$\bigcup_{s \geq 3} [2s(n-2)+4, 2s(n-1)+4].$$

These bands have slopes $3d, 4d, 5d, 6d, \ldots$

Watanabe's classes lie roughly along the middle of these bands.



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$\pi_*(BTop(2n))_{\mathbb{Q}}$

A pattern: $Top(2n+1)$

Using different techniques, Manuel Krannich and I investigated

$$\pi_*(BTop(2n+1))_{\mathbb{Q}} \text{ or equivalently } \pi_*(BDiff_c(\mathbb{R}^{2n+1}))_{\mathbb{Q}}$$

outside of the Igusa stable range.

Krannich–R-W '21. For $d = 2n+1 \geq 5$ we have

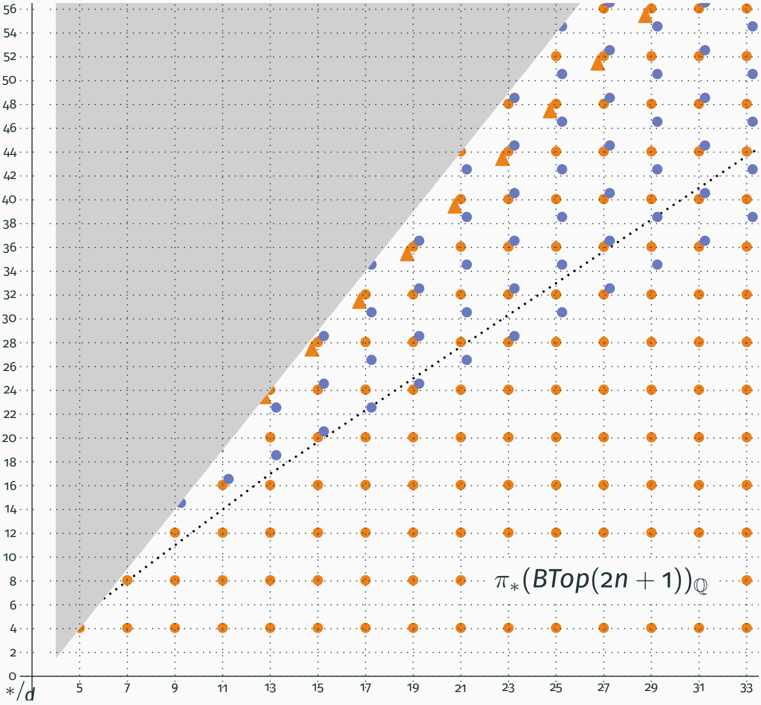
$$\pi_*(BTop(2n+1))_{\mathbb{Q}} = \bigoplus_{i \geq 1} \mathbb{Q}[4i] \oplus \bigoplus_{j \geq 1} \mathbb{Q}[d+4j+1] \oplus \mathbb{Q}[2d-2]$$

in degrees $* \leq 5n - 6 \sim \frac{5}{2}d$.

Results here are less complete, and obtained by a different method.

$$\mathbb{Q}[4i] = \text{“topological Pontrjagin classes”}$$
$$\mathbb{Q}[d+4j+1] = K_{4j+1}(\mathbb{Z})_{\mathbb{Q}}$$

$\mathbb{Q}[2d-2]$ = Watanabe's class corresponding to 



$\pi_*(BTop(2n+1))_{\mathbb{Q}}$
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A proposal

Proposal

The "band" picture suggests

$\pi_*(BTop(d))_{\mathbb{Q}}$ is a superposition of
phenomena happening on different "wavelengths"

The kinds of phenomena that occur depend only on the parity of d, but the rth phenomenon contributes to degrees around $r \cdot d$.

There is a mechanism from homotopy theory that could explain this:

Weiss' Orthogonal Calculus

This involves considering all $BTop(\mathbb{R}^d)$ at once, as the functor

$$\begin{aligned} \mathrm{Bt} : \left\{ \begin{matrix} \text{category of finite-dimensional} \\ \text{inner product spaces} \end{matrix} \right\} &\longrightarrow \left\{ \begin{matrix} \text{category of based} \\ \text{topological spaces} \end{matrix} \right\} \\ V &\longmapsto BTop(V). \end{aligned}$$

Orthogonal calculus

Orthogonal calculus considers continuous functors

$$\mathsf{F} : \left\{ \begin{matrix} \text{category of finite-dimensional} \\ \text{inner product spaces} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{category of based} \\ \text{topological spaces} \end{matrix} \right\}$$

as though they were functions, and develops a notion of Taylor expansions for them.

Notion of derivative $\mathsf{F}^{(1)}(V) := \text{fibre}(\mathsf{F}(V) \to \mathsf{F}(V \oplus \mathbb{R}))$ of such a functor, and hence of being polynomial of degree $\leq r$.

Any functor F has a best approximation $\mathsf{F} \to T_r\mathsf{F}$ by a polynomial functor of degree $\leq r$, assembling to a "Taylor tower".

$$\begin{array}{ccc} & & \vdots \\ & & \downarrow \\ & \nearrow & T_2\mathsf{F} \\ & & \downarrow \\ & \nearrow & T_1\mathsf{F} \\ & & \downarrow \\ \mathsf{F} & \longrightarrow & T_0\mathsf{F} \end{array}$$

Homogeneous polynomials $\text{fibre}(T_r\mathsf{F} \to T_{r-1}\mathsf{F})$ have a very particular structure: they are

$$V \longmapsto \Omega^\infty(\Theta\mathsf{F}^{(r)} \wedge_{O(r)} (\mathbb{R}^r \otimes V)^+)$$

for an $O(r)$-spectrum $\Theta F^{(r)}$, the rth *derivative*.

Orthogonal calculus for $V \mapsto \mathrm{Bt}(V) = BTop(V)$

(0) $T_0\mathrm{Bt}(V) \simeq BTop$, with $\pi_*(BTop)_{\mathbb{Q}} = \bigoplus_{i \geq 1} \mathbb{Q}[4i]$.

(1) **Waldhausen '81.**

$$\Theta\mathrm{Bt}^{(1)} \simeq K(\mathbb{S}),$$

the algebraic K-theory of the sphere spectrum; have $K(\mathbb{S})_{\mathbb{Q}} \xrightarrow{\sim} K(\mathbb{Z})_{\mathbb{Q}}$.

(2) **Krannich–R-W '21.**

$$\Theta\mathrm{Bt}^{(2)} \simeq_{\mathbb{Q}} map(S^1_+, \mathbb{S}^{-1}).$$

Simpler than the zeroth and first derivatives: finitely many nonzero rational homotopy groups.

The "band" pattern in $\pi_*(BTop(2n))_{\mathbb{Q}}$ suggests this is the case for all the higher derivatives too.

How to describe them?

Higher derivatives

The connection to Kontsevich's configuration space integrals suggests that one might study $Top(d)$ by its action on the spaces of finite configurations of distinct points in $\mathbb{R}^d$.

Not individually: should also remember how configurations can degenerate by points colliding. This can be packaged into the little d-discs operad E_d:

$$BTop(d) \longrightarrow BAut(E_d)$$

Fresse–Turchin–Willwacher '17 have determined $\pi_*(BAut(E_d^{\mathbb{Q}}))$ in terms of Kontsevich's graph cohomology.

Assuming we can form an orthogonal functor $\mathrm{Ba}(V) := BAut(E_V^{\mathbb{Q}})$:

$\pi_*(\Theta(-)^{(i)})_{\mathbb{Q}}$	Bt	Ba
$i = 0$	$\bigoplus_{i \geq 1} \mathbb{Q}[4i]$	0
$i = 1$	$K_*(\mathbb{Z})_{\mathbb{Q}}$	$HC_*^-(\mathbb{Z})$
$i > 1$	supported in finitely-many degrees	supported in finitely-many degrees, described by graph cohomology

The proposal

It seems plausible that the higher derivatives of Bt and Ba are rationally equivalent.

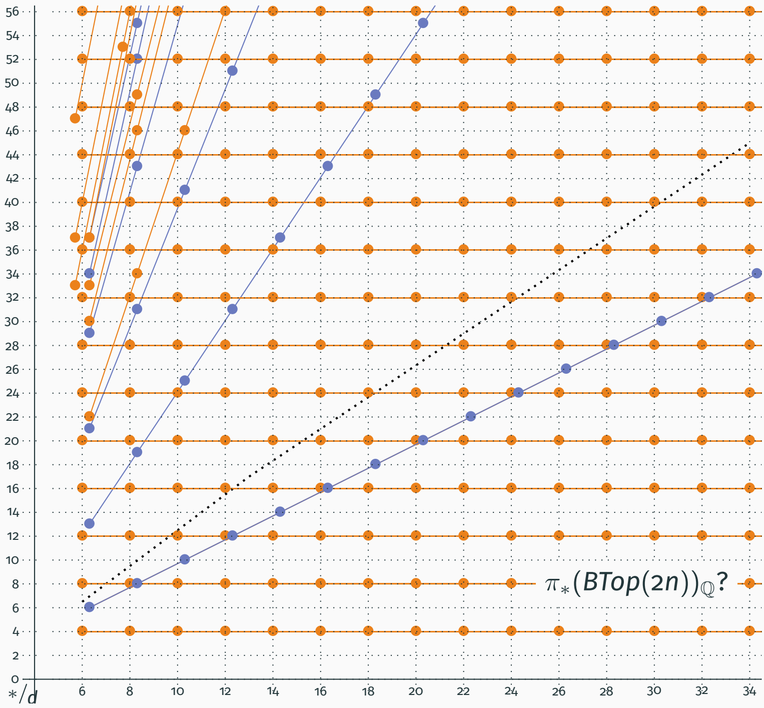
Proposal. The only difference between Bt and Bt is the zeroth and first derivatives, i.e. the square

$$\begin{array}{ccccc} BTop(d) & = & \mathrm{Bt}(\mathbb{R}^d) & \longrightarrow & T_1\mathrm{Bt}(\mathbb{R}^d) \\ & & \downarrow & & \downarrow \\ BAut(E_d^{\mathbb{Q}}) & = & \mathrm{Ba}(\mathbb{R}^d) & \longrightarrow & T_1\mathrm{Ba}(\mathbb{R}^d). \end{array}$$

is rationally a pullback for all large enough d (say $d \geq 5$).

Krannich–R-W '21. We indeed have $\Theta\mathrm{Bt}^{(2)} \xrightarrow{\sim} \Theta\mathrm{Ba}^{(2)}$.

This would be a remarkable relationship between homeomorphisms of Euclidean space, algebraic K- and L-theory, cyclic homology, and graph cohomology.



π_*(BTop(2n))_ℚ?
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