CORRIGENDUM TO: THE POSITIVE SCALAR CURVATURE COBORDISM CATEGORY

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We wish to explain and fill a gap in our paper [ERW22]. Theorem C of that paper identifies the homotopy fibre of the map $B\mathcal{P}_\theta^{2,1,\mathrm{psc}} \to B\mathcal{C}_\theta^{2,1}$ induced on classifying spaces from a certain cobordism category with positive scalar curvature metrics to the ordinary cobordism category, and a variant where the decoration psc is replaced by the decoration rst (for precise definitions, we refer to the published paper).

Here $\theta : B \to B\mathrm{O}(d)$ is a tangential structure, and Theorem C is stated without any further hypothesis concerning θ. However, the proof of Theorem C invokes Theorem 2.3.8 (via Lemma 4.2.2), and this result includes the hypothesis that θ be *once-stable* (see Definition 2.3.7 or [GRW14, Definition 5.4]). Therefore, the proof of Theorem C in the published paper is only complete for once-stable θ.

To fix this gap we explain how to deduce the following sharper version of Theorem 2.3.8, valid for all tangential structures, from Theorem 2.3.8.

Theorem 2.3.8′. *Let $\theta : B \to B\mathrm{O}(d)$ and let $d \geq 6$. Let $(W, \ell_W) : (M_0, \ell_{M_0}) \rightsquigarrow (M_1, \ell_{M_1})$ be a d-dimensional θ-cobordism. Assume that the inclusion $i_1 : M_1 \to W$ and the structure maps $\ell_{M_i} : M_i \to B$ are 2-connected. Then the following statements hold.*

(i) A psc metric on W is left stable if and only if it is right stable.
(ii) For each $g_0 \in \mathcal{R}^+(M_0)$, there exists $g_1 \in \mathcal{R}^+(M_1)$ and a stable $h \in \mathcal{R}^+(W)_{g_0,g_1}$.
(iii) For each $g_1 \in \mathcal{R}^+(M_1)$, there exists $g_0 \in \mathcal{R}^+(M_0)$ and a stable $h \in \mathcal{R}^+(W)_{g_0,g_1}$.

Proof. Let

$$W \xrightarrow{\ell'_W} B' \xrightarrow{\theta'} B\mathrm{O}(d)$$

be the second Moore–Postnikov factorization of the Gauss map of TW. The tangential structure θ' is once-stable, by [GRW14, Lemma 5.5]. Consider the diagram

$$\begin{array}{ccc} W & \xrightarrow{\ell'_W} & B' \\ {\scriptstyle \ell_W}\downarrow & \overset{\sigma}{\nearrow} & \downarrow{\scriptstyle \theta'} \\ B & \xrightarrow{\theta} & B\mathrm{O}(d). \end{array}$$

The hypotheses imply that $\ell_W : W \to B$ is 2-connected, so without loss of generality we may suppose that B is obtained by attaching cells of dimension ≥ 3 to W. We may also suppose that θ' is a fibration. Since θ' is 2-co-connected by construction,

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the lift σ exists by elementary obstruction theory. Then σ is 2-connected since ℓ_W and ℓ'_W are.

Therefore, the θ'-cobordism (W, ℓ'_W) satisfies the hypotheses of Theorem 2.3.8 in the published paper, and that theorem gives the desired conclusion. (The point being that the conclusion does not involve the tangential structure at all). □

The above addresses the gap, but in fact the gap did not affect the validity of any of the other results stated in the introduction of the paper:

- Theorem B uses Theorem C in its proof, but only for a once-stable θ. Theorem A is a special case of Theorem B.
- Theorem D is independent of Theorem C.
- Theorem E uses Theorem C where $\theta : B \to B\mathrm{O}(d)$ is the tangential 2-type of a d-manifold. Since by default d is large, such θ are once-stable by [GRW14, Lemma 5.5]. Theorem F is an application of Theorem E.
- Theorem G uses Theorem C, but only for the special tangential structure $B\mathrm{Spin}(d) \times BG \to B\mathrm{O}(d)$ which is once-stable (in fact stable).

References

[ERW22] J. Ebert and O. Randal-Williams, *The positive scalar curvature cobordism category*, Duke Math. J. **171** (2022), no. 11, 2275–2406.

[GRW14] S. Galatius and O. Randal-Williams, *Stable moduli spaces of high-dimensional manifolds*, Acta Math. **212** (2014), no. 2, 257–377.

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