Lecture notes on Cellular E_k-algebras

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Abstract

These are the collected lecture notes for the Oberwolfach seminar on *Cellular E_k-algebras* in 2021.

Contents

Chapter 1

Introduction and outline

The goal of this seminar is to explain the applications of cellular E_k-algebras to homological stability, through the concrete example of mapping class groups. In this first lecture, we will state the main results and explain the strategy as well as outline the remaining lectures. Recommended additional is the introduction of [GKRW18a] and Sections 1 and 2 of [GKRW19].

1.1 The statement

For a smooth surface $\Sigma_{g,1}$ of genus g with one boundary component, we let $\mathrm{Diff}_\partial(\Sigma_{g,1})$ denote the topological group of diffeomorphisms of $\Sigma_{g,1}$ fixing a neighborhood of the boundary pointwise. All of its path components are contractible [Gra73], so the map

$$\mathrm{Diff}_\partial(\Sigma_{g,1}) \longrightarrow \pi_0(\mathrm{Diff}_\partial(\Sigma_{g,1})) \tag{1.1}$$

is a homotopy equivalence. The right side is the group of isotopy classes of diffeomorphisms fixing a neighborhood of the boundary pointwise.

Definition 1.1.1. The *mapping class group* $\Gamma_{g,1}$ is given by $\pi_0(\mathrm{Diff}_\partial(\Sigma_{g,1}))$.

Since (1.1) is a homotopy equivalence, the classifying space $B\Gamma_{g,1}$ classifies manifold bundles with fiber $\Sigma_{g,1}$ and trivialised boundary bundle, which we will refer to as *surface bundles* in this lecture. This bijection is given as follows: for nice enough X, pulling back a universal surface bundle over $B\Gamma_{g,1}$ to X induces a bijection between the set $[X, B\Gamma_{g,1}]$ of homotopy classes of maps $X \to B\Gamma_{g,1}$ and the set of isomorphism classes of such surface bundles over X. As a consequence, understanding the cohomology groups $H^*(B\Gamma_{g,1}; \Bbbk)$ amounts to understanding the characteristic classes of surface bundles. By the universal coefficient theorem, we can equivalently try to understand the homology groups.

Question 1.1.2. What are the homology groups of mapping class groups?

When attempting to answer this question, it is advantageous to let g go to ∞. This may seem counter-intuitive but it is the underlying idea of *stability phenomena*. Fixing once and for all a standard surface $\Sigma_{1,1} \subset [0,1]^3$ as in Fig. 1.1, the surface $\Sigma_{g,1}$ can be

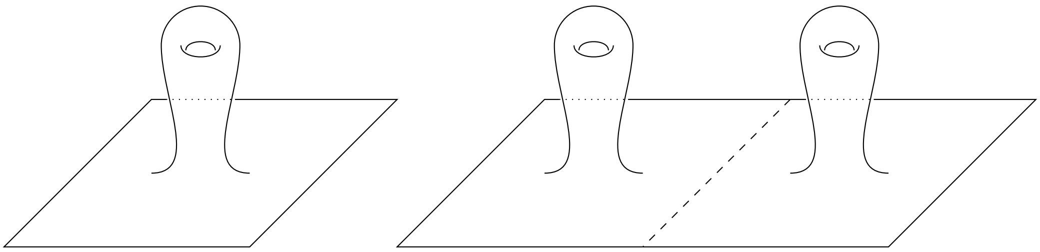


Figure 1.1: The standard surface $\Sigma_{1,1}$, and the surface $\Sigma_{g,1}$ for $g = 2$ obtained from two copies of $\Sigma_{1,1}$.

obtained as the union $\bigcup_{j=0}^{g-1}(\Sigma_{1,1} + j \cdot \vec{e}_1)$. In particular, there is an inclusion $\Sigma_{g-1,1} \subset \Sigma_{g,1}$ and any isotopy class of diffeomorphism of $\Sigma_{g-1,1}$ fixing a neighborhood of the boundary pointwise can be extended by the identity to such a diffeomorphism of $\Sigma_{g,1}$. This yields a homomorphism $\Gamma_{g-1,1} \to \Gamma_{g,1}$ and hence a map on classifying spaces

$$\sigma\colon B\Gamma_{g-1,1} \longrightarrow B\Gamma_{g,1},$$

called the *stabilisation map*. The *Harer stability theorem* say that this map is an isomorphism in a range tending to ∞ with g:

Theorem 1.1.3 (Homological stability for mapping class groups)**.** *The map*

$$\sigma_*\colon H_d(B\Gamma_{g-1,1};\mathbb{Z}) \longrightarrow H_d(B\Gamma_{g,1};\mathbb{Z})$$

is a surjection for $d \leq \frac{2g-1}{3}$ *and an isomorphism for* $d \leq \frac{2g-4}{3}$.

Remark 1.1.4. This result goes back to Harer [Har85] with improvements by Ivanov [Iva93], Boldsen [Bol12], and Randal-Williams [RW16] (see [Wah13] for an exposition, and [HV17] for a more "standard" proof along the lines of [RWW17] but with a worse range). The above statement is Theorem B (i) of [GKRW19]; the range it gives is optimal.

To visualise Theorem 1.1.3, one should draw the homology groups of mapping class groups as a grid, with $H_d(B\Gamma_{g,1};\mathbb{Z})$ in the (g,d)-entry so that stabilisation increases the first coordinate. Then the above homological stability result says that below a line of slope $\frac{2}{3}$ the entries are independent of g. In this *stable range*, the values are equal to the *stable homology* $\mathrm{colim}_{g\to\infty} H_d(B\Gamma_{g,1};\mathbb{Z})$ given by Madsen–Weiss theorem as $H_d(\Omega_0^\infty MTSO(2);\mathbb{Z})$ [MW07]. In particular, rationally the stable cohomology is the free graded-commutative algebra on the Miller–Morita–Mumford classes.

Question 1.1.5. What are the homology groups outside the stable range?

The main result discussed in this seminar is the existence of a *metastable range* above the stable range, in which it is not the case that the relative groups $H_d(B\Gamma_{g,1}, B\Gamma_{g-1,1};\mathbb{Z})$ vanish but there are maps between them that are isomorphisms [GKRW19, Theorem A]. Here the precise statement:

Theorem 1.1.6 (Secondary homological stability for mapping class groups)**.** *There are maps*

$$\varphi_*\colon H_{d-2}(B\Gamma_{g-3,1}, B\Gamma_{g-4,1};\mathbb{Z}) \longrightarrow H_d(B\Gamma_{g,1}, B\Gamma_{g-1,1};\mathbb{Z})$$

which are surjections for $d \leq \frac{3g-1}{4}$ *and isomorphisms for* $d \leq \frac{3g-5}{4}$.

Remark 1.1.7*.* This formulation is precise; the maps φ are *not* unique (see Lemma 13.1.2 and the remark following it).

The crucial observation is that $\frac{3}{4} > \frac{2}{3}$, so this is a statement about possibly non-zero groups. Rationally, the ranges can be improved to surjections for $d \leq \frac{4g-1}{5}$ and isomrphisms for $d \leq \frac{4g-6}{5}$. Fig. 1.2 reproduces a figure from [GKRW19]: the orange region is the metastable range, below it you find in blue the stable range, and the region above it remains mysterious.

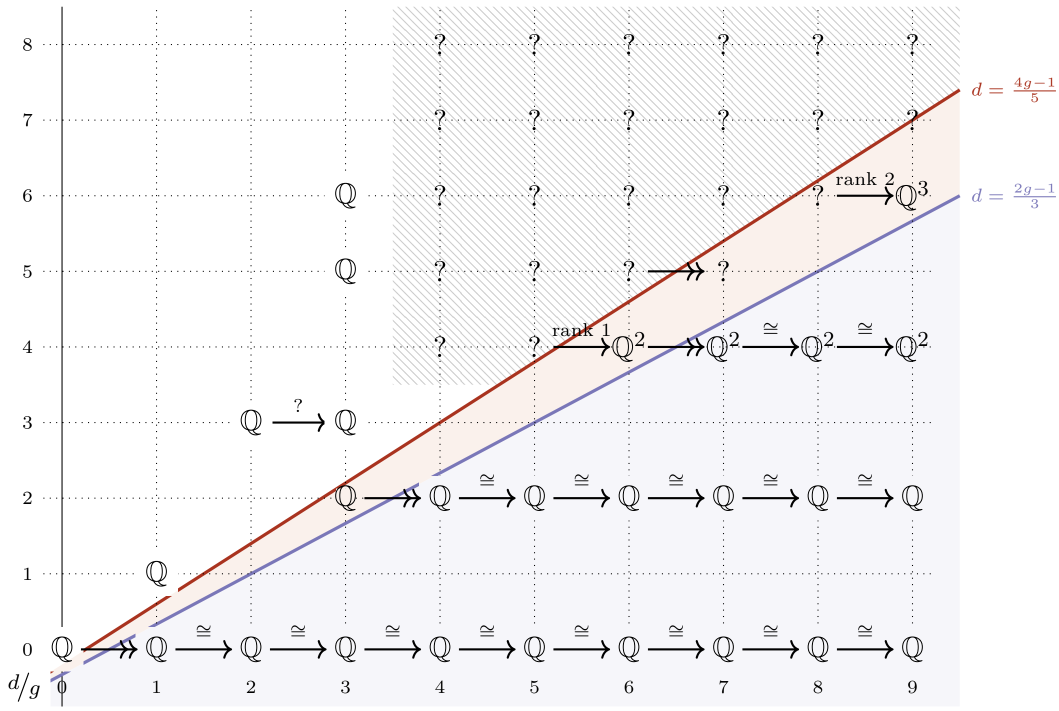


Figure 1.2: A summary of the low-degree low-genus rational homology of $\Gamma_{g,1}$ and the stabilisation maps. We have colored the stable range, the metastable range, and dashed the currently unknown remaining unstable homology.

Remark 1.1.8*.* The homology groups for low g are obtained by (algebro-)geometric methods or computer calculations (see Section 6.4 of [GKRW19] for references); see Chapter 6 for some examples using geometric techniques.

Example 1.1.9*.* Secondary homological stability is barely visible in Fig. 1.2: the stabilisation maps into $H_{2k}(B\Gamma_{3k,1};\mathbb{Q})$ are not surjections because the relative homology groups always have rank 1.

It remains an open question to determine the *secondary stable homology groups*

$$\operatorname*{colim}_{k\to\infty} H_{d+2k}(B\Gamma_{g+3k,1}, B\Gamma_{g+3k-1,1}; \mathbb{Z}).$$

1.2 The strategy

Let us now explain the strategy for proving Theorem 1.1.6 (we will obtain Theorem 1.1.3 along the way), simultaneously giving the outline of the upcoming lectures.

1.2.1 Operads, algebras, and indecomposables

The crucial observation is that the disjoint union $\bigsqcup_{g\geq 1} B\Gamma_{g,1}$ comes equipped with an additional algebraic structure. This structure is of a homotopy-theoretic nature, and encoded by the *little* 2*-cubes operad.* That is, $\bigsqcup_{g\geq 1} B\Gamma_{g,1}$ is an E_2-algebra. Let us explain this statement in more detail.

Operads and algebras

Many of the technical foundations of our arguments will go through for an arbitrary operad (or even a monad) on a symmetric monoidal category C. Recall that an *operad* $\mathcal{O}$ in symmetric monoidal category C is a collection $\{\mathcal{O}(n)\}_{n\geq 0}$ of objects $\mathcal{O}(n)$ in C with an action of the symmetric group $\mathfrak{S}_n$, together with a unit map $\mathbb{1} \to \mathcal{O}(1)$, and composition maps

$$\mathcal{O}(n) \otimes \mathcal{O}(i_1) \otimes \cdots \otimes \mathcal{O}(i_n) \longrightarrow \mathcal{O}(i_1 + \ldots + i_n)$$

which are equivariant, associative, and unital. You should think of $\mathcal{O}(n)$ as a space of n-ary operations. This is clear when we consider the definition of an $\mathcal{O}$*-algebra* in C; it is an object A of C with action maps

$$\mathcal{O}(n) \otimes A^{\otimes n} \longrightarrow A$$

which are equivariant, associative, and unital. This can be found in Sections 2, 3, and 4 of [GKRW18a].

Cellular algebras and indecomposables

The strategy will for proving Theorem 1.1.6 is to give a "homotopical presentation" of $\bigsqcup_{g\geq 1} B\Gamma_{g,1}$ in a category of algebras over the little k-cubes operad.

More generally, for an operad $\mathcal{O}$, such a presentation of an $\mathcal{O}$-algebra A is given by a weak equivalence to A from a *cellular algebra*; this is an $\mathcal{O}$-algebra obtained by iterated pushouts on free $\mathcal{O}$-algebras on inclusions of the form $S^{k-1} \hookrightarrow D^k$. Even better are CW-algebras, which come a specified skeletal filtration. We will explain the theory of CW approximation for a general operad $\mathcal{O}$, but it proceeds among the same lines as CW approximation of 1-connected topological spaces. One attaches $\mathcal{O}$-cells to obtain increasingly accurate approximations. To understand which $\mathcal{O}$-cells are needed, the crucial input is a Hurewicz theorem and the appropriate replacement of homology in its statement: this is the homology of the derived $\mathcal{O}$-indecomposables. This can be found in Sections 3, 8, and 11 of [GKRW18a].

There will be two lectures about this topic:

- **Monday**: Indecomposables I – indecomposables of E_k-algebras in simplicial sets (Chapter 2).
- **Tuesday**: Indecomposables II – indecomposables in other categories (Chapter 5).

1.2.2 E_k-algebras and their properties

Let us now return to the study of mapping class groups.

The E_k-operad and E_k-algebras

The point has come to define, for $k \geq 1$, the (non-unitary) little k-cubes operad $\mathcal{C}_k$, which is an operad in spaces. We will often refer to it as "the" E_k-operad, but it is but one of many weakly equivalent choices; you may have seen the little k-discs instead. We will similarly refer to a $\mathcal{C}_k$-algebra as an E_k-algebra.

Definition 1.2.1. Let $\mathrm{Emb}^{\mathrm{rect}}(\bigsqcup_n I^k, I^k)$ denote the space of n-tuples of rectilinear embeddings $I^k \to I^k$ (that is, compositions of scaling and translation) whose interiors are disjoint. Then the *little k-cubes operad* $\mathcal{C}_k$ has space of n-ary operations given by

$$\mathcal{C}_k(n) := \begin{cases} \varnothing & \text{if } n = 0, \\ \mathrm{Emb}^{\mathrm{rect}}(\bigsqcup_n I^k, I^k) & \text{if } n > 0. \end{cases}$$

The symmetric group $\mathfrak{S}_n$ acts on $\mathcal{C}_k(n)$ by permuting the cubes, the unit $* \to \mathcal{C}_k(1)$ picks out the identity, and the composition maps are induced by composition of rectilinear embeddings.

See Fig. 1.3 and Fig. 1.4 for examples in the case $k = 2$.

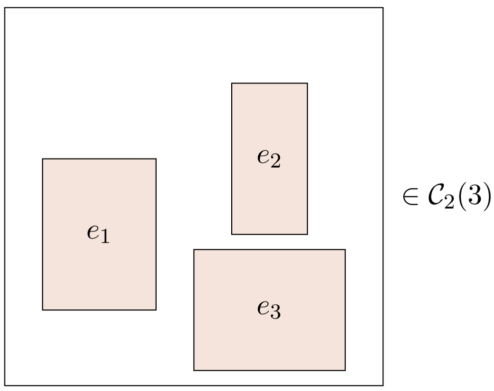


Figure 1.3: An element of $\mathcal{C}_2(3)$.

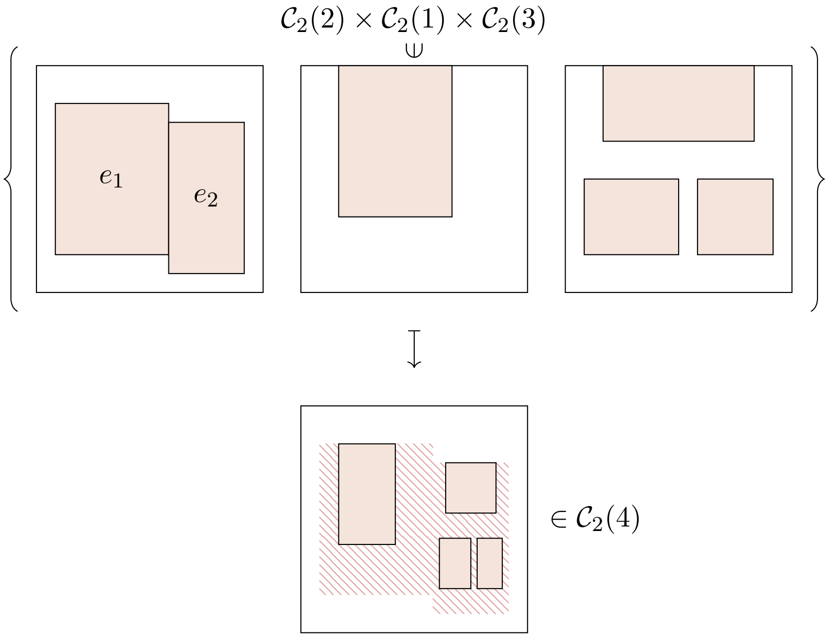


Figure 1.4: An example of composition in $\mathcal{C}_2$. We have left out the labels on the inner cubes for readability.

In the following definition, we can work in the category of spaces or more generally any category with a suitable copowering over spaces (the structure on C that allows you to define the product of a space with an object of C).

Definition 1.2.2. A *(non-unital) E_k-algebra* is an algebra over the operad $\mathcal{C}_k$.

Example 1.2.3 (Iterated loop spaces)*.* The prototypical examples of E_k-algebras in spaces are *iterated loop spaces*. Let $\Omega^k X$ be the space of maps of pairs $(I^k, \partial I^k) \to (X, x_0)$. The action maps

$$\mathcal{C}_k(n) \times (\Omega^k X)^n \longrightarrow \Omega^k X$$

are given by "inserting" the jth map f_j into the image of the jth cube and extending to the remainder of the domain by the constant map with value x_0. See Fig. 1.5 for an example in the case $k = 2$. The recognition principle says that any E_k-algebra Y in spaces with $\pi_0(Y)$ a group (under the multiplication induced by the E_k-algebra structure), is weakly equivalent as an E_k-algebra to a k-fold loop space [May72].

Example 1.2.4 (Moduli spaces of manifolds)*.* Let us consider the space of unparametrised compact submanifolds of $[0,1]^k \times \mathbb{R}^\infty$ which coincide with $[0,1]^k \times \{0\}$ with $\partial[0,1]^k \times \mathbb{R}^\infty$, topologised as in [GRW10]. If we require they are diffeomorphic relative to the boundary (and up to smoothing corners) to one of a fixed collection of compact pairwise non-diffeomorphic manifolds $M_1, M_2, \ldots$, we obtain a model $\mathcal{M}$ for $\bigsqcup_i B\mathrm{Diff}_\partial(M_i)$. (To prove this, we may assume we have a single manifold M and observe that the space $\mathcal{E}$ of parametrised compact submanifolds diffeomorphic to M is a contractible space with free properly discontinuous action of $\mathrm{Diff}_\partial(M)$ and $\mathcal{M}$ is the quotient $\mathcal{E}/\mathrm{Diff}_\partial(M)$.)

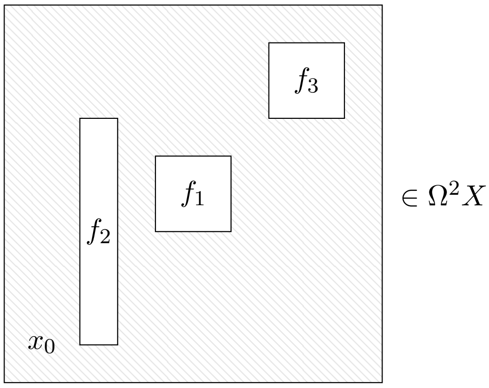


Figure 1.5: The result of combining three elements $f_1, f_2, f_3 \in \Omega^2 X$ with an element of $\mathcal{C}_2(3)$.

If the collection $M_1, M_2, \ldots$ is closed under boundary connected sum, the space $\mathcal{M}$ is an E_k-algebra. The action maps

$$\mathcal{C}_k(n) \times \mathcal{M}^n \longrightarrow \mathcal{M}$$

are given by "inserting" the jth submanifold W_j into the image of the jth cube times $\mathbb{R}^\infty$ and extending to the remainder of $[0,1]^k \times \mathbb{R}^\infty$ by $[0,1]^k \times \{0\}$.

In particular, we may take $k = 2$ and the collection of manifolds $\Sigma_{1,1}, \Sigma_{2,1}, \ldots$ and get an E_2-algebra structure on

$$\mathcal{M} \simeq \bigsqcup_{g \geq 1} B\mathrm{Diff}_\partial(\Sigma_{g,1}) \simeq \bigsqcup_{g \geq 1} B\Gamma_{g,1}.$$

Later, in Chapter 9 we will give a more algebraic construction of this E_2-algebra structure, using the fact that the mapping class groups $\Gamma_{g,1}$ are the automorphism groups in a certain braided-monoidal groupoid MCG; classifying spaces of braided monoidal categories are always E_2-algebras.

This appears in Section 12 of [GKRW18a].

Special properties of the E_k-operad

E_k-algebras have two properties that distinguish the theory of CW-approximation for E_k-algebras from that for $\mathcal{O}$-algebras:

(1) We understand very well the homology of the free E_k-algebras which are the building blocks of CW approximations. Indeed, a theorem of F. Cohen describes it in terms of certain homology operations [CLM76]. This makes computations for cellular or CW E_k-algebras particularly tractable.

(2) There are alternative methods for computing E_k-indecomposables. Indeed, they are given by a k-fold iterated bar construction,which is particularly tractable for the examples we are interested in. As this terminology indicates, it can be computed iteratively. This appears in Sections 13, 14 and 16 of [GKRW18a].

There will be two lectures about this topic:

- **Tuesday**: E_k-algebras I – homology of free E_k-algebras (Chapter 4).
- **Wednesday**: E_k-algebras II – the iterated bar construction (Chapter 7).

1.2.3 Generic homological stability

The techniques which prove Theorem 1.1.6 are applicable to many examples, and before proving that result, we give a criterion for an E_k-algebra to have homological stability; this will yield Theorem 1.1.3 as an example.

This "generic homological stability result" applies to E_2-algebras which arise from braided-monoidal groupoids G satisfying some mild conditions. In this case, one can understand the bar construction which computes the E_1-indecomposables in terms of certain combinatorial objects—the *E_1-splitting complexes*—and then understand the E_2-indecomposables by the iterative procedure mentioned above. If the connectivity of E_1-splitting complexes increases sufficiently fast, then our knowledge of CW approximation and the homology of free E_k-algebras can be used to prove that one can read off homological stability from a few low degree homology groups. This appears in Sections 17 and 18 of [GKRW18a].

There will be three lectures about this topic:

- **Wednesday**: Generic homological stability I – bounded symmetric powers (Chapter 8).
- **Wednesday**: Generic homological stability II – E_2-algebras from braided monoidal groupoids (Chapter 9)
- **Thursday**: Generic homological stability III – a generic homological stability result (Chapter 10).

1.2.4 Facts about mapping class groups

To apply the generic homological stability to the E_2-algebra $\bigsqcup_{g\geq 1} B\Gamma_{g,1}$, obtain Theorem 1.1.3, and make the improvements necessary to prove Theorem 1.1.6, we will need some input.

The first is the connectivity of the E_1-splitting complexes, which is an argument about arc complexes. The second is knowledge of the homology groups $H_d(B\Gamma_{g,1};\mathbb{Z})$ for low d and g. Both are provided by classical techniques, but since they are the fuel for the machine developed in the other lectures we explain how you prove them. This appears in Sections 3 and 4 of [GKRW19].

There will be two lectures about this topic:

- **Monday**: Facts about mapping class groups I – arc complexes (Chapter 3).
- **Tuesday**: Facts about mapping class groups II – low-degree homology (Chapter 6).

1.2.5 Secondary homological stability for mapping class groups

Once we have set up the machinery and provided as input the relevant facts about mapping class groups, we can prove Theorem 1.1.6. We first do so with rational coefficients, because it is then significantly easier to construct the maps φ_* in the statement of this theorem. After that we shall explain how to address the difficulties which arise when working with integer coefficients. The argument are essentially elaborations of the generic homological stability result. This appears in Section 5 of [GKRW19].

There will be two lectures about this topic:

- **Thursday**: Secondary homological stability I – rational argument (Chapter 11).
- **Friday**: Secondary homological stability II – integral argument (Chapter 13)

1.2.6 Outlook

Finally, we look at other results which can be obtained using similar techniques, in particular for general linear groups [GKRW18b, GKRW20]. We will also discuss open problems.

There will be two lectures about this topic:

- **Thursday**: Outlook I: General linear groups (Chapter 12).
- **Friday**: Outlook II.

Chapter 2

Indecomposables I: E_k-algebras in simplicial sets

2.1 Summary/recollection

In Chapter 1 we defined the little k-cubes operad $\mathcal{C}_k$, whose nth space $\mathcal{C}_k(n)$ consists of ordered n-tuples of rectilinear embeddings $I^k \to I^k$ whose interiors are disjoint. Let us take the singular simplicial set of the spaces $\mathcal{C}_k(n)$ to turn them into simplicial sets, yielding an operad in the category sSets of simplicial sets which we will denote by the same letter $\mathcal{C}_k$.

Similarly, we defined an E_k algebra (in the category sSets) to be a simplicial set A together with maps

$$\mathcal{C}_k(n) \times A^n \to A, \tag{2.1}$$

for each $n \in \mathbb{Z}_{\geq 1}$, satisfying some properties including invariance under the evident action of the symmetric group S_n on the domain. We also saw two types of examples: the k-fold loop space of a based space (or rather the singular simplicial set thereof), and examples based on moduli spaces of manifolds.

A good way to encode the data (2.1) and the properties it is required to satisfy, is to encode it as A being an algebra for a *monad*. This is a monoid in the category of functors $s\text{Sets} \to s\text{Sets}$. Indeed, the operad $\mathcal{C}_k$ gives rise to a functor

$$E_k \colon s\text{Sets} \longrightarrow s\text{Sets}$$

defined by

$$E_k(X) = \coprod_{n=1}^{\infty} (\mathcal{C}_k(n) \times X^n)/S_n, \tag{2.2}$$

where the symmetric group S_n acts on X^n by permuting factors, and on $\mathcal{C}_k(n)$ by permuting the embeddings $I^k \to I^k$. As mentioned in Chapter 1, composition of embeddings $I^k \to I^k$ gives rise to maps

$$\mathcal{C}_k(n) \times \mathcal{C}_k(i_1) \times \cdots \times \mathcal{C}_k(i_n) \longrightarrow \mathcal{C}_k(i_1 + \cdots + i_k)$$

which in turn induce maps $\mu \colon E_k(E_k(X)) \to E_k(X)$ that are natural in the simplicial set X. There is also a natural injection $1 \colon X \to E_k(X)$ as the $n = 1$ summand in (2.2).

The endofunctor E_k and the natural transformations μ and 1 forms a *monad* on sSets, expressing an associativity and unitality property of the natural transformations μ and 1.

The data of the maps (2.1) is equivalent to a single map $\mu\colon E_k(A) \to A$. Given such a map there are two ways to construct a map of simplicial sets $E_k(E_k(A)) \to E_k(A)$, either by applying the functor E_k to the map $\mu\colon E_k(A) \to A$, or by the natural transformation $E_k \circ E_k \Rightarrow E_k$. The required properties can be expressed concisely as these two maps $E_k(E_k(A)) \to E_k(A)$ becoming equal after composing with $E_k(A) \to A$. (Actually there is a further property about the unit.)

2.2 Free algebras and cell attachments

E_k-algebras form a category in an evident way, which we denote by $\mathrm{Alg}_{E_k}(s\mathrm{Sets})$ and forgetting the E_k-algebra structure gives a forgetful functor

$$\begin{aligned} \mathrm{Alg}_{E_k}(s\mathrm{Sets}) &\longrightarrow s\mathrm{Sets} \\ (A, \mu) &\longmapsto A \end{aligned}$$

which admits a left adjoint *free E_k algebra* functor that we denote F^{E_k}. Explicitly, $F^{E_k}(X)$ has underlying simplicial set $E_k(X)$, equipped with E_k-algebra structure given by the map $E_k(E_k(X)) \to E_k(X)$ mentioned above.

A somewhat important example is the *free E_k-algebra on a point*, whose underlying simplicial set is

$$E_k(\mathrm{point}) = \coprod_{n=1}^{\infty} \mathcal{C}_k(n)/S_n \simeq \coprod_{n=1}^{\infty} \mathrm{Conf}_n(\mathbb{R}^k).$$

Here, $\mathrm{Conf}_n(\mathbb{R}^k)$ denotes the unordered configuration space of n points in $\mathbb{R}^k$, and the homotopy equivalence uses that rectilinear embeddings $I^k \to I^k$ are determined by their centers of mass, up to contractible data.

Let us choose a triangulation of the d-dimensional disk D^d, and use the same notation D^d for the corresponding simplicial set, and $\partial D^d \subset D^d$ for the simplicial set corresponding to the triangulation of the boundary. Suppose $(A, \mu) \in \mathrm{Alg}_{E_k}(s\mathrm{Sets})$ and we are given a map of simplicial sets

$$e\colon \partial D^d \longrightarrow A.$$

To this data we can associate the diagram

$$F^{E_k}(D^d) \longleftarrow F^{E_k}(\partial D^d) \longrightarrow (A, \mu)$$

of E_k-algebras, using that F^{E_k} is left adjoint to the forgetful functor. By standard methods one shows that the category of E_k-algebras has all colimits—that is, it is cocomplete—so we may define a new E_k-algebra as the pushout

$$\begin{array}{ccc} F^{E_k}(\partial D^d) & \xrightarrow{\ e\ } & (A, \mu) \\ \downarrow & & \downarrow \\ F^{E_k}(D^d) & \longrightarrow & A \cup_e^{E_k} D^d, \end{array}$$

in $\mathsf{Alg}_{E_k}(s\mathrm{Sets})$. We call this the E_k-algebra obtained by attaching an E_k-cell to (A, μ) along e. There is a universal property for maps out of this new E_k-algebra; we leave it to the reader to formulate it.

More informally, the cell attachment can be described in two steps: first form the pushout $D^d \leftarrow \partial D^d \to A$ in simplicial sets, which is a "partially defined E_k-algebra" in the sense that points in A can be multiplied in the required ways, but products involving simplices of $D^d \setminus \partial D^d$ are undefined. The E_k-cell attachment is the result of freely adding new simplices to this partially defined E_k-algebra for each undefined operation.

2.3 Indecomposables (non-derived)

One point of view on indecomposables is that they are trying to answer the answer the following question:

Question 2.3.1. If we know that an E_k-algebra (A, μ) is free, can we find out what it is free on?

It turns out that this is possible, at least up to adding a basepoint. More precisely, we will define a functor Q^{E_k} fitting in the diagram

$$\begin{array}{ccc} s\mathrm{Sets} & \xrightarrow{F^{E_k}} & \mathrm{Alg}_{E_k}(s\mathrm{Sets}) \\ & \searrow^{+} & \downarrow Q^{E_k} \\ & & s\mathrm{Sets}_* \end{array} \tag{2.3}$$

where $s\mathrm{Sets}_*$ denotes the category of pointed simplicial sets, and "+" is the functor that adds a disjoint basepoint.

It is in fact not difficult to define a functor with this property. We will do this, and discuss some of its formal properties. This will only become useful after we discuss how to derive these functors, though.

2.3.1 Definition and behavior on free E_k algebras

We define the *decomposables* subspace of an E_k-algebra (A, μ_A) as the image of the natural map

$$\bigsqcup_{n=2}^{\infty} (\mathcal{C}_k(n) \times A^n)/S_n \longrightarrow A.$$

Notice that we omitted the subspace $\mathcal{C}_k(1) \times A \subset E_k(A)$ in the domain. In the paper we use a more elaborate notation, but in this lecture I will write $\mathrm{Dec}(A, \mu) \subset A$ for the decomposables of (A, μ).

In other words, A comes with multiplication maps (2.1), and the decomposables subspace is the union of their images over $n \geq 2$.

Example 2.3.2. Let $(A, \mu) = F^{E_k}(X)$ be a free E_k-algebra on a simplicial set X. Then

$$A = \bigsqcup_{n=1}^{\infty} (\mathcal{C}_k(n) \times X^n)/S_n,$$

and by inspecting how the E_k-structure on A works, we see that

$$\mathrm{Dec}(A,\mu) = \coprod_{n=2}^{\infty} (\mathcal{C}_k(n) \times X^n)/S_n.$$

As a consequence,

$$\frac{A}{\mathrm{Dec}(A,\mu)} = \frac{F^{E_k}(X)}{\mathrm{Dec}(F^{E_k}(X))} \cong (\mathcal{C}_k(1) \times X)_+, \tag{2.4}$$

where as usual the subscript denotes a disjoint basepoint.

This almost answers the question! Forming quotient by $\mathrm{Dec}(A,\mu)$ gave us back what (A,μ) was free on, except for the basepoint and except for the factor of $\mathcal{C}_k(1)$.

In fact $\mathcal{C}_k(1)$ is contractible so it does not matter very much that it appeared in (2.4), but aesthetic reasons we get rid of it in the following way. The contractible space $\mathcal{C}_k(1)$ has a natural monoid structure given by composition of embeddings, and as part of the structure map $\mu : E_k(A) \to A$ we have an action of $\mathcal{C}_k(1)$ on A. It is easy to verify that this action preserves the subset $\mathrm{Dec}(A,\mu) \subset A$, so the following is well defined.

Definition 2.3.3. For $(A,\mu) \in \mathrm{Alg}_{E_k}(s\mathrm{Sets})$, the *indecomposables* are defined as the orbit space

$$\mathrm{Q}^{E_k}(A,\mu) = (\frac{A}{\mathrm{Dec}(A,\mu)})/\mathcal{C}_k(1).$$

Comparing with the calculation (2.4), we get

$$Q^{E_k}(F^{E_k}(X)) = (\mathcal{C}_k(1) \times X)_+/\mathcal{C}_k(1) \cong X_+,$$

fulfilling the desired (2.3) up to natural isomorphism of pointed simplicial sets.

2.3.2 A right adjoint

Rather than trying to be an "inverse functor" to F^{E_k}, indecomposables is often presented as *left adjoint* to another functor, sometimes known as square-zero extension (in our paper we call this the *trivial* E_k-algebra structure). For any simplicial set X there is a "trivial" way to define an E_k structure on X_+, namely

$$\mathcal{C}_k(n) \times (X_+)^n \to X_+$$

sends $(\alpha, x) \mapsto x$ for $n = 1$ and $(\alpha, x_1, \dots, x_n) \mapsto +$ for all $n > 1$. We will not make much use of this "trivial E_k-algebra" functor, other than point out that its existence implies that Q^{E_k} preserves all colimits.

2.3.3 Behavior under cell attachments

What happens when one applies Q^{E_k} to an E_k-cell attachment?

Lemma 2.3.4. *Let $e\colon \partial D^d \to A$ be as above. Then there is a pushout square of simplicial sets*

$$\begin{array}{ccc} \partial D^d & \longrightarrow & Q^{E_k}(A,\mu) \\ \downarrow & & \downarrow \\ D^d & \longrightarrow & Q^{E_k}(A \cup_e^{E_k} D^d). \end{array}$$

Therefore, $(A,\mu) \mapsto |Q^{E_k}(A,\mu)|$ takes E_k cell attachments to ordinary cell attachments (as in usual CW complexes, for instance).

Proof. We have already explained that Q^{E_k} admits a right adjoint, so it preserves all colimits and in particular pushouts. Combine this with the calculation of indecomposables of free E_k-algebras. □

We want to iterate E_k-cell attachments.

Definition 2.3.5. An E_k-algebra is *cellular*, if it is isomorphic to a (possibly transfinite) iteration of cell attachments. That is, given an ordinal κ and E_k-algebras A_i for $i \leq \kappa$ such that A_0 is the initial E_k-algebra, A_{i+1} is obtained from A_i by a cell attachment as above, and $A_i = \mathrm{colim}_{j<i} A_j$ when $i \leq \kappa$ is a limit ordinal, then A_κ is cellular.

The main use of indecomposables in our papers is to answer questions of the form: given A, how many cell attachments of each dimension d is necessary for building a cellular E_k-algebra A' with an E_k map $A' \to A$ which is a weak equivalence. In order to answer that, we need the *derived* indecomposables, which I'll define in my next talk.

Using underived indecomposables we can answer a different, and admittedly artificial, question: if we know that A is cellular (not just up to homotopy) and built using finitely many cells of each dimension, what can we say about how many cells were used? Indeed, $A \mapsto |Q^{E_k}(A)|$ takes each E_k cell attachment to an ordinary cell attachment, as explained above. Therefore, $\widetilde{H}_d(Q^{E_k}(A))$ is a finitely generated abelian group, and we see that there must be at least

$$\mathrm{rank}(\widetilde{H}_d(Q^{E_k}(A)))$$

many cell attachments of dimension d.

2.4 Derived indecomposables

The above definitions of E_k-algebra, indecomposables, cell attachments, etc., are not really useful notions without introducing some homotopy theory. The most important notion is:

Definition 2.4.1. A map $f\colon A \to B$ in $\mathrm{Alg}_{E_k}(s\mathrm{Sets})$ is a *weak equivalence* if the underlying map in sSets is a weak equivalence.

The indecomposables behave well with respect to this notion of weak equivalence, in the following sense:

Lemma 2.4.2. *Let $f\colon A \to A'$ be a weak equivalence in $\mathrm{Alg}_{E_k}(s\mathrm{Sets})$ and assume both A and A' are cellular. Then the induced map*

$$Q^{E_k}(f)\colon Q^{E_k}(A) \to Q^{E_k}(A')$$

is also a weak equivalence.

Lemma 2.4.3. *Let $A \in \mathrm{Alg}_{E_k}(s\mathrm{Sets})$. Then*

- *There exists a cellular approximation $A' \to A$.*
- *For any two cellular approximations $A' \to A$ and $A'' \to A$, there exists a cellular approximation $A''' \to A$ and a zig-zag $A' \to A''' \leftarrow A''$ over A.*

About proofs. In fact it is better to prove slightly more than what is stated, namely to construct a *model category* structure on $\mathrm{Alg}_{E_k}(s\mathrm{Sets})$ in which weak equivalences and fibrations are detected on underlying simplicial sets. Cellular objects are cofibrant in this model structure. The functor $Q^{E_k} : \mathrm{Alg}_{E_k}(s\mathrm{Sets}) \to s\mathrm{Sets}_*$ is a *left Quillen functor*. The previous lemmas then follow from standard model category theory. We do not plan to say more about that in the lectures, because either you've seen it before; or this is not the right moment to learn it. In the latter case, it hopefully suffices to take the statements of the lemmas on faith for now. □

Using these lemmas we see that for any $A \in \mathrm{Alg}_{E_k}(s\mathrm{Sets})$, the homotopy type of $Q^{E_k}(A')$ for a cellular approximation $A' \to A$ is independent of the choice of cellular approximation. This homotopy type is the *derived indecomposables*

$$Q^{E_k}_{\mathbb{L}}(A) \simeq Q^{E_k}(A') \quad \text{for any cellular approximation } A' \to A. \tag{2.5}$$

Remark 2.4.4. In fact the equivalence (2.5) is essentially the definition of $Q^{E_k}_{\mathbb{L}}$, except that we have not explained how to make $Q^{E_k}_{\mathbb{L}}$ into a *functor*. This is again standard methods from model categories: there is a *functorial* way to factor the unique map $\emptyset \to A \in \mathrm{Alg}_{E_k}(s\mathrm{Sets})$ as a cofibration $\varnothing \to A'$ followed by an acyclic fibration $A' \to A$, and any such choice leads to a functor $Q^{E_k}_{\mathbb{L}}$ satisfying (2.5).

Corollary 2.4.5. *Assume that $A \in \mathrm{Alg}_{E_k}(s\mathrm{Sets})$ admits a cellular approximation $A' \to A$ which has only finitely many cells in each dimension. Then A' must have at least*

$$\mathrm{rank}(\widetilde{H}_d(Q^{E_k}_{\mathbb{L}}(A)))$$

many cells of dimension d. □

In the paper, we write

$$H^{E_k}_d(A) = \widetilde{H}_d(Q^{E_k}_{\mathbb{L}}(A))$$

and call it *E_k-homology*. The rank of the E_k-homology groups therefore gives a lower bound for the number of cell attachments necessary for a cellular approximation. To make use of this lower bound, we must

- Find an effective way to *calculate*, or at least estimate $Q^{E_k}_{\mathbb{L}}(A)$ for the A's that we are interested in. (This is the content of Chapter 7).
- Ideally, find criteria for when the lower bound can be realized. (This is not quite realistic for the E_k-algebras in sSets, but will be quite realistic when passing to other categories, as we discuss in Chapter 5.)

Chapter 3

Facts about mapping class groups I: arc complexes

This is the first of two chapters that provide the eventual input about mapping class group that is needed to prove Theorem 1.1.3 and Theorem 1.1.6.

3.1 Statements

Recall the standard surface $\Sigma_{g,1} = \bigcup_{j=0}^{g-1}(\Sigma_{1,1} + j \cdot \vec{e}_1)$ and choose two distinguished points $b_0, b_1 \in \partial\Sigma_{g,1}$ on its boundary.

Definition 3.1.1. Let $S(\Sigma_{g,1}, b_0, b_1)_p$ denote the set of $(p+1)$-tuples $([s_0], \ldots, [s_p])$ of isotopy classes of arcs in $\Sigma_{g,1}$ from b_0 to b_1, such that there are representatives $s_0, \ldots, s_p$ of these isotopy classes

- (i) which are disjoint except for their endpoints,
- (ii) whose order $s_0, \ldots, s_p$ agrees with the clockwise order of the s_i at b_0,
- (iii) such that each s_i splits $\Sigma_{g,1}$ into two subsurfaces both having strictly positive genus, and the region between each pair s_i and s_{i+1} also has strictly positive genus.

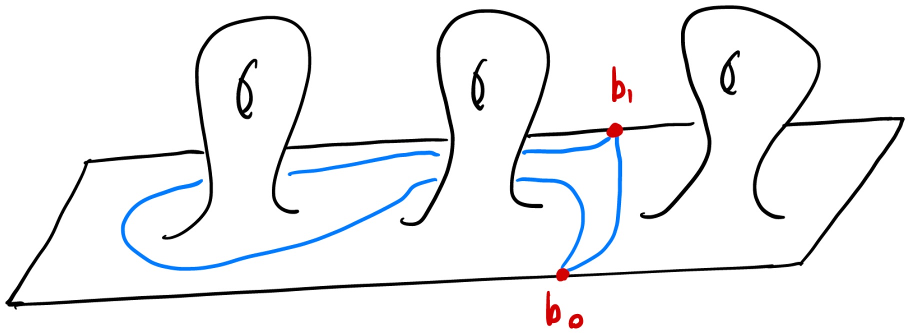
These form the p-simplices of a semi-simplicial set $S(\Sigma_{g,1}, b_0, b_1)_\bullet$, with i-th face map given by forgetting the isotopy class $[s_i]$.

Remark 3.1.2. It is always worrying when one imposes conditions on representatives of equivalence classes. Fortunately, by a theorem of Gramain [Gra73, Théorème 5] the space of arcs on a surface in a given isotopy class is contractible: the same then follows for tuples of arcs disjoint except at the endpoints, which shows that these conditions do not actually depend on the choice of representatives.

The goal of this lecture is to give an idea of [GKRW19, Theorem 3.4]:

Theorem 3.1.3. *$|S(\Sigma_{g,1}, b_0, b_1)_\bullet|$ is $(g-3)$-connected.*

This semi-simplicial set is clearly $(g-2)$-dimensional, as the largest number of arcs there can be is $(g-1)$, splitting the surface into g pieces of genus 1 each. It follows that it is homotopy equivalent to a wedge of $(g-2)$-spheres.



There is a long history of the study of (simplicial complexes or) semi-simplicial sets of (curves or) arcs on surfaces, and they come in many flavours. Typically one considers systems of (curves or) arcs which *do not* separate the surface, making our example a bit unusual (though it is analogous to the curve complex studied in [Loo13] and indeed the connectivity is identical). One can usually show that such complexes are "highly-connected" (typically that they are either $\frac{g-c}{2}$- or $(g-c)$-connected for some small constant c) but, while there are some general principles, these arguments are usually ad hoc, long, and difficult. In [GKRW19] we deduced the Theorem from the connectivity of a different arc complex, of "ordered nonseparating arcs" whose connectivity had already been established (see [Wah13, Section 4] for a detailed account). Unfortunately this makes the overall argument rather involved, so I will instead explain the same strategy in a simpler situation.

3.2 A nerve theorem

Let X and P be posets, and

$$F\colon \mathsf{P}^{op} \longrightarrow \{\text{downward closed subposets of } \mathsf{X},\ \text{inclusions}\}$$

be a map of posets: we think of the poset as indexing a cover of $|\mathsf{X}|$, and want to understand the relationship between the homotopy types $|\mathsf{P}|$ and $|\mathsf{X}|$. There are many results of this flavour: Borsuk's Nerve Theorem [Bor48], Quillen's Poset Fibre Lemma [Qui78], and generalisations [vdKL11].

We let $\mathsf{X}_{<x}$ and $\mathsf{P}_{<p}$ denote the under-posets as usual, and set $\mathsf{P}_x := \{p \in \mathsf{P}^{op} \text{ s.t. } x \in F(p)\}$.

Theorem 3.2.1 (Nerve Theorem)**.** *Suppose that* X *has no infinite descending chains and* P *has no infinite ascending chains. Then there is a zig-zag*

$$|\mathsf{X}| \overset{\phi}{\longleftarrow} ? \overset{\psi}{\longrightarrow} |\mathsf{P}|$$

where the map ϕ *is* $\min_{x\in\mathsf{X}}(\text{conn}(|\mathsf{X}_{<x}|) + \text{conn}(|\mathsf{P}_x|) + 3)$*-connected and the map* ψ *is* $\min_{p\in\mathsf{P}}(\text{conn}(|\mathsf{P}_{<p}|) + \text{conn}(|F(p)|) + 3)$*-connected.*

3.3 Arcs for punctures

Consider the surface $\Sigma_{0,1}^n$, that is the disc with n punctures, with distinguished points $b_0, b_1 \in \partial\Sigma_{0,1}^n$ on its boundary.

Definition 3.3.1. Let $S(\Sigma_{0,1}^n, b_0, b_1)_p$ denote the set of $(p+1)$-tuples $([s_0], \ldots, [s_p])$ of isotopy classes of arcs in $\Sigma_{0,1}^n$ from b_0 to b_1, such that there are representatives $s_0, \ldots, s_p$ of these isotopy classes

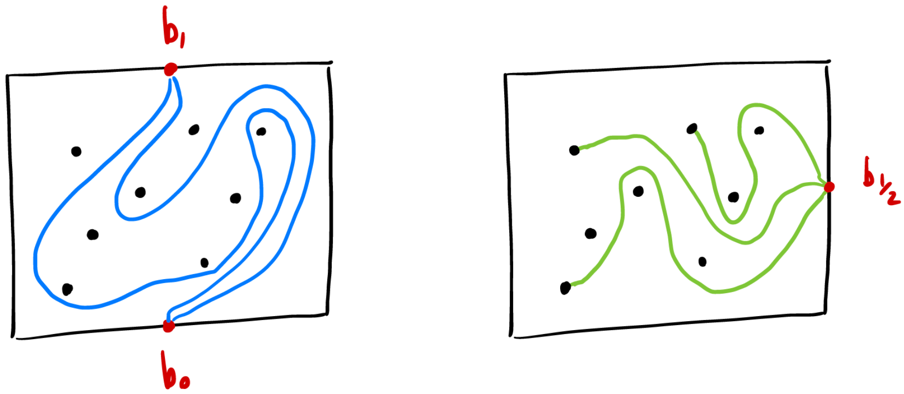
(i) which are disjoint except for their endpoints,

(ii) whose order $s_0, \ldots, s_p$ agrees with the clockwise order of the s_i at b_0,

(iii) such that each s_i splits $\Sigma_{0,1}^n$ into two subsurfaces both having strictly positive number of punctures, and the region between each pair s_i and s_{i+1} also has strictly positive number of punctures.

These form the p-simplices of a semi-simplicial set $S(\Sigma_{0,1}^n, b_0, b_1)_\bullet$, with i-th face map given by forgetting the isotopy class $[s_i]$. It is the nerve of a poset $\mathsf{S}(\Sigma_{0,1}^n, b_0, b_1)$.

Definition 3.3.2. Let $b_{1/2}$ be a further distinguished point on the boundary, and let $\mathcal{A}(\Sigma_{0,1}^n, b_{1/2})$ be the simplicial complex with vertices the isotopy classes $[a]$ of arcs in $\Sigma_{0,1}^n$ from $b_{1/2}$ to one of the punctures, and where a collection $[a_0], \ldots, [a_p]$ of distinct vertices spans a p-simplex if they go to distinct punctures and there are representatives that are disjoint except at their endpoints.



Theorem 3.3.3 (Hatcher–Wahl [HW10]). *$\mathcal{A}(\Sigma_{0,1}^n, b_{1/2})$ is $(n-2)$-connected.*

Using this we prove:

Theorem 3.3.4. *$|S(\Sigma_{0,1}^n, b_0, b_1)_\bullet|$ is $(n-3)$-connected.*

Proof. We proceed by strong induction on n: if $n = 2$ then it is indeed non-empty, i.e. (-1)-connected; if $n < 2$ there is nothing to show.

We will consider the $(n-2)$-skeleton $\mathcal{A}(\Sigma_{0,1}^n, b_{1/2})^{(n-2)}$, consisting of those systems of arcs which do not reach every puncture. Let $\mathsf{P} := \mathsf{simp}(\mathcal{A}(\Sigma_{0,1}^n, b_{1/2})^{(n-2)})$ be the poset of simplices of this skeleton, and consider the map

$$F : \mathsf{simp}(\mathcal{A}(\Sigma_{0,1}^n, b_{1/2})^{(n-2)})^{\mathrm{op}} \longrightarrow \{\text{downward closed subposets of } \mathsf{S}(\Sigma_{0,1}^n, b_0, b_1)\}$$

with $F([a_0, \ldots, a_p])$ given by the subposet of those isotopy classes of separating arcs from b_0 to b_1 which can be represented disjointly from the a_i. We try to apply the Nerve Theorem with $|\mathsf{S}(\Sigma^h_{0,1}, b_0, b_1)| \leftarrow ? \rightarrow |\mathsf{simp}(\mathcal{A}(\Sigma^n_{0,1}, b_{1/2})^{(n-2)})|$.

If $s \in \mathsf{S}(\Sigma^n_{0,1}, b_0, b_1)$ has h punctures to its left and $(n-h)$ to its right, then

$$\mathsf{S}(\Sigma^n_{0,1}, b_0, b_1)_{<s} = \mathsf{S}(\Sigma^h_{0,1}, b_0, b_1)$$

which by induction (as $h, n-h < n$) is $(h-3)$-connected, and

$$\mathsf{simp}(\mathcal{A}(\Sigma^n_{0,1}, b_{1/2})^{(n-2)})_s = \mathsf{simp}(\mathcal{A}(\Sigma^{n-h}_{0,1}, b_{1/2}))$$

which is $(n-h-2)$-connected by Hatcher–Wahl's theorem. This shows that the map $|\mathsf{S}(\Sigma^n_{0,1}, b_0, b_1)| \leftarrow ?$ is $(n-2)$-connected.

On the other hand if $a = [a_0, \ldots, a_p] \in \mathsf{simp}(\mathcal{A}(\Sigma^n_{0,1}, b_{1/2})^{(n-2)})$ then the poset $\mathsf{simp}(\mathcal{A}(\Sigma^n_{0,1}, b_{1/2}))_{<a}$ is the face poset of $\partial\Delta^{p-1}$ so is $(p-3)$-connected. As we only took the $(n-2)$-skeleton, we must have $p+1 < n$, and it follows that the poset $F(a)$ has a top element, given by the arc which follows the boundary of the surface and the a_i, so $F(a)$ is contractible. Thus the map $? \rightarrow |\mathsf{simp}(\mathcal{A}(\Sigma^n_{0,1}, b_{1/2})^{(n-2)})| = |\mathcal{A}(\Sigma^n_{0,1}, b_{1/2})^{(n-2)}|$ is an equivalence. The latter is $(n-3)$-connected by Hatcher–Wahl's theorem.

Combining these shows that $|\mathsf{S}(\Sigma^n_{0,1}, b_0, b_1)|$ is $(n-3)$-connected as required. □

3.4 Addendum: proof of the nerve theorem

We proceed by forming the homotopy colimit $\mathrm{hocolim}_{\mathsf{P}^{op}} |F|$, concretely given by the coequaliser

$$\coprod_{p \geq q \in \mathsf{P}} |\mathsf{P}_{\leq q}| \times |F(p)| \rightrightarrows \coprod_{p \in \mathsf{P}} |\mathsf{P}_{\leq p}| \times |F(p)| \longrightarrow \mathrm{hocolim}_{\mathsf{P}^{op}} |F|$$

of the two natural maps. There are maps

$$|\mathsf{X}| \xleftarrow{\phi} \mathrm{hocolim}_{\mathsf{P}^{op}} |F| \xrightarrow{\psi} \mathrm{hocolim}_{\mathsf{P}^{op}} * = |\mathsf{P}|$$

induced by $|\mathsf{X}| \leftarrow |F(p)| \rightarrow *$, and we try to estimate their connectivities. Using the assumptions, define the *height* and *depth* as

$$h(x) = \max\{r \in \mathbb{N} \text{ s.t. there is a chain } x = x_0 > x_1 > \cdots > x_r \in \mathsf{X}\}$$
$$d(p) = \max\{r \in \mathbb{N} \text{ s.t. there is a chain } p = p_0 < p_1 < \cdots < p_r \in \mathsf{P}\}.$$

Filtering $|\mathsf{P}|$ and $\mathrm{hocolim}_{\mathsf{P}^{op}} |F|$ by $d(-)$, there are cocartesian squares

$$\begin{array}{ccc} \coprod_{d(p)=d} |\mathsf{P}_{<p}| \times |F(p)| & \longrightarrow & (\mathrm{hocolim}_{\mathsf{P}^{op}} |F|)^{\geq d+1} \\ \downarrow & & \downarrow \\ \coprod_{d(p)=d} |\mathsf{P}_{\leq p}| \times |F(p)| & \longrightarrow & (\mathrm{hocolim}_{\mathsf{P}^{op}} |F|)^{\geq d} \end{array} \qquad \begin{array}{ccc} \coprod_{d(p)=d} |\mathsf{P}_{<p}| & \longrightarrow & |\mathsf{P}|^{\geq d+1} \\ \downarrow & & \downarrow \\ \coprod_{d(p)=d} |\mathsf{P}_{\leq p}| & \longrightarrow & |\mathsf{P}|^{\geq d}. \end{array}$$

The squares

$$\begin{array}{ccc} |\mathsf{P}_{<p}| \times |F(p)| & \longrightarrow & |\mathsf{P}_{<p}| \\ \downarrow & & \downarrow \\ |\mathsf{P}_{\leq p}| \times |F(p)| & \longrightarrow & |\mathsf{P}_{\leq p}| \end{array}$$

are $(\mathrm{conn}(|\mathsf{P}_{<p}|) + \mathrm{conn}(|F(p)|) + 3)$-cocartesian[1], and it then follows by standard manipulations with cubes that the map ψ is

$$\min_{p \in \mathsf{P}} (\mathrm{conn}(|\mathsf{P}_{<p}|) + \mathrm{conn}(|F(p)|) + 3)\text{-connected.}$$

Similarly, filtering $|\mathsf{X}|$ and $\mathrm{hocolim}_{\mathsf{P}^{op}} |F|$ by $h(-)$, there are cocartesian squares

$$\begin{array}{ccc} \coprod_{h(x)=h} \mathrm{hocolim}_{p \in \mathsf{P}_x} |F(p)_{<x}| & \longrightarrow & (\mathrm{hocolim}_{\mathsf{P}^{op}} |F|)^{\leq h-1} \\ \downarrow & & \downarrow \\ \coprod_{h(x)=h} \mathrm{hocolim}_{p \in \mathsf{P}_x} |F(p)_{\leq x}| & \longrightarrow & (\mathrm{hocolim}_{\mathsf{P}^{op}} |F|)^{\leq h} \end{array} \qquad \begin{array}{ccc} \coprod_{h(x)=h} |\mathsf{X}_{<x}| & \longrightarrow & |\mathsf{X}|^{\leq h-1} \\ \downarrow & & \downarrow \\ \coprod_{h(x)=h} |\mathsf{X}_{\leq x}| & \longrightarrow & |\mathsf{X}|^{\leq h}. \end{array}$$

For $p \in \mathsf{P}_x$ the poset $F(p)_{\leq x}$ has a top element so is contractible, and hence $\mathrm{hocolim}_{p \in \mathsf{P}_x} |F(p)_{\leq x}| \simeq |\mathsf{P}_x|$. As the posets $F(p)$ are downwards closed, if $x \in F(p)$ then $F(p)_{<x} = \mathsf{X}_{<x}$, and so $\mathrm{hocolim}_{p \in \mathsf{P}_x} |F(p)_{<x}| \simeq |\mathsf{P}_x| \times |\mathsf{X}_{<x}|$. Thus the squares

$$\begin{array}{ccc} \mathrm{hocolim}_{p \in \mathsf{P}_x} |F(p)_{<x}| & \longrightarrow & |\mathsf{X}_{<x}| \\ \downarrow & & \downarrow \\ \mathrm{hocolim}_{p \in \mathsf{P}_x} |F(p)_{\leq x}| & \longrightarrow & |\mathsf{X}_{\leq x}| \end{array}$$

are $(\mathrm{conn}(|\mathsf{X}_{<x}|) + \mathrm{conn}(|\mathsf{P}_x|) + 3)$-cocartesian, and it then follows by standard manipulations with cubes that the map ϕ is

$$\min_{x \in \mathsf{X}} (\mathrm{conn}(|\mathsf{X}_{<x}|) + \mathrm{conn}(|\mathsf{P}_x|) + 3)\text{-connected.}$$

[1] The homotopy pushout is the join $|\mathsf{P}_{<p}| * |F(p)|$ which is $(\mathrm{conn}(|\mathsf{P}_{<p}|) + \mathrm{conn}(|F(p)|) + 2)$-connected, but furthermore $|\mathsf{P}_{\leq p}| \simeq *$ and so the map $|\mathsf{P}_{<p}| * |F(p)| \to *$ is $(\mathrm{conn}(|\mathsf{P}_{<p}|) + \mathrm{conn}(|F(p)|) + 3)$-connected.

Chapter 4

E_k-algebras I: The homology of free E_k-algebras

The purpose of this lecture is to give a description of the homology of free E_k-algebras in $\mathsf{sSet}^{\mathbb{N}}$ (i.e. spaces with an additional grading) which is sufficiently detailed for use in the proof of the generic homological stability theorem and that of Theorem 1.1.6.

4.1 Additional remarks on E_k-algebras

4.1.1 Unital E_k-algebras

Previously, we have considered (non-unital) E_k-algebras, which are by definition the algebras over the operad $\mathcal{C}_k$ given as in Definition 1.2.1. These E_k-algebras do not come with a specified unit. For the sake of describing the homology of free E_k-algebras it is more convenient to add this in; we are more used to free graded-commutative algebras with unit than without. This is done by replacing $\mathcal{C}_k$ by the following operad:

Definition 4.1.1. The *unital little k-cubes operad* $\mathcal{C}_k^+$ has as space of n-ary operations given by

$$\mathcal{C}_k^+(n) \coloneqq \mathrm{Emb}^{\mathrm{rect}}(\sqcup_n I^k, I^k)$$

for all $n \geq 0$. The symmetric group $\mathfrak{S}_n$ acts on $\mathcal{C}_k(n)$ by permuting the cubes, the unit $* \to \mathcal{C}_k(1)$ picks out the identity, and the composition maps are given by composition of rectilinear embeddings.

Definition 4.1.2. A *unital E_k-algebra* (also called a *E_k^+-algebra*) is an algebra over the operad $\mathcal{C}_k^+$.

Example 4.1.3. The map $\mathcal{C}_k^+(n) \otimes \mathbf{R}^{\otimes n} \to \mathbf{R}$ for $n = 0$ yields a map $\mathbb{1} \to \mathbf{R}$ specifying a unit for the E_k-algebra structure. It turns out that there is not much of a difference between units for E_k-algebras as a structure (as in our case) and as a property; this is the content of [Lur, Theorem 5.4.4.5].

Any E_k-algebra $\mathbf{R}$ can be unitalised to an E_k^+-algebra $\mathbf{R}^+$ by formally adding in a unit; this is a functor

$$\begin{aligned} \mathsf{Alg}_{\mathsf{E_k}}(\mathsf{C}) &\longrightarrow \mathsf{Alg}_{E_k^+}(\mathsf{C}) \\ \mathbf{R} &\longmapsto \mathbf{R}^+ \end{aligned}$$

left adjoint to the forgetful functor. On underlying objects, we have $U^{E_k^+}\mathbf{R}^+ = \mathbb{1} \sqcup U^{E_k}\mathbf{R}$.

4.1.2 Additional gradings

For the remainder of this lecture we will work in the category $\mathsf{sSet}^{\mathbb{N}}$. Here $\mathbb{N}$ is the symmetric monoidal groupoid with objects given by the non-negative integers, only identity morphisms, and monoidal structure given by addition; $\mathsf{sSet}^{\mathbb{N}}$ is then the category of functors $\mathbb{N} \to \mathsf{sSet}$ with symmetric monoidal structure given by Day convolution:

$$(X \otimes Y)(g) = \bigsqcup_{g_1+g_2=g} X(g_1) \times X(g_2).$$

In this case working with functors is a tool to keep track of an additional "genus" grading, so we denote the objects of $\mathbb{N}$ by g. (Functor categories are also useful in certain constructions, as we will see when we construct E_2-algebras from braided monoidal groupoids in Chapter 9.)

Example 4.1.4. The assignment $g \mapsto B\Gamma_{g,1}$ refines $\bigsqcup_{g \geq 0} B\Gamma_{g,1}$ to a functor $\mathbb{N} \to \mathsf{sSet}$.

Remark 4.1.5. There is no additional work in replacing sSet with another sufficiently nice category ($\mathsf{sMod}_{\mathbb{k}}$ and Sp are good choices) or $\mathbb{N}$ with another symmetric monoidal groupoid G. Both will appear later in this seminar.

4.2 Free E_k-algebras

4.2.1 The free-forgetful adjunction

Let us recall a notion from Chapter 2. For an operad $\mathcal{O}$ in simplicial sets, let $\mathsf{Alg}_{\mathcal{O}}(\mathsf{sSet}^{\mathbb{N}})$ denote the category of $\mathcal{O}$-algebras in $\mathsf{sSet}^{\mathbb{N}}$. Taking the underlying objects yields a *forgetful functor* $U^{\mathcal{O}} \colon \mathsf{Alg}_{\mathcal{O}}(\mathsf{sSet}^{\mathbb{N}}) \longrightarrow \mathsf{sSet}^{\mathbb{N}}$ with left adjoint given by the *free $\mathcal{O}$-algebra functor*

$$F^{\mathcal{O}} \colon \mathsf{sSet}^{\mathbb{N}} \longrightarrow \mathsf{Alg}_{\mathcal{O}}(\mathsf{sSet}^{\mathbb{N}}).$$

Recalling that $U^{\mathcal{O}}F^{\mathcal{O}}$ is the underlying functor of the monad $\mathbf{O}$ associated to $\mathcal{O}$, we see that

$$U^{\mathcal{O}}F^{\mathcal{O}}(X) = \mathbf{O}(X) = \bigsqcup_{n \geq 0} \mathcal{O}(n) \times_{\mathfrak{S}_n} X^{\otimes n}.$$

4.2.2 Specialising to the E_k-operad

Let us now take $\mathcal{O} = \mathcal{C}_k^+$, the unital little k-cubes operad of Definition 4.1.1; we get a functor

$$F^{E_k^+} \colon \mathsf{sSet}^{\mathbb{N}} \longrightarrow \mathsf{Alg}_{E_k^+}(\mathsf{sSet}^{\mathbb{N}})$$

and a formula of the underlying objects of its image. We will often denote $F^{E_k^+}(X)$ by $\mathbf{E}_k^+(X)$ for brevity and $E_k^+(X)$ for its underlying object.

Remark 4.2.1. By comparing the right adjoints, one sees that $\mathbf{E}_k^+(X) \cong \mathbf{E}_k(X)^+$.

There is a more geometric description of free unital E_k-algebras in terms of configuration spaces, generalising a remark in Chapter 2. Let us write $\dot{I} := \mathrm{int}(I) = (0,1)$.

Definition 4.2.2. Let $\mathrm{Conf}_n(\dot{I}^k)$ be the *configuration space of n ordered points*, given by

$$\mathrm{Conf}_n(\dot{I}^k) \coloneqq \{(x_1, \ldots, x_n) \mid x_i \neq x_j \text{ if } i \neq j\} \subset (\dot{I}^k)^n.$$

There is a map $E_k^+(n) \to \mathrm{Conf}_n(\dot{I}^k)$ which records the centers of the n cubes; this is a $\mathfrak{S}_n$-equivariant homotopy equivalence. Since the action of $\mathfrak{S}_n$ is free on both terms, we get an induced weak equivalence

$$E_k^+(X) = \bigsqcup_{n \geq 0} E_k^+(n) \times_{\mathfrak{S}_n} X^{\otimes n} \longrightarrow \bigsqcup_{n \geq 0} \mathrm{Conf}_n(\dot{I}^k) \times_{\mathfrak{S}_n} X^{\otimes n} =: \mathrm{Conf}(\dot{I}^k; X) \tag{4.1}$$

in $\mathsf{sSet}^{\mathbb{N}}$. You can think of the right side as configuration spaces of unordered points in $\mathbb{R}^k$ (as $\dot{I}^k$ is of course homeomorphic to $\mathbb{R}^k$) with labels in X.

Example 4.2.3. For $k = 2$ and $X = *$, the right side is the disjoint union over n of the configuration spaces of n unordered points in $\dot{I}^2$; this is homotopy equivalent to the classifying space of the nth braid group.

The left side of (4.1), being the underlying objects of a free E_k^+-algebra, comes with an E_k^+-algebra structure and so the right side, given by inserting configuration of labeled points into the cubes (see Fig. 4.1). Using these, the map is a weak equivalence of E_k^+-algebras in $\mathsf{sSet}^{\mathbb{N}}$.

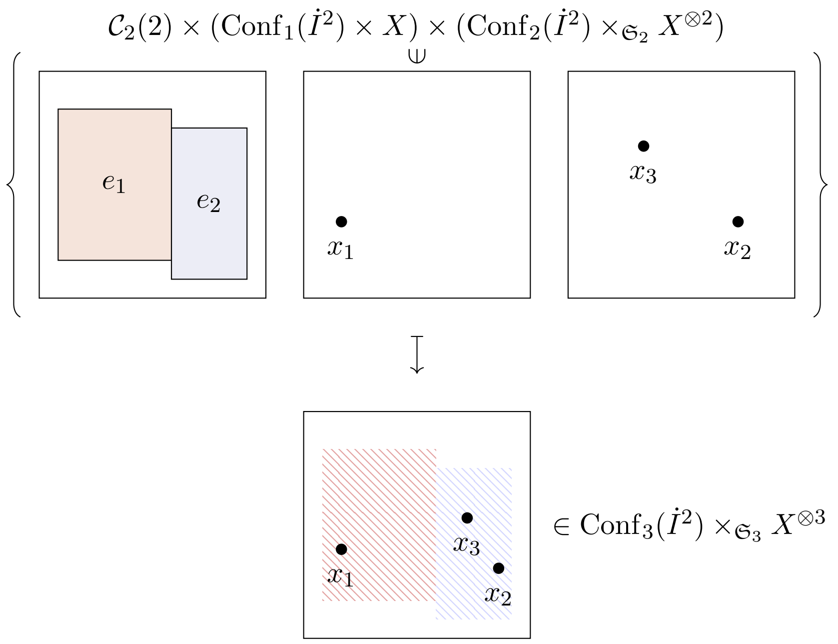


Figure 4.1: An example of E_2-algebra structure on $\bigsqcup_{n \geq 0} \mathrm{Conf}_n(\dot{I}^k) \times_{\mathfrak{S}_n} X^{\otimes n}$.

4.3 The homology of free E_k-algebras

We will now study the homology of free E_k^+-algebras for $k \geq 2$. An object $\mathbf{R} \in \mathsf{sSet}^{\mathbb{N}}$ has bigraded homology groups with coefficients in a commutative ring $\Bbbk$:

$$H_{g,d}(X;\Bbbk) \coloneqq H_d(X(g);\Bbbk).$$

If X is a unital E_k-algebra $\mathbf{R}\mathsf{Alg}_{E_k^+}(\mathsf{sSet}^{\mathbb{N}})$, then the E_k^+-algebra structure endows its homology groups with certain operations. These arise evaluating the map induced on homology by

$$C_*(\mathcal{C}_k(n);\Bbbk) \otimes_{\mathfrak{S}_n} C_*(\mathbf{R};\Bbbk)^{\otimes n} \xrightarrow{\simeq} C_*(\mathcal{C}_k(n) \times_{\mathfrak{S}_n} \mathbf{R}^{\otimes n};\Bbbk) \longrightarrow C_*(\mathbf{R};\Bbbk) \tag{4.2}$$

on certain elements of the domain. By the work of F. Cohen, we have a complete understanding of these operations when $\Bbbk$ is a field $\mathbb{Q}$ or $\mathbb{F}_\ell$ with ℓ prime.

4.3.1 The product and Browder bracket

The easiest operations arise from the case $n = 2$ (that is, binary operations), and are constructed in the same manner for all $\Bbbk$. We start by precomposing the map induced on homology by (1.1) with the external product on homology to get

$$(\theta_2)_*\colon H_*(\mathcal{C}_k(2);\Bbbk) \otimes H_{*,*}(\mathbf{R};\Bbbk)^{\otimes 2} \longrightarrow H_{*,*}(\mathbf{R};\Bbbk),$$

where H_* is shorthand for $H_{0,*}$. Letting two small cubes circle each other describes a homotopy equivalence

$$S^{k-1} \xrightarrow{\sim} \mathcal{C}_k(2)$$

and thus we have two distinguished generators $u_0 \in H_0(\mathcal{C}_k(2);\mathbb{Q})$ and $u_{k-1} \in H_{k-1}(\mathcal{C}_k(2);\Bbbk)$ from which we derive a pair of homology operations.

Definition 4.3.1.

(i) The *product*

$$- \cdot -\colon H_{g_1,d_1}(\mathbf{R};\Bbbk) \otimes H_{g_2,d_2}(\mathbf{R};\Bbbk) \longrightarrow H_{g_1+g_2,d_1+d_2}(\mathbf{R};\Bbbk)$$

is given by $\theta_*(u_0 \otimes - \otimes -)$.

(ii) The *Browder bracket*

$$[-,-]\colon H_{g_1,d_1}(\mathbf{R};\Bbbk) \otimes H_{g_2,d_2}(\mathbf{R};\Bbbk) \longrightarrow H_{g_1+g_2,d_1+d_2+k-1}(\mathbf{R};\Bbbk)$$

is given by $(-1)^{(k-1)d+1}\theta_*(u_{k-1} \otimes - \otimes -)$.

Remark 4.3.2. The sign on the Browder bracket is hard to justify, but makes the relations more palatable (or does it?).

More generally, we can consider the maps

$$(\theta_n)_*\colon H_*(\mathcal{C}_k(n);\Bbbk) \otimes_{\mathfrak{S}_n} H_{*,*}(\mathbf{R};\Bbbk)^{\otimes n} \longrightarrow H_{*,*}(\mathbf{R};\Bbbk),$$

which exhibit $H_{*,*}(\mathbf{R};\Bbbk)$ as an algebra over the *homology operad* $H_*(\mathcal{C}_k;\Bbbk)$. We can in particular understand some properties of these operations by studying this homology operad, resulting in relations satisfied by the product and Browder bracket.

Example 4.3.3. The product is linear in both entries, unital, associative, and graded-commutative. For example, associativity follows from the fact that $\mathcal{C}_k(3)$ is path-connected.

Example 4.3.4. The Browder bracket is linear in both entries, symmetric up to a sign, and satisfies the Jacobi relation up to a sign. As an example, let us prove symmetry and determine the exact sign. For $x_i \in H_{g_i,d_i}(\mathbf{R};\Bbbk)$ with $i = 1, 2$, we have

$$\theta_*(u_{k-1} \otimes x_1 \otimes x_2) = (-1)^{k+d_1d_2}\theta_*(u_{k-1} \otimes x_2 \otimes x_1).$$

where one part of the sign comes from the involution on S^{k-1} and the other from the Koszul sign rule upon switching x_1 and x_2. We leave it to the reader to insert the signs in the definition of the Browder bracket.

Example 4.3.5. The bracket acts a derivation of the product up to a sign.

4.3.2 The Dyer–Lashof operations

When $\Bbbk$ is such that the the external product maps

$$H_*(\mathcal{C}_k(n);\Bbbk) \otimes_{\mathfrak{S}_n} H_{*,*}(\mathbf{R};\Bbbk)^{\otimes n} \longrightarrow H_{*,*}(\mathcal{C}_k(n) \times_{\mathfrak{S}_n} \mathbf{R}^{\otimes n};\Bbbk)$$

are always isomorphisms, the previous section tells the complete story. But this is only the case if $\Bbbk$ is a field of characteristic 0, e.g. $\Bbbk = \mathbb{Q}$.

When $\Bbbk = \mathbb{F}_\ell$ for a prime ℓ, one can construct further operations that do not arise from an application of the map induced by the E_k^+-algebra structure to an element in the image of the external product.

Example 4.3.6. Let's try to understand this for the term $\mathcal{C}_k(2) \times_{\mathfrak{S}_2} (S^i)^{\times 2}$ in $\mathbf{E}_k^+(S^i)$, working with coefficients in $\mathbb{F}_2$ and taking $k \geq 2$. Are there elements in its homology that do not arise by applying products and Browder brackets to the class $[S^i] \otimes [S^i]$?

To understand this, we replace it with a labeled configuration space. Consider the Serre spectral sequence in $\mathbb{F}_2$-homology for the fibration sequence

$$S^i \times S^i \longrightarrow \mathrm{Conf}_2(\dot{I}^k) \times_{\mathfrak{S}_2} (S^i)^{\times 2} \longrightarrow \mathrm{Conf}_2(\dot{I}^k)/\mathfrak{S}_2 \simeq \mathbb{R}P^{k-1}.$$

Its E^2-page has three non-zero rows, given by $H_p(\mathbb{R}P^k;\mathbb{F}_2)$ for $q = 0$, $H_p(S^{k-1};\mathbb{F}_2)$ for $q = i$ (observe that $H_i(S^i \times S^i;\mathbb{F}_2)$ is $\mathbb{F}_2[C_2]$ as a representation of $\pi_1(\mathbb{R}P^k) = C_2$), and $H_p(\mathbb{R}P^{k-1};\mathbb{F}_2)$ for $q = 2i$. Of these, only the terms on the 0th and $(k-1)$st column can be in the image of the external product map. (However, the class in $(p,q) = (k-1, 2i)$ turns out not to be; a full rotation of two of the same class around each around is twice a half rotation.)

In this example, the crucial elements are those on the top row: there are k of these, obtained by combining the $\mathbb{F}_2$-homology of BC_2 in a range (note $\mathbb{R}P^k \to BC_2$ is k-connected) with the square of the fundamental class $[S^i]$ of S^i. In general, a similar construction replacing this fundamental class by a chain representing a homology class of $\mathbf{R}$ yields *Araki–Kudo–Dyer–Lashof operations*

$$Q^s \colon H_{g,d}(\mathbf{R};\mathbb{F}_2) \longrightarrow H_{2g,d+s}(\mathbf{R} : \mathbb{F}_2)$$

where s must satisfy $d \leq s \leq d+k-1$ [KA56, DL62]. For $s = d$, this is—as the example suggests– equal to squaring using the product, while for $s = d+k-1$ this is called *top operation* and denoted ζ because it satisfies slightly different relations than the other Araki–Kudo–Dyer–Lashof operations. One may think of these operations as higher operations derived from the square, not unlike the Steenrod squares [May70].

For odd primes ℓ, the story is similar with $\mathcal{C}_k(\ell)$ replacing $\mathcal{C}_k(2)$ and we get higher operations derived from the ℓth power map. These are the *Dyer–Lashof operations*

$$Q^s\colon H_{g,d}(\mathbf{R};\mathbb{F}_\ell) \longrightarrow H_{\ell g, d+2s(\ell-1)}(\mathbf{R}:\mathbb{F}_\ell)$$
$$\beta Q^s\colon H_{g,d}(\mathbf{R};\mathbb{F}_\ell) \longrightarrow H_{\ell g, d+2s(\ell-1)-1}(\mathbf{R}:\mathbb{F}_\ell)$$

where s must satisfy $d \leq 2s \leq d+k-1$. Once more, for $2s = d$ the map Q^s is the ℓth power map and for $2s = d+k-1$ these are called the top operations ζ and ξ respectively. As the notation suggests, for E_k^+-algebras in spaces βQ^s is obtained by applying the Bockstein to Q^s but this is neither true by definition nor true in general (e.g. for E_k^+-algebras in chain complexes).

The (Araki–Kudo)–Dyer–Lashof and top operations satisfy a variety of relations, both amongst themselves and with the product and bracket. We will not give them here (see [GKRW18a, Chapter 16]), as we will not require the details but only easily stated qualitative consequences.

Notation 4.3.7. If we need to stress the prime ℓ in the Dyer–Lashof operations, we write Q^s_ℓ or βQ^s_ℓ.

4.3.3 F. Cohen's theorem

The previous section explains that $H_{*,*}(\mathbf{R};\Bbbk)$ comes with a generous amount of operations with $\Bbbk = \mathbb{Q}$ or $\mathbb{F}_\ell$. In particular, when $\mathbf{R}$ is a free E_k^+-algebra $\mathbf{E}_k^+(X)$ the identity of the operad $\mathcal{C}_k^+$ provides a canonical map

$$X \longrightarrow E_k^+(X) = \bigsqcup_{n\geq 0} \mathcal{C}_k^+(n) \times_{\mathfrak{S}_n} X^{\otimes n}$$

and thus an induced map $H_{*,*}(X;\Bbbk) \to H_{*,*}(E_k^+(X);\Bbbk)$. We can apply homology operations to its image, and informally F. Cohen's theorem says that all homology classes are obtained in this manner. It is proven in [CLM76] (but see [Wel82] and [GKRW18a, Chapter 16] for corrections).

Rational case

If $\Bbbk = \mathbb{Q}$, then we have only described the product and the Browder bracket. The relations among these make $H_{*,*}(\mathbf{R};\mathbb{Q})$ into a so-called $(k-1)$-*Poisson algebra*. Let $\mathrm{Pois}_{k-1}(V)$ denote the free $(k-1)$-Poisson algebra on a bigraded $\mathbb{Q}$-vector space V (one homological grading and one "genus grading"); this is obtained by iteratively taking products and Browder brackets starting with V, and enforcing all relations that hold among these in the homology of any E_k^+-algebra. The relations are such that we can write everything as a linear combination of products of brackets. However, for the sake of consistent notion we will write $W_{k-1}(V) \coloneqq \mathrm{Pois}_{k-1}(V)$.

Theorem 4.3.8 (F. Cohen). *The map*

$$W_{k-1}(H_{*,*}(X;\mathbb{Q})) \longrightarrow H_{*,*}(E_k^+(X);\mathbb{Q})$$

is an isomorphism.

Finite field case

If $\Bbbk = \mathbb{F}_\ell$, then in addition to the product and Browder bracket we have the (Araki–Kudo)–Dyer–Lashof and top operations. The relations among these make $H_{*,*}(\mathbf{R};\mathbb{F}_\ell)$ into a so-called W_{k-1}*-algebra*. Let $W_{k-1}(V)$ denote the free W_{k-1}-algebra on a bigraded $\mathbb{F}_\ell$-vector space V; this is obtained by iteratively applying products, Browder brackets, (Araki–Kudo)–Dyer–Lashof operations and top operations starting with V, and enforcing all relations among these. The relations are such that we can write everything as linear combinations of products of Dyer–Lashof operations applied to brackets.

Theorem 4.3.9 (F. Cohen). *The map*

$$W_{k-1}(H_{*,*}(X;\mathbb{F}_\ell)) \longrightarrow H_{*,*}(E_k^+(X);\mathbb{F}_\ell)$$

is an isomorphism.

Remark 4.3.10. It may be helpful to remind you that what $W_{k-1}(-)$ means depends on the field we are using as coefficients. Hence this is a different theorem, even though it looks identical to the previous one. In terms of qualitative behavior, you should distinguish between the three cases $\mathbb{Q}$, $\mathbb{F}_2$, and $\mathbb{F}_\ell$ for ℓ odd.

4.4 Iterated mapping cones of E_k-algebras

That you might expect E_k-algebra structure to be related to homological stability and secondary homological stability, is justified by these phenomena being present in the homology of free E_k-algebras. We will work this out in an example. We will then explain a technique that allows us to phrase these homological stability properties in a more robust manner.

4.4.1 An example

Let us look at the homology with $\mathbb{F}_2$-coefficients of the free E_2^+-algebra $E_2^+(D^{1,0}\sigma)$. This uses the following notation:

Notation 4.4.1. For $X \in \mathsf{sSet}$ we let $X^g \in \mathsf{sSet}^{\mathbb{N}}$ denote X placed in "genus" grading g.

In particular, $D^{1,0}$ is the point D^0 placed in "genus" grading 1. Recall that the relations allow you to rewrite every generator as one where the brackets occur before all other operations; but $[\sigma,\sigma] = 0$ (for $\ell = 2$ this is in fact one of the relations). Thus we get that the $\mathbb{F}_2$-homology of $E_2^+(D^{1,0}\sigma)$ is a free graded-commutative bigraded algebra on iterated Araki–Kudo–Dyer–Lashof operations on σ. The general formula is that

$$H_{*,*}(E_2^+(D^{1,0}\sigma);\mathbb{F}_2) \cong \mathbb{F}_2[Q^I\sigma \mid I \text{ admissible}],$$

where I runs over sequences $(s_1, \ldots, s_r)$ with $2s_j \geq s_{j-1}$, $\sum_{j=2}^{r}(2s_j - s_{j-1}) > 0$ and $s_r = 1$, and we write the top operations as a Araki–Kudo–Dyer–Lashof operation. But this simplifies quite a bit: all I are of the form $(s_1, \ldots, s_{k+1}) = (2^k, 2^{k-1}, \ldots, 1)$ and in this case $Q^I\sigma = Q^{s_1} \cdots Q^{s_{k+1}}\sigma$ has bidegree $(2^{k+1}, 2^{k+1} - 1)$. This is proven by induction: if I is admissible then so is any subsequence from the right, and on an class of degree $2^{k+1} - 1$ we can only apply $Q^{2^{k+1}-1}$ and $Q^{2^{k+1}}$ but the former is a square. (The reader might want to compare this to cohomology computations of braid groups as [Fuk70].)

The upshot is Fig. 4.2. This lists the generators in each bidegree. Taking the free graded-commutative algebra on these generators, one observes that all elements in bidegree $d < \frac{g}{2}$ are multiples of σ—*homological stability*—and once we quotient out the submodule generated by σ all elements in bidegree $d < \frac{2g}{3}$ are multiples of $Q^1\sigma$—*secondary homological stability*. We can quotient out the submodule generated by $Q^1\sigma$ and observe *tertiary homological stability*, etc.

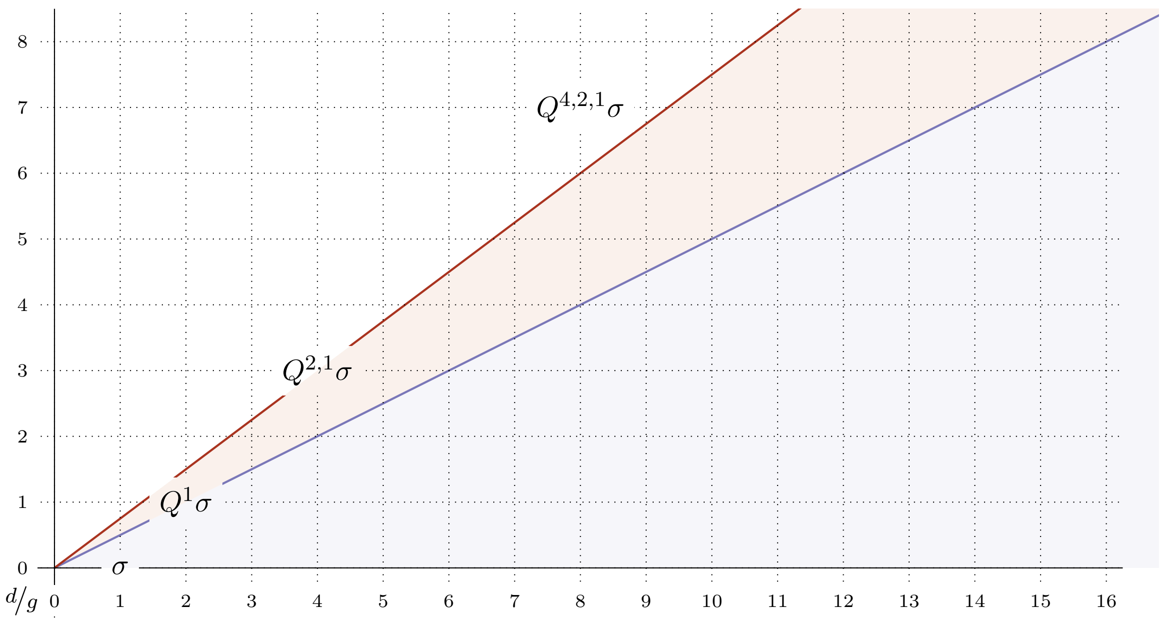


Figure 4.2: The generators of the bigraded $\mathbb{F}_2$-algebra $H_{*,*}(E_2^+(D^{1,0}\sigma); \mathbb{F}_2)$. The first index is the horizontal axis, the second index the vertical axis. The stable range is shaded blue, the metastable range orange.

4.4.2 Iterated mapping cones and adapters

In the previous example, we took the various quotients *after* taking homology. This is usually a bad idea, and we would rather take a quotient *before* taking homology.

To do so, we observe that taking the quotient of an associative algebra A by an ideal generated by an element x has the following universal property. Consider A as an A–A-bimodule. The element $x \in A$ produces by adjunction a map $x \cdot - \colon A \to A$ of

A-modules, and we can form the pushout

$$\begin{array}{ccc} A & \xrightarrow{x\cdot -} & A \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & A/(x). \end{array}$$

Mimicking this homotopy-theoretically for an E_k^+-algebra $\mathbf{R}$ for $k \geq 1$, we switch to working in $\mathsf{sMod}_{\Bbbk}^{\mathbb{N}}$ instead of $\mathsf{sSet}^{\mathbb{N}}$; this is pointed so there is a zero object 0, and is harmless because we intend to take homology anyway. Consider $\mathbf{R}$ as a $\mathbf{R}$−$\mathbf{R}$-bimodule, or if you prefer you can rectify $\mathbf{R}$ to an associative algebra $\overline{\mathbf{R}}$ first. Let $S^{h,k}$ denote the pointed $(k-1)$-sphere S^k in "genus" grading h. A map $f\colon \partial S^{h,k} \to \mathbf{R}$ corresponding to an element of $H_k(\mathbf{R}(h))$ (if we are working with simplicial $\Bbbk$-algebras a homotopy class is the same as a homology class of the corresponding chain complex under Dold–Kan, and we prefer the latter notaton). This produces by adjunction a map $f \cdot -\colon S^{h,k} \otimes \mathbf{R} \to \mathbf{R}$ of $\mathbf{R}$-modules and we can form the homotopy pushout

$$\begin{array}{ccc} \mathbf{R} \otimes S^{h,k} & \xrightarrow{f\cdot -} & \mathbf{R} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbf{R}/(f). \end{array}$$

By construction, there is a long exact sequence

$$\cdots \longrightarrow H_{g,d}(\mathbf{R}) \xrightarrow{f\cdot -} H_{g+h,d+k}(\mathbf{R}) \longrightarrow H_{g+h,d+k}(\mathbf{R}/(f)) \longrightarrow \cdots$$

so whether $f \cdot -$ is a surjection or isomorphism in a range can be deduced by studying $\mathbf{R}/(f)$.

The difficulty with this construction is that it can not be iterated, because we "use up" one of the module structures each time we take a quotient. This can be resolved for $k \geq 2$: $\mathbf{R}$ admits as many commuting $\mathbf{R}$-module structures as you would like. Explicitly, this may be achieved by the device of an *adapter*, analogously to using Moore loops to construct rectify an E_1^+-algebra to an associative algebra. The construction of an adapter uses the geometry of the k-dimensional cube with $k \geq 2$ to produce three or more commuting $\mathbf{R}$-module structures; Fig. 4.3 should give you an impression of how to get three such structures.

We may then take iterated cofibers to our heart's content, writing

$$\mathbf{R}/(f_1, f_2, \cdots) \coloneqq ((\mathbf{R}/(f_1))/(f_2))/\cdots).$$

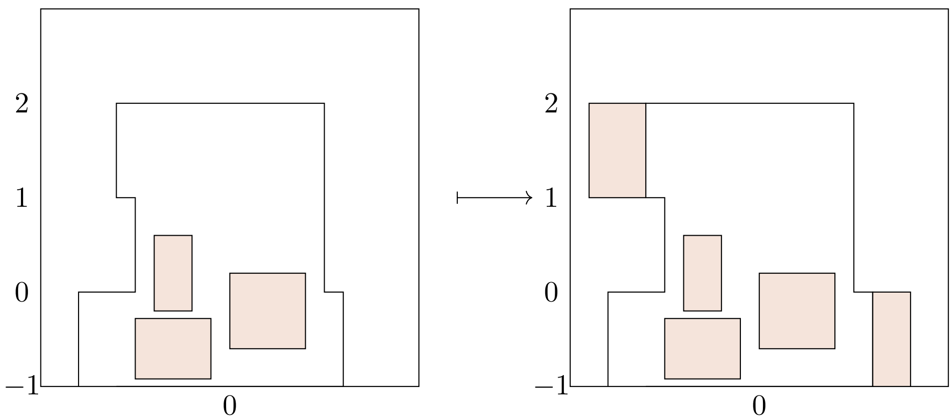


Figure 4.3: Heuristically, the result of using simultaneously an "upper left" $\mathbf{R}$-module structure and a right $\mathbf{R}$-module structure in an adapter.

Chapter 5

Indecomposables II: Indecomposables in other categories

In this lecture we discuss some generalizations of the setup from the first lecture. First we replace simplicial sets by simplicial modules over a ring, and how to transport definitions to that setting. Then we discuss how to do something similar in functor categories, which is the setting we really need for applications.

In [GKRW18a] we take an axiomatic approach to these different "settings": write a list of axioms, prove theorems based on those axioms, and then observe that simplicial sets, simplicial modules, and functor categories satisfy the axioms. In the lectures we will take the opposite approach, presenting the main ideas in a slightly simplified setting before generalizing. Hopefully that will help keeping new concepts apart somewhat.

5.1 Simplicial modules

Let $\Bbbk$ be a commutative ring (typically: $\mathbb{Z}$ or a field) and let $s\mathrm{Mod}_{\Bbbk}$ be the category of simplicial $\Bbbk$-modules. Given $X, Y \in s\mathrm{Mod}_{\Bbbk}$ we define $X \otimes Y \in s\mathrm{Mod}_{\Bbbk}$ to be the tensor product over $\Bbbk$ (in each simplicial degree). Given a simplicial set K we let $\Bbbk[K] \in s\mathrm{Mod}_{bk}$ be the free simplicial $\Bbbk$-module on K. Using this we define a functor

$$\begin{aligned} E_k \colon s\mathrm{Mod}_{\Bbbk} &\longrightarrow s\mathrm{Mod}_{\Bbbk} \\ X &\longmapsto \coprod_{n=1}^{\infty} (\Bbbk[\mathcal{C}_k(n)] \otimes X^{\otimes n})/S_n. \end{aligned}$$

Clarification: we have written $\coprod$ for the coproduct in the category $s\mathrm{Mod}_{\Bbbk}$, which more explicitly is the direct sum. Likewise, the quotient by the action of S_n should be formed in the category $s\mathrm{Mod}_{\Bbbk}$, which more explicitly means passing to coinvariants.

As in sSets, the operad structure on $\mathcal{C}_k$ again gives the functor $E_k \colon s\mathrm{Mod}_{\Bbbk} \to s\mathrm{Mod}_{\Bbbk}$ the structure of a monad, and we define $\mathrm{Alg}_{E_k}(s\mathrm{Mod}_{\Bbbk})$ to be the category of algebras for this monad. We again have a free functor

$$F^{E_k} \colon s\mathrm{Mod}_{\Bbbk} \longrightarrow \mathrm{Alg}_{E_n}(s\mathrm{Mod}_{\Bbbk})$$

left adjoint to the forgetful functor in the other direction. Using this, we define cell attachments $A \cup_e^{E_k} D^d$ for $A \in \mathrm{Alg}_{E_k}(s\mathrm{Mod}_{\Bbbk})$, along an attaching map $e \colon \partial D^d \to A$ of underlying simplicial sets, as the pushout

$$\begin{array}{ccc} F^{E_k}(\Bbbk[\partial D^d]) & \xrightarrow{e} & A \\ \downarrow & & \downarrow \\ F^{E_k}(\Bbbk[D^d]) & \longrightarrow & A \cup_e^{E_k} D^d \end{array}$$

in the category $\mathrm{Alg}_{E_k}(s\mathrm{Mod}_\Bbbk)$.

Using this, we define cellular algebras and derived indecomposables as indecomposables of a cellular approximation, as we did for E_k-algebras in simplicial sets: first define underived indecomposables of $A \in \mathrm{Alg}_{E_k}(s\mathrm{Mod}_\Bbbk)$ as the cokernel of $\bigsqcup_{n=2}^{\infty}(\Bbbk[\mathcal{C}_k(n)] \otimes A^{\otimes n})/S_n \to A$, then derived indecomposables is indecomposables of a cellular approximation.

5.1.1 Linearization

The functor $K \mapsto \Bbbk[K]$ from sSets to $s\mathrm{Mod}_\Bbbk$ is symmetric monoidal (and left adjoint and compatible with the copowering over simplicial sets). Therefore, an E_k-algebra structure on $A \in s$Sets gives rise to an E_k-algebra structure on $\Bbbk[A] \in s\mathrm{Mod}_\Bbbk$. This is compatible with indecomposables, in the sense that the diagram

$$\begin{array}{ccc} \mathrm{Alg}_{E_k}(s\mathrm{Sets}) & \longrightarrow & \mathrm{Alg}_{E_k}(s\mathrm{Mod}_\Bbbk) \\ \downarrow{\scriptstyle Q^{E_k}} & & \downarrow{\scriptstyle Q^{E_k}} \\ s\mathrm{Sets}_* & \longrightarrow & s\mathrm{Mod}_\Bbbk \end{array}$$

commutes up to natural isomorphism. Here the bottom horizontal map denotes free $\Bbbk$-module relative to the basepoint. The proof is purely formal, using nothing but colimits commuting with other colimits...

For $A \in \mathrm{Alg}_{E_k}(s\mathrm{Sets})$ we shall often be interested in the *homology* of the underlying simplicial set (e.g. for proving homological stability). Since the homology of $X \in s$Sets is the homotopy of $\Bbbk[X]$ (or equivalently, the homology of the chain complex associated to the simplicial abelian group $\Bbbk[X]$), we have not lost too much information by passing from A to $\Bbbk[A]$.

5.2 A Hurewicz theorem and a Whitehead theorem

The indecomposables of $A \in \mathrm{Alg}_{E_k}(s\mathrm{Mod}_\Bbbk)$ are defined as the quotient of $A \in s\mathrm{Mod}_\Bbbk$ by something. In particular, there is a canonical quotient map

$$A \longrightarrow Q^{E_k}(A).$$

For a cellular approximation $A' \to A$ there is therefore a zig-zag

$$A \xleftarrow{\simeq} A' \to Q^{E_k}(A') \simeq Q^{E_k}_{\mathbb{L}}(A),$$

inducing a well defined map on homotopy groups

$$\pi_d(A) \to \pi_d(Q_{\mathbb{L}}^{E_k}(A)) = H_d^{E_k}(A).$$

By passing to mapping cones, there is also a relative version of this "Hurewicz map" for a morphism $f : A \to B$ in $\mathrm{Alg}_{E_k}(s\mathrm{Mod}_{\Bbbk})$

The main result about this map is the following "Hurewicz theorem" which we will use but not prove in these lectures (similar results obtained by Basterra–Mandell and Harper–Hess; recent results of Heuts give sharp conditions under which generalizations of "indecomposables detect weak equivalences" hold):

Theorem 5.2.1. *Let $A, A \in \mathrm{Alg}_{E_k}(s\mathrm{Mod}_{\Bbbk})$ satisfy*[1] $\pi_0(A) = 0 = \pi_0(A)$*. Let $f : A \to B$ be morphism, and assume the underlying map of simplicial sets is n-connective, i.e. that $\pi_i(B, A) = 0$ for $i < n$. Then the map*

$$\pi_n(B, A) \to \pi_n(Q_{\mathbb{L}}^{E_k}(B), Q_{\mathbb{L}}^{E_k}(A))$$

is an isomorphism.

Here we define $\pi_n(A', A)$ as the homotopy[2] group of the mapping cone of f in $s\mathrm{Mod}_{\Bbbk}$.

Corollary 5.2.2. *Let $f : A \to A'$ be as in the previous theorem, and assume that $\pi_i(Q_{\mathbb{L}}^{E_k}(A'), Q_{\mathbb{L}}^{E_k}(A)) = 0$ for $i < n$. Then $\pi_i(A', A) = 0$ for $i < n$. In particular, f is a weak equivalence if and only if $Q_{\mathbb{L}}^{E_k}(f)$ is a weak equivalence, under this assumption.*

5.3 Minimal cell structures in simplicial modules

For E_k-algebras in simplicial sets, we discussed how homology of relative indecomposables give a lower bound on the numbers of d-cells needed in a cellular approximation. For E_k-algebras in simplicial modules over a field $\Bbbk$, the same argument applies, showing that any cellular approximation $A' \to A$ must have at least

$$\dim_{\Bbbk}(\pi_d(Q_{\mathbb{L}}^{E_k}(A))) = \dim_{\Bbbk} H^{E_k}(A)$$

many cells of dimension d. Sometimes this bound is optimal:

Proposition 5.3.1. *Let $\Bbbk$ be a field and let $A \in \mathrm{Alg}_{E_k}(s\mathrm{Mod}_{\Bbbk})$ have $\pi_0(A) = 0$. Assume that $A' \in \mathrm{Alg}_{E_k}(s\mathrm{Mod}_{\Bbbk})$ is cellular, is built using precisely $\dim_{\Bbbk} H_d^{E_k}(A)$ many cells of dimension d for $d < n$ and no cells of dimension $\geq n$, and that there is given a map $A' \to A$ inducing an isomorphism in $H_d^{E_k}$ for $d < n$.*

Then there exists a cellular $A'' \in \mathrm{Alg}_{E_k}(s\mathrm{Mod}_{\Bbbk})$ obtained by attaching precisely $\dim_{\Bbbk}(H_n^{E_k}(A))$ many d-cells to A' and no other cells, and a morphism $A'' \to A$ inducing an isomorphism in $H_d^{E_k}$ for $d \leq n$.

[1] Recall that we work with E_k-algebras which don't have "units"

[2] In the paper we write this as $H_n(A', A)$, thinking of it as relative homology of the chain complexes associated to A' and A.

By induction, there exists a cellular approximation $A' \to A$ with precisely $\dim_{\Bbbk} H_d^{E_k}(A)$ many d-cells for all d.

About proof. The idea is to choose a basis for the $\Bbbk$-vector space

$$\pi_n(A, A') \xrightarrow{\cong} H_n^{E_k}(A, A') \xleftarrow{\cong} H_n^{E_k}(A).$$

Here the first isomorphism is the Hurewicz theorem above, and the second follows from the long exact sequence of (A', A) in E_k-homology.

Then we represent each basis element by a map of simplicial sets $(\Delta^n, \partial\Delta^n) \to (A', A)$. Such a map is precisely the data needed for attaching a cell to A' and extending the map to A. If $A'' \to A$ is the result of attaching these cells, one then checks that $H_d^{E_k}(A'', A) = 0$ for $d \leq n+1$ because the cells precisely kill a basis for $H_{n+1}^{E_k}(A', A)$. The result then follows from long exact sequences. □

5.4 Functor categories

In this section we discuss how the categories $s\mathrm{Sets}$ and $s\mathrm{Mod}_{\Bbbk}$ may be replaced by functor categories, for example

$$s\mathrm{Mod}_{\Bbbk}^{\mathsf{G}} = \text{functors } \mathsf{G} \to s\mathrm{Mod}_{\Bbbk}.$$

We have a ($\Bbbk$-linearized) Yoneda embedding

$$\begin{aligned} \mathsf{G}^{\mathrm{op}} &\to s\mathrm{Mod}_{\Bbbk}^{\mathsf{G}} \\ g &\mapsto \Bbbk[\mathsf{G}(g, -)] \end{aligned} \tag{5.1}$$

as well as

$$\begin{aligned} s\mathrm{Sets} &\to s\mathrm{Mod}_{\Bbbk}^{\mathsf{G}} \\ K &\mapsto \Bbbk[K \times \mathsf{G}(1_{\mathsf{G}}, -)], \end{aligned} \tag{5.2}$$

where $1_{\mathsf{G}} \in \mathsf{G}$ is the monoidal unit.

If G is given a monoidal structure $\oplus$, there is an essentially unique monoidal structure $\otimes$ on $s\mathrm{Mod}_{\Bbbk}^{\mathsf{G}}$ making (5.1) into a monoidal functor, and such that $\otimes$ preserves colimits in each variable separately. This is the *Day convolution*, given explicitly by the formula

$$(X \otimes Y)(g) = \mathrm{colim}_{g_1 \oplus g_2 \to g} X(g_1) \otimes_{\Bbbk} Y(g_2), \tag{5.3}$$

where the colimit is over the category whose objects are triples (g_1, g_2, f) consisting of $g_1, g_2 \in \mathsf{G}$ and a map $g_1 \oplus g_2 \to g$. Similarly when G is braided monoidal or symmetric monoidal.

A simple example which is relevant to us is $\mathsf{G} = \mathbb{N}$, the category whose objects are the non-negative integers, and which has only identity morphisms. In this case the formula spells out to $(X \otimes Y)(n) = X(n) \otimes Y(0) \amalg \cdots \amalg X(0) \otimes Y(n)$. In this case we are essentially just "keeping track of an extra grading". (Clarification: $\amalg$ denotes the coproduct in $s\mathrm{Mod}_{\Bbbk}$, i.e. the direct sum.)

Using (5.3) and (5.2), we define a functor $E_k : s\mathrm{Mod}_{\mathbb{k}}^{\mathsf{G}} \to s\mathrm{Mod}_{\mathbb{k}}^{\mathsf{G}}$ as

$$E_k(X) = \coprod_{n=1}^{\infty} (\mathbb{k}[\mathcal{C}_k(n) \times \mathsf{G}(1,-)] \otimes X^{\otimes n})/S_n. \tag{5.4}$$

As before, composition of embeddings $I^k \to I^k$ gives this functor E_k the structure of a monad, and we define E_k-algebras in $s\mathrm{Mod}_{\mathbb{k}}^{\mathsf{G}}$ as algebras for this monad: i.e., functors $A : \mathsf{G} \to s\mathrm{Mod}_{\mathbb{k}}$ equipped with a map $\mu : E_k(A) \to A$ satisfying some condition. We will write

$$\mathrm{Alg}_{E_k}(s\mathrm{Mod}_{\mathbb{k}}^{\mathsf{G}})$$

for the category of E_k-algebras in $s\mathrm{Mod}_{\mathbb{k}}^{\mathsf{G}}$.

Remark 5.4.1. The formula (5.4) only makes sense in the symmetric monoidal case, since otherwise we don't have a well defined action of S_n on $X^{\otimes n}$. For $k = 2$ the formula can be rewritten as

$$E_2(X) = \coprod_{n=1}^{\infty} (\mathbb{k}[\widetilde{\mathcal{C}_k(n)} \times \mathsf{G}(1,-)] \otimes X^{\otimes n})/B_n,$$

where $\widetilde{\mathcal{C}_k(n)}$ denotes a certain universal cover, and B_n denotes the braid group. This formula also makes sense when G is only braided monoidal, and defines a monad. Hence E_2 algebras in $s\mathrm{Mod}_{\mathbb{k}}^{\mathsf{G}}$ make sense in this case, which we shall use. (But E_3 and higher does not make sense.)

For example, if we are interested in homology of the mapping class groups $\Gamma_{g,1}$, we study the object

$$(g \mapsto \mathbb{k}[N\Gamma_{g,1}]) \in s\mathrm{Mod}_{\mathbb{k}}^{\mathbb{N}}.$$

Finally, there is a notion attaching a cell to $A \in \mathrm{Alg}_{E_k}(s\mathrm{Mod}_{\mathbb{k}}^{\mathsf{G}})$, whose input is an "attaching" map $e : \partial D^d \to A(g)$ for some $g \in \mathsf{G}$ and $d \in \mathbb{N}$. The object G represents a functor $\mathsf{G}(g,-) \in \mathrm{Sets}^{\mathsf{G}}$ and the attaching map corresponds to a map

$$\partial D^d \times \mathbb{k}[\mathsf{G}(g,-)] \to A$$

in $s\mathrm{Mod}_{\mathbb{k}}^{\mathsf{G}}$. Cell attachment along e is then defined as the pushout in $\mathrm{Alg}_{E_k}(s\mathrm{Mod}_{\mathbb{k}}^{\mathsf{G}})$

$$\begin{array}{ccc} F^{E_k}(\partial D^d \times \mathbb{k}[\mathsf{G}(g,-)]) & \xrightarrow{e} & A \\ \downarrow & & \downarrow \\ F^{E_k}(D^d \times \mathbb{k}[\mathsf{G}(g,-)]) & \longrightarrow & A \cup_e^{E_k} D^{g,d}, \end{array}$$

In the paper, we write the object $D^d \times \mathbb{k}[\mathsf{G}(g,-)]$ as $D^{g,d}$, and similarly for $\partial D^{g,d}$. Thus we see that each cell has a *bidegree* $(g,d) \in \mathsf{G} \times \mathbb{N}$.

Indecomposables and derived indecomposables are defined by the same method as before, and are functors

$$\mathrm{Alg}_{E_k}(s\mathrm{Mod}_{\mathbb{k}}^{\mathsf{G}}) \to s\mathrm{Mod}_{\mathbb{k}}^{\mathsf{G}},$$

and E_k homology is defined as

$$H_{g,d}^{E_k}(A) = \pi_d(Q_{\mathbb{L}}^{E_k}(A)(g)).$$

For the same formal reasons as before, when $\Bbbk$ is a field, the dimension of this vector space is a lower bound on the numbers of (g,d)-cells in a cellular approximation to A.

There is again a convenient criterion for when this may be realized:

Proposition 5.4.2. *Assume* G *is a groupoid*[3]*,* $\Bbbk$ *is a field, and that* $\pi_0(A(g)) = 0$ *when* g *is invertible in the monoidal structure (i.e. that there exists* g' *such that* $g \oplus g'$ *is isomorphic to the monoidal unit). Assume furthermore that there exists*[4] *a function*[5] $\omega : \mathsf{G}/iso \to \mathbb{N}$ *such that* $\omega(g \oplus g') \geq \omega(g) + \omega(g')$ *and such that* $\omega(g) > 0$ *when* g *is not invertible in the monoidal structure.*

Then there exists a cellular $A' \in \mathrm{Alg}_{E_k}(s\mathrm{Mod}_{\Bbbk}^{\mathsf{G}})$ *built using precisely* $\dim H^{E_k}_{g,d}(A)$ *many cells of dimension* (g,d)*.*

About proof. One first proves a Hurewicz theorem, asserting that

$$\pi_d(A(g), B(g)) \to H^{E_k}_{g,d}(A, B)$$

is an isomorphism if these groups vanish in "smaller" bidegrees (in a suitable ordering on $\mathsf{G} \times \mathbb{N}$). The proof then produces a cellular approximation inductively on bidegrees, in each step using this Hurewicz theorem to produce attaching maps. The assumptions about existence of ω ensure that this works. □

5.5 Filtrations

As a special case of functor categories, we consider functors out of $\mathbb{Z}_{\leq}$: the category with objects $n \in \mathbb{Z}$, a single morphism $n \to m$ if $n \leq m$ and none otherwise, with the symmetric monoidal structure given by addition. Functors $\mathbb{Z}_{\leq} \to s\mathrm{Mod}_{\Bbbk}$ are *filtered objects* of $s\mathrm{Mod}_{\Bbbk}$. Beware that we do not require $n \to n+1$ to go to an "injection", as one sometimes does when considering filtered objects. We also write $\mathbb{Z}_{=}$ for the category with only identity morphisms. Then we have two important functors

$$\mathrm{gr} : s\mathrm{Mod}_{\Bbbk}^{\mathbb{Z}_{\leq}} \to s\mathrm{Mod}_{\Bbbk}^{\mathbb{Z}_{=}}$$
$$\mathrm{colim} : s\mathrm{Mod}_{\Bbbk}^{\mathbb{Z}_{\leq}} \to s\mathrm{Mod}_{\Bbbk}.$$

The first is *associated graded*, which to $X \in s\mathrm{Mod}_{\Bbbk}^{\mathbb{Z}_{\leq}}$ associates

$$\mathrm{gr}(X) : \mathbb{Z}_{=} \to s\mathrm{Mod}_{\Bbbk}$$
$$n \mapsto X(n)/X(n-1).$$

The second, denoted colim, simply takes colimit over $\mathbb{Z}_{\leq}$. Both are compatible with the symmetric monoidal structures given by Day convolution, and induce functors on algebra

[3] I don't think this is completely essential

[4] In the paper we call the existence of such a functor ω that G is "Artinian". It plays a role somewhat similar to that of a discrete valuation on a local ring.

[5] In the paper we express this as a symmetric monoidal *functor* $\mathsf{G} \to \mathbb{N}_{\leq}$

categories fitting into diagrams

$$\begin{array}{ccc} \mathrm{Alg}_{E_k}(s\mathrm{Mod}_{\Bbbk}^{\mathbb{Z}_{\leq}}) & \xrightarrow{\mathrm{gr}} & \mathrm{Alg}_{E_k}(s\mathrm{Mod}_{\Bbbk}^{\mathbb{Z}_{=}}) \\ \big\downarrow{\scriptstyle Q^{E_k}} & & \big\downarrow{\scriptstyle Q^{E_k}} \\ s\mathrm{Mod}_{\Bbbk}^{\mathbb{Z}_{\leq}} & \xrightarrow{\quad\mathrm{gr}\quad} & s\mathrm{Mod}_{\Bbbk}^{\mathbb{Z}_{=}}. \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Alg}_{E_k}(s\mathrm{Mod}_{\Bbbk}^{\mathbb{Z}_{\leq}}) & \xrightarrow{\mathrm{colim}} & \mathrm{Alg}_{E_k}(s\mathrm{Mod}_{\Bbbk}) \\ \big\downarrow{\scriptstyle Q^{E_k}} & & \big\downarrow{\scriptstyle Q^{E_k}} \\ s\mathrm{Mod}_{\Bbbk}^{\mathbb{Z}_{\leq}} & \xrightarrow{\quad\mathrm{colim}\quad} & s\mathrm{Mod}_{\Bbbk}. \end{array}$$

We will use this in the following way: Given $A \in \mathrm{Alg}_{E_k}(s\mathrm{Mod}_{\Bbbk})$, we look for a $C \in \mathrm{Alg}_{E_k}(s\mathrm{Mod}_{\Bbbk}^{\mathbb{Z}_{\leq}})$ and an $X \in s\mathrm{Mod}_{\Bbbk}^{\mathbb{Z}_{=}}$, and weak equivalences

$$\begin{aligned} \mathrm{colim}(C) &\simeq A \\ \mathrm{gr}(C) &\simeq F^{E_k}(X). \end{aligned} \tag{5.5}$$

Indeed, this situation gives a spectral sequence whose E^1-page is homotopy of the free E_k-algebra $\mathrm{gr}(C)$, converging to homotopy of $\mathrm{colim}(C)$. This can be useful because the homotopy of free E_k algebras is well understood. (Recall that homotopy of an object of $s\mathrm{Mod}_{\Bbbk}$ is homology of the corresponding chain complex.)

The above discussion can[6] be repeated with $s\mathrm{Mod}_{\Bbbk}$ replaced by $s\mathrm{Mod}_{\Bbbk}^{\mathsf{G}}$ and hence $s\mathrm{Mod}_{\Bbbk}^{\mathbb{Z}_{\leq}}$ replaced by $s\mathrm{Mod}_{\Bbbk}^{\mathsf{G}\times\mathbb{Z}_{\leq}}$. We then have the following result about when such a "multiplicative filtration" of A with the minimal number of cells may be achieved.

Theorem 5.5.1. *In the situation of Proposition 5.4.2, there exists a* $C \in \mathrm{Alg}_{E_k}(s\mathrm{Mod}_{\Bbbk}^{\mathsf{G}\times\mathbb{Z}_{\leq}})$, *an* $X \in s\mathrm{Mod}_{\Bbbk}^{\mathsf{G}\times\mathbb{Z}_{=}}$ *and weak equivalences* (5.5)*, such that*

$$\dim_{\Bbbk}(\pi_d(X(g,n))) = \begin{cases} \dim_{\Bbbk}(H_{g,d}^{E_k}(A)) & \text{for } d = n \\ 0 & \text{otherwise.} \end{cases}$$

About proof. We construct C by a filtered analogue of cell attachments. The main new idea is to give D^d filtration d and $\partial D^d \subset D^d$ filtration $d-1$. That the disk and its boundary have different filtration causes the attaching maps to be trivial in the associated graded, which makes the associated graded free: it is obtained by iterated cell attachments along trivial attaching maps.

That this is possible using the minimal number of cell attachments consistent with the E_k-homology of A again uses a Hurewicz theorem, similar to what we discussed before. □

[6] in the actual lecture I'll probably state this only for trivial G

Chapter 6

Facts about mapping class groups II: low-degree homology

6.1 Statements

A lot is known about the homology groups $H_d(\Gamma_{g,1};\mathbb{Z})$ in low degrees or low genus, due to the work of many people[1], which in the range we will need can be summarised as follows.

d/g	0	1	2	3	4
2			$\mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}$
1		$\mathbb{Z}$	$\mathbb{Z}/10$		
0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$

In this talk I want to give some idea of the following, a slight simplification of Lemma 3.6 of [GKRW19], which explains the information about the homology of mapping class groups that we shall need. Write $\sigma \in H_0(\Gamma_{1,1};\mathbb{Z})$ for the generator, and I will explain the rest of the notation along the way.

Theorem 6.1.1.

(i) $H_1(\Gamma_{1,1};\mathbb{Z}) = \mathbb{Z}\{\tau\}$,

(ii) $H_1(\Gamma_{2,1};\mathbb{Z}) = \mathbb{Z}/10\{\sigma\tau\}$,

(iii) $H_1(\Gamma_{g,1};\mathbb{Z}) = 0$ *for* $g \geq 3$,

(iv) $H_2(\Gamma_{1,1};\mathbb{Z})$ *is zero,*

(v) $H_2(\Gamma_{2,1};\mathbb{Z})$ *is torsion,*

(vi) $H_2(\Gamma_{3,1};\mathbb{Z})/\mathrm{Im}(\sigma \cdot -) = \mathbb{Z}\{\lambda\}$,

[1] Abhau, Benson, Bödigheimer, Boes, F. Cohen, Ehrenfried, Godin, Harer, Hermann, Korkmaz, Looijenga, Meyer, Morita, Mumford, Pitsch, Sakasai, Stipsicz, Tommasi, Wang, ...

(vii) $H_2(\Gamma_{4,1};\mathbb{Z}) = \mathbb{Z}\{\sigma\lambda\}$,

(viii) *the extension*

$$0 \longrightarrow H_2(\Gamma_{3,1};\mathbb{Z})/\mathrm{Im}(\sigma\cdot -) \longrightarrow H_2(\Gamma_{3,1},\Gamma_{2,1};\mathbb{Z}) \longrightarrow H_1(\Gamma_{2,1};\mathbb{Z}) \longrightarrow 0$$

is isomorphic to

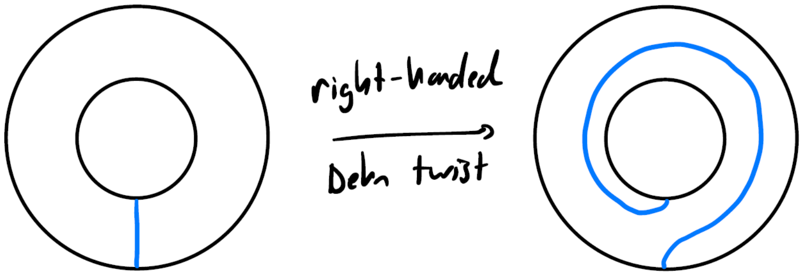
$$0 \longrightarrow \mathbb{Z}\{\lambda\} \xrightarrow{\lambda\mapsto 10\mu} \mathbb{Z}\{\mu\} \xrightarrow{\mu\mapsto u\cdot\sigma\tau} \mathbb{Z}/10\{\sigma\tau\} \longrightarrow 0$$

for some unit $u \in (\mathbb{Z}/10)^{\times}$ *(in fact* $u = 1$*, but we will not need this).*

6.2 Presentations of mapping class groups.

The most basic diffeomorphism of an oriented surface is the (right-handed) *Dehn twist*: the diffeomorphism of the cylinder relative to its boundary as shown below.

Any simple closed curve c on an oriented surface Σ has a neighbourhood oriented diffeomorphic to the cylinder, and the (right-handed) Dehn twist along c is the (isotopy class of) diffeomorphism τ_c obtained by implanting the Dehn twist in this neighbourhood. If ϕ is another diffeomorphism, this description makes it clear that $\phi\tau_c\phi^{-1} = \tau_{\phi(c)}$.



As the figure above shows, the best way of thinking about diffeomorphisms of surfaces is to consider how they act on (curves and) arcs. The action of τ_c on an arc $a \subset \Sigma$ is simple: after putting these in general position, $\tau_c(a)$ is the arc obtained by following a and taking a detour along c at each intersection point.

Lemma 6.2.1. *The stabiliser of an isotopy class (with fixed endpoints) of arc* $[a]$ *for the* $\Gamma(\Sigma)$*-action is the subgroup* $\Gamma(\Sigma \setminus \mathrm{nbhd}(a)) \leq \Gamma(\Sigma)$.

This gives a method for proving identities in the mapping class group: to show that a diffeomorphism ϕ is trivial act on some arc a, and show that $\phi(a)$ is isotopic to a; if so, ϕ may be isotoped to fix a, then removing a reduces us to the analogous question on a simpler surface; finally, use that the mapping class group of a disc is trival. The relations below may all be proved using this method.

6.2.1 The braid relation.

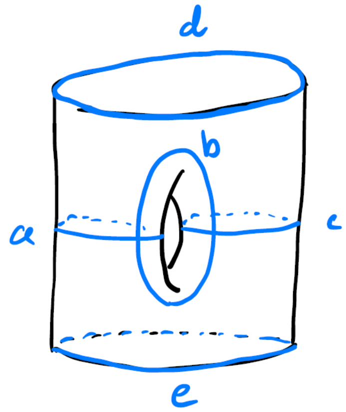
If a and b are disjoint simple closed curves, then

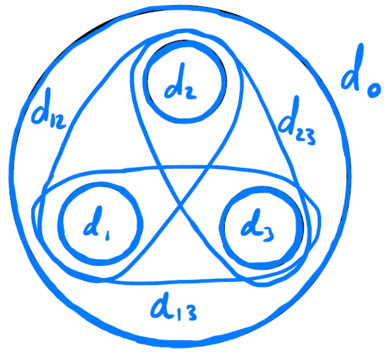
$$[\tau_a, \tau_b] = \tau_a\tau_b\tau_a^{-1}\tau_b^{-1} = \tau_{\tau_a(b)}\tau_b^{-1} = e$$

using that $\tau_a(b) = b$ because the support of the diffeomorphism τ_a is disjoint from b. If instead a and b intersect at precisely one point then one can see $\tau_a \tau_b(a) = b$ and so

$$\tau_a \tau_b \tau_a = (\tau_a \tau_b \tau_a (\tau_a \tau_b)^{-1}) \tau_a \tau_b = \tau_{\tau_a \tau_b(a)} \tau_a \tau_b = \tau_b \tau_a \tau_b.$$

(These are completely analogous to the relations in the braid group, for elementary braids which are (i) not adjacent and (ii) adjacent.)





6.2.2 The two-holed torus relation.

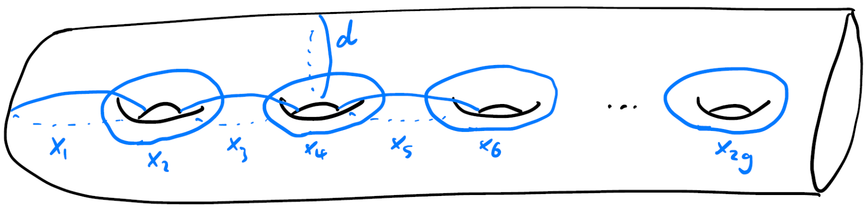
On the torus with two discs removed, let a, b, c, d, e be the simple closed curves shown in in the left-hand figure above. Then

$$(\tau_a \tau_b \tau_c)^4 = \tau_d \tau_e.$$

The lantern relation. On a sphere with four discs removed, let d_0, d_1, d_2, d_3, d_{12}, d_{13}, d_{23} be the simple closed curves shown in the right-hand figure above. Then

$$\tau_{d_0} \tau_{d_1} \tau_{d_2} \tau_{d_3} = \tau_{d_{12}} \tau_{d_{13}} \tau_{d_{23}}.$$

Theorem 6.2.2 (Wajnryb [Waj83]). *The group $\Gamma_{g,1}$ is generated by the τ_{x_i} and, for $g \geq 2$, τ_d. A complete set of relations is given by: the braid relations among these generators; the two-holed torus relation for the curves x_1, x_2, x_3; the lantern relation for the curves x_1, x_3, x_5.* □



Note that the two-holed torus and lantern relations for the indicated curves also involve other curves which are not in the listed generators: they can however to be expressed in terms of the generators, and Wajnryb does so explicitly.

In particular, as there are diffeomorphisms sending any non-separating curve to any other, all Wajnryb's generators are conjugate, so the abelianisation of the mapping class group is always cyclic, generated by any non-separating Dehn twist. More precisely, as the two-holed torus relation imposes "12 non-separating Dehn twists = 2 non-separating Dehn twists" and the lantern relation imposes "4 non-separating Dehn twists = 3 non-separating Dehn twists" we have

$$\begin{aligned} H_1(\Gamma_{1,1};\mathbb{Z}) &= \mathbb{Z} \\ H_1(\Gamma_{2,1};\mathbb{Z}) &= \mathbb{Z}/10 \\ H_1(\Gamma_{g,1};\mathbb{Z}) &= 0 \text{ for } g \geq 3 \end{aligned}$$

in all cases generated by any non-separating Dehn twist.

6.3 Second homology.

Wajnryb's presentation shows that

$$\Gamma_{1,1} = \langle \tau_{x_1}, \tau_{x_2} \,|\, \tau_{x_1}\tau_{x_2}\tau_{x_1} = \tau_{x_2}\tau_{x_1}\tau_{x_2} \rangle,$$

which is the Artin presentation of the braid group on 3 strands. Thus

$$B\Gamma_{1,1} \simeq (S^1 \vee S^1) \cup_{\tau_{x_1}\tau_{x_2}\tau_{x_1}\tau_{x_2}^{-1}\tau_{x_1}^{-1}\tau_{x_2}^{-1}} D^2 \cup \{\text{cells of dimension} \geq 3\}$$

whose cellular chains is

$$\mathbb{Z} \overset{0}{\longleftarrow} \mathbb{Z} \oplus \mathbb{Z} \overset{(1,-1)}{\longleftarrow} \mathbb{Z} \longleftarrow \cdots$$

giving $H_1(\Gamma_{1,1};\mathbb{Z}) = \mathbb{Z}$ (as we saw above), and $H_2(\Gamma_{1,1};\mathbb{Z}) = 0$.

More generally, recall that for a group G given by a presentation F/R, there is *Hopf's formula* $H_2(G;\mathbb{Z}) = \frac{[F,F]\cap R}{[F,R]}$ for the second homology of G. Using this, Korkmaz–Stipsicz [KS03] have shown that $H_2(\Gamma_{g,1};\mathbb{Z}) \cong \mathbb{Z}$ for $g \geq 4$, that $H_2(\Gamma_{2,1};\mathbb{Z}) = \mathbb{Z}/2$, and that $H_2(\Gamma_{3,1};\mathbb{Z})$ is either[2] $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}/2$ with the $\mathbb{Z}/2$ coming from $H_2(\Gamma_{2,1};\mathbb{Z}) = \mathbb{Z}/2$ (in fact the second case occurs).

Furthermore, it follows from their calculation that stabilisation gives

$$H_2(\Gamma_{3,1};\mathbb{Z}) \overset{\text{epi}}{\longrightarrow} H_2(\Gamma_{4,1};\mathbb{Z}) \overset{\sim}{\longrightarrow} H_2(\Gamma_{5,1};\mathbb{Z}) \overset{\sim}{\longrightarrow} \cdots .$$

Finally, using a similar presentation for Γ_g they show that for $g \geq 2$ the map $H_1(\Gamma_{g,1};\mathbb{Z}) \to H_1(\Gamma_g;\mathbb{Z})$ is iso and the map $H_2(\Gamma_{g,1};\mathbb{Z}) \to H_2(\Gamma_g;\mathbb{Z})$ is epi, and is iso for $g \geq 4$. (These also follow from homological stability done classically [Bol12, RW16].)

6.3.1 The Hodge class and second cohomology.

The composition

$$B\Gamma_{g,1} \longrightarrow B\Gamma_g \longrightarrow B\mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow B\mathrm{Sp}_{2g}(\mathbb{R}) \simeq BU(g)$$

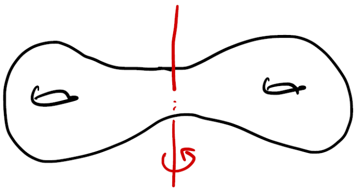
pulls back the first Chern class to a class $\lambda_1 \in H^2(\Gamma_{g,1};\mathbb{Z})$, known as the *Hodge class*. By construction the stabilisation maps $\Gamma_{g-1,1} \to \Gamma_{g,1}$ pull back λ_1 to λ_1.

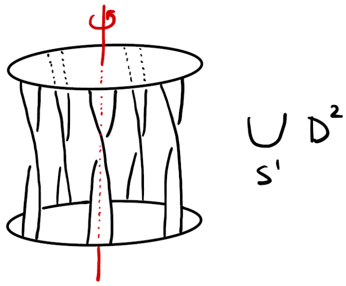
[2] Hopf's formula is not an algorithm.

Theorem 6.3.1. *λ_1 generates $H^2(\Gamma_{g,1};\mathbb{Z}) \cong \mathbb{Z}$ for all $g \geq 3$.*

Given this, there is a class $\lambda \in H_2(\Gamma_{3,1};\mathbb{Z})/\text{torsion}$ uniquely characterised by $\langle \lambda_1, \lambda \rangle = 1$, and $H_2(\Gamma_{4,1};\mathbb{Z}) = \mathbb{Z}\{\sigma\lambda\}$.

Proof sketch. By the calculations of the last section $H^2(\Gamma_{g,1};\mathbb{Z}) \cong \mathbb{Z}$ for all $g \geq 3$, and these groups are sent isomorphically to each other by stabilisation; furthermore, $H^2(\Gamma_g;\mathbb{Z}) \xrightarrow{\sim} H^2(\Gamma_{g,1};\mathbb{Z})$ for $g \geq 4$: thus it suffices to prove that λ_1 generates $H^2(\Gamma_g;\mathbb{Z})$ for $g \gg 0$, or in other words that stably it is not divisible by any prime.





For $p = 2$, consider the Riemannn surface obtained as a simply-branched double cover of an elliptic curve branched at two points. It has genus 2 by Riemann–Hurwitz, and the deck transformation gives an action of μ_2 on Σ_2, which is as shown in the left-hand figure above. Then $H^1(\Sigma_2;\mathbb{R})$, with its complex structure given by Poincaré duality and a choice of inner product, is isomorphic to

$$L^{\otimes 0} \oplus L^{\otimes 1},$$

where L is the standard $\mathbb{C}$-representation of μ_2 with $c_1(L) =: x$. The first Chern class of this representation is then $x \in H^*(\mu_2;\mathbb{Z}) = \mathbb{Z}[x]/(2x)$ so is not zero.

For p odd, consider the Riemann surface C obtained as the simply-branched double cover of $\mathbb{CP}^1$ branched over 0 and the pth roots of unity μ_p. It has genus $\frac{p-1}{2}$ by Riemann–Hurwitz, and the action of μ_p on $\mathbb{CP}^1$ can be lifted to an action on C: topologically, this is shown in the right-hand figure above. The vector space $H^1(C;\mathbb{R})$ with the complex structure given by Poincaré duality can be identified with the space of holomorphic 1-forms on C. The decomposition of this $\mathbb{C}$-vector space into irreducible μ_p-representations can be obtained from the fixed-point data (the "Eichler trace formula", cf. see [FK92, V.2]) and is

$$L^{\otimes 1} \oplus L^{\otimes 2} \oplus \cdots \oplus L^{\otimes \frac{p-1}{2}},$$

where L is the standard $\mathbb{C}$-representation of μ_p with $c_1(L) =: x$. The first Chern class is then $(\sum_{i=1}^{(p-1)/2} i)x = \frac{1}{2}\frac{p-1}{2}\frac{p+1}{2}x = -\frac{1}{8}x \in H^*(\mu_p;\mathbb{Z}) = \mathbb{Z}[x]/(px)$ which is not zero.

In both cases examples or arbitrarily large genus can be obtained by starting with Riemann surfaces of higher genus: thus λ_1 is stably not divisible by p. ☐

6.3.2 Identifying the extension.

By the Universal Coefficient Theorem, describing the extension is the same as describing the image of λ_1 under

$$H^2(\Gamma_{3,1};\mathbb{Z}) = \mathbb{Z}\{\lambda_1\} \longrightarrow H^2(\Gamma_{2,1};\mathbb{Z}) \cong \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/10\{\sigma\tau\},\mathbb{Z}).$$

For the statement we have made it suffices to show that $\lambda_1 \in H^2(\Gamma_{2,1};\mathbb{Z}) \cong \mathbb{Z}/10$ is not divisible by 2 or 5.

Just as above, using $H^2(\Gamma_2;\mathbb{Z}) \xrightarrow{\sim} H^2(\Gamma_{2,1};\mathbb{Z})$ it suffices to show that $\lambda_1 \in H^2(\Gamma_2;\mathbb{Z})$ is not divisible by 2 or 5. The actions of μ_2 and μ_5 on Σ_2 from the proof above show that it is not.

Chapter 7

E_k-algebras II: iterated bar constructions

Chapter 2 introduced the notion of the derived E_k-decomposables of (non-unital) E_k-algebras in sSet, and Chapter 5 generalised this to other categories. In this lecture we explain how these derived E_k-indecomposables can also be computed by an iterated bar construction, and give some applications.

Remark 7.0.1. So as to be more agnostic about the homotopy-theoretic foundations than in the paper, I will not be talking about cofibrancy conditions and derived functors; all objects are implicitly replaced and all functors implicitly derived.

7.1 Iterated bar constructions

We will be similarly agnostic towards the category we are working in, but you should keep in mind the examples sSet_*, $\mathsf{sSet}_*^{\mathsf{G}}$, and $\mathsf{sMod}_k^{\mathsf{G}}$. The reason we want a pointed category (i.e. the initial and terminal object coincide) will become clear momentarily.

7.1.1 Augmentations

The categories of E_k-algebras and E_k^+-algebras are not equivalent, just like how non-unital and unital commutative algebras are not.

This can be resolved by adding to the latter the data of an augmentation (at least if the category is stable). Observe that $\mathbb{1}$ is canonically an E_k^+-algebra, through the map $\mathbf{E}_k^+(\mathbb{1}) \to \mathbb{1}$ which takes each $\mathcal{C}_k(n)$ to a point. An E_k^+-algebra $\mathbf{R}$ receives a canonical map $\mathbb{1} \to \mathbf{R}$ of E_k^+-algebras, and an *augmentation* is a map $\epsilon\colon \mathbf{R} \to \mathbb{1}$ of E_k^+-algebras such that the composition

$$\mathbb{1} \longrightarrow \mathbf{R} \longrightarrow \mathbb{1}$$

is the identity.

Example 7.1.1. $\mathbb{1}$ is augmented by the identity.

Example 7.1.2. If C is pointed, then the unitalisation $\mathbf{S}^+$ of a non-unital E_k-algebra $\mathbf{S}$ has a *canonical augmentation* given on underlying objects by

$$\mathbf{S}^+ \cong \mathbf{S} \sqcup \mathbb{1} \longrightarrow \mathbb{1}$$

where the map is the identity on $\mathbb{1}$ and the zero map on $\mathbf{S}$ (this only makes sense because C is pointed).

The *augmentation ideal* $I(\mathbf{R})$ is the fiber of ϵ, which is an E_k-algebra; we say that the augmentation is *split* if the canonical map $\mathbb{1} \sqcup I(\mathbf{R}) \to \mathbf{R}$ is a weak equivalence; this is always true if C is stable. It is also true for the canonical augmentation of a unitalisation.

7.1.2 The E_k-bar construction

We will work towards the E_k-bar construction by starting with associative algebras, generalising to E_1-algebras, and finishing with E_k-algebras.

The bar construction

Recall that a semisimplicial object is a functor out of $\Delta_{\mathrm{inj}}^{\mathrm{op}}$, where $\Delta_{\mathrm{inj}} \subset \Delta$ the subcategory with the same objects but only injective morphisms. The homotopy theory of semisimplicial objects is similar to that of simplicial objects, but it will spare us having to define degeneracy maps; there are only face maps. The geometric realisation of such objects is the usual coend, and is sometimes referred to as the thick geometric realisation.

Suppose that $\mathbf{A}$ is a unital strictly associative algebra with an augmentation $\epsilon\colon \mathbf{A} \to \mathbb{1}$. Then we can form the semisimplicial object

$$[p] \longmapsto B_p(\mathbf{A}, \epsilon) \coloneqq \mathbf{A}^{\otimes p}, \tag{7.1}$$

where the ith face map uses the multiplication for $0 < i < p$ and the augmentation followed by the unit isomorphism for $i = 0, p$: the augmentation is crucial for this construction and the result will depend greatly on it.

Definition 7.1.3. The *bar construction* $B(\mathbf{A}, \epsilon)$ is the geometric realisation of (7.1).

Example 7.1.4. Maybe the following diagram for $B_\bullet(\mathbf{A}, \epsilon)$ is instructive:

$$\mathbb{1} \;\underset{\epsilon}{\overset{\epsilon}{\leftleftarrows}}\; \mathbf{A} \;\underset{\epsilon}{\overset{\epsilon}{\overset{\mu}{\Lleftarrow}}}\; \mathbf{A}^{\otimes 2} \;\Lleftarrow\!\!\leftarrow \;\cdots$$

The E_1-bar construction

If we are given an augmented E_1^+-algebra $\mathbf{R}$, we can not directly perform the previous construction as there is no canonical choice of a multiplication map making it into a unital strictly associative algebra. This can be resolved by rectifying $\mathbf{R}$ to a unital strictly associative algebra $\overline{\mathbf{R}}$ (e.g. by a Moore loops construction) but it is better to modify the construction to incorporate all operations of the E_1^+-operad.

To do so, let $\mathcal{P}(\bullet)$ be the semisimplicial space given by

$$[p] \longmapsto \{(t_0 < \ldots < t_p) \in (0,1)^{p+1}\},$$

where the ith face map deletes t_i. You should think of this as demarcating intervals in $[0,1]$. Then we can form a semisimplicial object

$$[p] \longmapsto B_p^{E_1}(\mathbf{R}, \epsilon) \coloneqq \mathcal{P}(p) \times \mathbf{R}^p, \tag{7.2}$$

where the ith face map uses a rescaled version of the rectilinear embedding $[t_{i-1}, t_i] \sqcup [t_i, t_{i+1}] \hookrightarrow [t_i, t_{i+2}]$, explicitly given by

$$I \ni t \overset{e_1}{\longmapsto} \tfrac{t_i - t_{i-1}}{t_{i+1} - t_{i-1}} t \in I \quad \text{and} \quad I \ni s \overset{e_2}{\longmapsto} \tfrac{t_i - t_{i-1}}{t_{i+1} - t_{i-1}} + \tfrac{t_{i+1} - t_i}{t_{i+1} - t_{i-1}} s \in I$$

to combine the ith and $(i+1)$st of the $\mathbf{R}$ terms using the E_1^+-algebra structure. The face maps for $i = 0, p$ are still given by the augmentation followed by the unit isomorphism.

Definition 7.1.5. The E_1*-bar construction* $B^{E_1}(\mathbf{R}, \epsilon)$ is the geometric realisation of (7.2).

The E_k-bar construction

The E_k-bar construction for augmented E_k^+-algebras is given by replacing the interval with demarcated interval with a k-dimensional cube and a grid. Recall that a k-fold semisimplicial object is a functor out of a k-fold product of $\Delta^{\mathrm{op}}_{\mathrm{inj}}$. We let $\mathcal{P}(\bullet, \dots, \bullet)$ be the k-fold semisimplicial space given by

$$[p_1, \dots, p_k] \longmapsto \mathcal{P}(p_1) \times \cdots \times \mathcal{P}(p_k),$$

consisting of elements $t_i^j \in \mathcal{P}(p_j)$. The ith face map in the j direction deletes t_i^j, see Fig. 7.1.

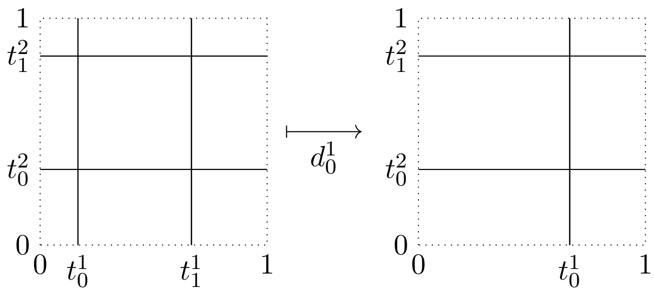


Figure 7.1: The face map of $d_0^1 \colon \mathcal{P}(1,1) \to \mathcal{P}(0,1)$.

Then we can form the k-fold semisimplicial object

$$[p_1, \dots, p_k] \longmapsto B_p^{E_k}(\mathbf{R}, \epsilon) := \mathcal{P}(p_1, \dots, p_k) \times \mathbf{R}^{p_1 \cdots p_k}, \tag{7.3}$$

One should think of this as a grid of hyperplanes in all k coordinates dividing I^k into little cubes. All of those not touching ∂I^k are labeled by $\mathbf{R}$—the *inner cubes*—the remaining ones by $\mathbb{1}$—the *outer cubes*—see Fig. 7.2.

As in the E_1-bar construction, the face maps either combine two $\mathbf{R}$'s using rescaled versions of the cubes or apply the augmentation and use the unit isomorphism. I will spare you the details, which can be found in Section 13.1 of [GKRW18a].

Definition 7.1.6. The E_k*-bar construction* is the geometric realisation of (7.3).

Example 7.1.7. We have $B^{E_k}(\mathbb{1}, \epsilon_{\mathbb{1}}) \simeq \mathbb{1}$.

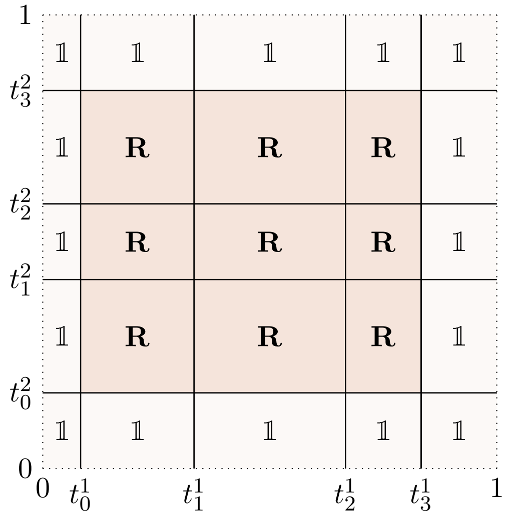


Figure 7.2: An illustration of $B^{E_2}_{3,3}(\mathbf{R}, \epsilon)$.

By definition, for any augmented E_k^+-algebra there are canonical augmented E_k^+-algebra maps $\mathbb{1} \to \mathbf{R} \to \mathbb{1}$ whose composition is the identity. Thus the E_k-bar construction of $\mathbb{1}$ is a retract of that for $\mathbf{R}$. Moreover, if the augmentation is split, then there is a splitting

$$B^{E_k}(\mathbf{R}, \epsilon) \simeq B^{E_k}(\mathbb{1}, \epsilon_{\mathbb{1}}) \sqcup \widetilde{B}^{E_k}(\mathbf{R}, \epsilon)$$

defining the right-most term, the *reduced E_k-bar construction*. In general this is the cofiber of the map in from $B^{E_k}(\mathbb{1}, \epsilon_{\mathbb{1}})$.

Remark 7.1.8. In fact, that one can form the E_k-bar construction is so inherent to notion of an E_k-algebra that one can define E_k-algebras in terms of the E_k-bar construction: this is contained in [Hau18].

7.1.3 The E_k-bar construction computes derived E_k-indecomposables

Let us now restrict to the pointed setting, so that we have a canonical augmentation $\epsilon_{\mathrm{can}} \colon \mathbf{R}^+ \to \mathbb{1}$ for any E_k^+-algebra $\mathbf{R}^+$ that is obtained as the unitalisation of a E_k-algebra $\mathbf{R}$; this has augmentation ideal $I(\mathbf{R}^+) = \mathbf{R}$ and always satisfies $\mathbf{R}^+ \simeq \mathbb{1} \vee \mathbf{R}$. We introduce the shorthand

$$\widetilde{B}^{E_k}(\mathbf{R}) \coloneqq \widetilde{B}^{E_k}(\mathbf{R}^+, \epsilon_{\mathrm{can}}).$$

The following is proven in [GKRW19, Chapter 14], but was known before in various forms (e.g. [BM11, Fra08, Fre11]).

Theorem 7.1.9. *There is a zigzag of natural weak equivalences of functors* $\mathsf{Alg}_{E_k}(\mathsf{C}) \to \mathsf{C}$

$$\widetilde{B}^{E_k}(-) \simeq S^k \wedge Q^{E_k}(-).$$

The proof proceeds along the following familiar lines: construct a zigzag of natural transformations so that all of the intermediate functors commute with geometric realisation, resolve the input by a geometric realistion of free E_k-algebras, and verify it by hand for those. It is the last two steps I want to expand upon here.

The case of a free E_k-algebra

Recall that $\mathbf{E}_k(-)$ serves as shorthand for $F^{E_k}(-)$.

Question 7.1.10. What is the functor $S^k \wedge Q^{E_k}(\mathbf{E}_k(-))$?

It suffices to understand just the term $Q^{E_k}(\mathbf{E}_k(-))$ and to do so, we observe it is a composition of two left adjoints and hence we can understand it through the composition of the corresponding right adjoints. This composition of right adjoints is $U^{E_k}(Z^{E_k}(-))$, which takes the underlying object of an object made into an E_k-algebra by endowing it with trivial E_k-algebra structure; this is just the identity functor. Hence its left adjoint $Q^{E_k}(\mathbf{E}_k(-))$ is the identity as well.

Question 7.1.11. What is the functor $\widetilde{B}^{E_k}(\mathbf{E}_k(-))$?

Let us evaluate it on $X_+ \in \mathsf{sSet}_*$; the choice of category and the disjoint basepoint is a simplification for exposition's sake. The general case is similar in spirit but has some additional technical details.

The unitalisation of $\mathbf{E}_k(X_+)$ is $\mathbf{E}_k^+(X_+)$. We next recall from the lecture on the homology of free E_k-algebras that there is a weak equivalence

$$\mathbf{E}_k^+(X_+) \longrightarrow \bigvee_{n\geq 0} \mathrm{Conf}_n(\dot{I}^k)_+ \wedge_{\mathfrak{S}_n} X_+^{\wedge n} = \mathrm{Conf}(\dot{I}^k; X)_+$$

of E_k^+-algebras, and replace $\mathbf{E}_k(X_+)$ by this labeled configuration space model. It remains to indicate why $\overline{B}^{E_k}(\mathrm{Conf}(\dot{I}^k; X)_+) \simeq S^k \wedge X$. I will be brief as Chapter 8 will give a similar proof in more detail (for bounded symmetric powers instead of labeled configuration spaces, but note that unordered configuration spaces are just instances of bounded symmetric powers).

To do so, we introduce a space Y of unordered configurations of distinct points in $\mathbb{R}^k$ labeled by X modulo the subspace where at least one point lies outside the open disc $\dot{D}^k_{10}$ of radius 10. It may be helpful to observe that this still splits as a wedge; if a particle leaves $\dot{D}^k_{10}$ it does not just disappear but takes the entire configuration to the basepoint. Define a k-fold semisimplicial pointed space

$$[p_1, \ldots, p_k] \longmapsto B_{p_1,\ldots,p_k}$$

with $B_{p_1,\ldots,p_k} \subset \mathcal{P}(p_1, \ldots, p_k)_+ \wedge Y$ consisting of those pairs of a grid and a configuration so that the configuration avoids the grids. There is a semisimplicial map

$$B_{p_1,\ldots,p_k} \longrightarrow B_{p_1,\ldots,p_k}(\mathrm{Conf}(\dot{I}^k; X)_+)$$

given by the basepoint if any of the complement of the inner cubes contains a point and recording the locations of the points in the inner squares otherwise. This is a levelwise weak equivalence by "pushing particles in outside the inner cubes out of the $\dot{D}^k_{10}$" and hence geometrically realises to a weak equivalence. Furthermore, forgetting all grids yields a map

$$|B_{\bullet,\dots,\bullet}| \xrightarrow{\sim} Y$$

which is a weak equivalence by a "microfibration argument". Finally, we compute the homotopy type of Y by a "scanning argument": we zoom in on the origin, pushing the particles out. There are three situations which can arise:

- If the configuration had more than one particle, this yields the basepoint.
- If it had no particles, this yields a copy of S^0.
- It if had one particle, this yields a $(I^k \times X)/(\partial I^k \times X) = S^k \wedge X_+$.

We conclude that

$$B^{E_k}(\mathrm{Conf}(\dot{I}^k; X)_+) \simeq S^0 \vee (S^k \wedge X_+)$$

and to obtain $\overline{B}^{E_k}(\mathrm{Conf}(\dot{I}^k; X)_+)$ we remove the first term.

Resolution by free E_k-algebras

To reduce the general case to that of free E_k-algebras, one uses the *monadic bar resolution.* This is a trick that is often in many other settings as well, and it is the homotopical version of the fact that every $\mathcal{O}$-algebra $\mathbf{A}$ can be presented as a reflexive coequaliser

$$F^{\mathcal{O}}(\mathbf{O}(A)) \underset{\text{act}}{\overset{\mathbf{O}(\text{act})}{\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}}} F^{\mathcal{O}}(A) \longrightarrow \mathbf{A},$$

with reflection given by $F^{\mathcal{O}}(\text{unit})$. This reflexive coequaliser is the beginning of an augmented simplicial object $B_\bullet(F^{\mathcal{O}}, \mathbf{O}, A)$ given by

$$[p] \longmapsto B_p(F^{\mathcal{O}}, \mathbf{O}, A) \coloneqq F^{\mathcal{O}}(\mathbf{O}^p(A)),$$

which has the same colimit: the augmentation provides an isomorphism

$$\underset{\Delta^{\mathrm{op}}}{\mathrm{colim}}\, B_\bullet(F^{\mathcal{O}}, \mathbf{O}, A) \xrightarrow{\cong} \mathbf{A}.$$

Definition 7.1.12. The *monadic bar resolution* of $\mathbf{A}$ is given by

$$\underset{\Delta^{\mathrm{op}}}{\mathrm{hocolim}}\, B_\bullet(F^{\mathcal{O}}, \mathbf{O}, A),$$

which is just the geometric realisation of this simplicial object.

The augmentation provides a map $\mathrm{hocolim}_{\Delta^{\mathrm{op}}} B_\bullet(F^{\mathcal{O}}, \mathbf{O}, A) \to \mathbf{A}$ and an extra degeneracy argument proves it is a weak equivalence.

7.2 Bar spectral sequences

One virtue of the E_k-bar construction is that it leads to geometric or combinatorial models for the derived E_k-indecomposables, at least if the E_k-algebra it is applied to is of a geometric or combinatorial origin. The latter is the case for the E_2-algebra built from mapping class groups, which arises from a braided monoidal groupoid as explained in Chapter 9.

A second virtue of the E_k-bar construction is that it is iterated. Let us also abbreviate the unreduced E_k-bar construction $B^{E_k}(\mathbf{R}, \epsilon_{\text{can}})$ to $B^{E_k}(\mathbf{R})$. If $\mathbf{R}$ is an E_k^+-algebra, then for $\ell < k$ we have that $B^{E_\ell}(\mathbf{R})$ is an $E_{k-\ell}^+$-algebra and we have a natural weak equivalence

$$B^{E_k}(\mathbf{R}) \simeq B^{E_\ell}(B^{E_{k-\ell}}(\mathbf{R})).$$

In the case $\ell = k-1$, we get $B^{E_1}(B^{E_{k-1}}(\mathbf{R}))$ where the first term is the geometric realisation of a semisimplicial object. Forgoing a "genus" grading for a moment, we get a geometric realisation spectral sequence

$$E^1_{pq} = H_q(B_p^{E_1}(B^{E_{k-1}}(\mathbf{R})); \Bbbk) \Longrightarrow H_{p+q}(B^{E_k}(\mathbf{R}); \Bbbk)$$

If $\Bbbk$ is a field $\mathbb{F}$, the right term can be identified using the Künneth theorem and the definition of the E_1-bar construction as

$$H_*(B_p^{E_1}(B^{E_{k-1}}(\mathbf{R})); \mathbb{F}) \cong H_*(B^{E_{k-1}}(\mathbf{R}); \mathbb{F})^{\otimes p}.$$

The d^1-differential is $\sum_{i=0}^p (-1)^i (d_i)_*$ so we see this is nothing but the bar complex for computing Tor of $\mathbb{F}$ against $\mathbb{F}$ over $H_*(B^{E_{k-1}}(\mathbf{R}); \mathbb{F})$. That is, we have

$$E^2_{pq} = \mathrm{Tor}_p^{H_*(B^{E_{k-1}}(\mathbf{R});\mathbb{F})}(\mathbb{F}, \mathbb{F})_q \Longrightarrow H_{p+q}(B^{E_k}(\mathbf{R}); \mathbb{F}).$$

Remark 7.2.1. Of course, instead of taking $\Bbbk$ to be a field $\mathbb{F}$, we could demand that the homology of $H_*(B^{E_{k-1}}(\mathbf{R}); \Bbbk)$ is a free $\Bbbk$-module.

This spectral sequence has one very useful consequence. Adding back in a "genus" grading, it can be used to transfer vanishing lines for E_{k-1}-homology to E_k-homology:

Theorem 7.2.2. *Let $\mathbf{R}$ be an E_k-algebra in $\mathsf{sSet}_*^{\mathbb{N}}$ and f a function $\mathbb{N} \to \mathbb{Z}$ such that $\inf\{f(g_1) + f(g_2) \mid g_1 + g_2 = g\} \geq f(g)$.*[1] *Then if $\ell < k$ is such that $H^{E_\ell}_{g,d}(\mathbf{R}; \Bbbk) = 0$ for $d < f(g) - \ell$ then $H^{E_k}_{g,d}(\mathbf{R}; \Bbbk) = 0$ for $d < f(g) - k$.*

Remark 7.2.3. To see where the shifts in the statement come from, recall that $H^{E_k}_{g,d}(\mathbf{R}; \Bbbk) = \widetilde{H}_{g,d}(Q^{E_k}(\mathbf{R}); \Bbbk) = \widetilde{H}_{g,d+k}(B^{E_k}(\mathbf{R}); \Bbbk)$ using Theorem 7.1.9 and the suspension isomorphism.

Remark 7.2.4. There is also a result for transferring vanishing lines downwards, i.e. from E_k-homology to E_{k-1}-homology. Its proof uses different techniques.

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[1] More general, one takes here a *abstract connectivity* as in [GKRW18a]. Usually f will be an affine-linear function.

Chapter 8

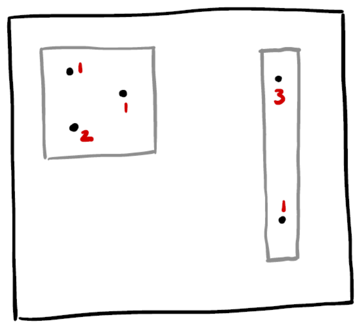
Generic homological stability I: Bounded symmetric powers

8.1 Bounded symmetric powers

We will consider the following E_2-algebra: configurations of points in $\dot{I}^2$ which are allowed to collide, but where $\leq k$ points may occupy the same position. More formally, let $\mathrm{Sym}_{\leq k}(n) \subset (\dot{I}^2)^n/\mathfrak{S}_n$ be the subspace of the nth symmetric power consisting of those unordered tuples $[x_1, x_2, \ldots, x_n]$ where no $(k+1)$ x_i's are equal. Then

$$\begin{aligned} \mathrm{Sym}_{\leq k} \colon \mathbb{N} &\longrightarrow \mathsf{Top} \\ n &\longmapsto \begin{cases} \mathrm{Sym}_{\leq k}(n) & n > 0, \\ \emptyset & n = 0, \end{cases} \end{aligned}$$

has the structure of a (nonunital) E_2-algebra in $\mathbb{N}$-graded spaces in an evident way:



8.2 Computing the E_2-homology

The goal of this lecture is to calculate $H^{E_2}_{*,*}(\mathrm{Sym}_{\leq k}; \mathbb{Z})$, in order to give an idea of how the tools developed so far may be applied.

8.2.1 The pointed setting

By adding a disjoint basepoint to each $\mathrm{Sym}_{\leq k}(n)$ we may work in the category of $\mathbb{N}$-graded (nonunital) E_2-algebras in Top_*. Let us do this without changing the notation $\mathrm{Sym}_{\leq k}$. In this case the unitalisation $\mathrm{Sym}^+_{\leq k}$ is given by $\mathrm{Sym}_{\leq k}(n)_+$ in every grading $n \geq 0$, and it has the canonical augmentation

$$\epsilon\colon \mathrm{Sym}^+_{\leq k} \longrightarrow \mathbb{1} = \begin{cases} S^0 & \text{when evaluated at } 0 \\ * & \text{otherwise} \end{cases}$$

given by sending all $\mathrm{Sym}_{\leq k}(n)$ with $n > 0$ to the basepoint.

8.2.2 The E_2 bar construction

By Theorem 7.1.9 there is a natural equivalence

$$S^2 \wedge Q^{E_2}_{\mathbb{L}}(\mathrm{Sym}_{\leq k}) \simeq \widetilde{B}^{E_2}(\mathrm{Sym}^+_{\leq k}, \epsilon)$$

relating the derived E_2-indecomposables of Chapter 2 and the reduced E_2-bar construction of Chapter 7; furthermore, the reduced and unreduced bar constructions are related by

$$S^0 \vee \tilde{B}^{E_2}(\mathrm{Sym}^+_{\leq k}, \epsilon) \simeq B^{E_2}(\mathrm{Sym}^+_{\leq k}, \epsilon).$$

It turns out that we can give a geometric description of the E_2 bar construction of $\mathrm{Sym}_{\leq k}$: in fact this is often possible for configuration-like examples, by the same argument as below which is often known as "scanning". For other kinds of examples, such as mapping class groups or general linear groups, it is usually not possible to give such a geometric description and one must proceed differently.

Lemma 8.2.1. *The space $B^{E_2}(\mathrm{Sym}^+_{\leq k}, \epsilon)(n)$ is weakly equivalent to the space of configurations of n points in $\mathbb{R}^2$ of multiplicity $\leq k$, modulo the subspace of those where at least one point lies outside the open disc $\dot{D}^2_{10}$ of radius 10.*

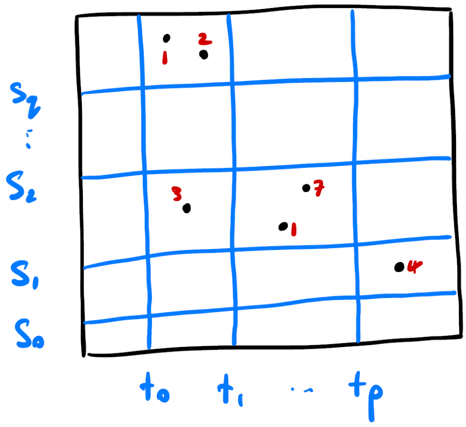
Proof. Let X denote the pointed space described. Form an augmented bi-semi-simplicial space $X_{\bullet,\bullet} \to X$ as follows. The space $X_{p,q} = X \wedge (0,1)^{p+1+q+1}_+$ is given by tuples

$$(\xi; t^1_0, t^1_1, \ldots, t^1_p; t^2_0, t^2_1, \ldots, t^2_q)$$

with $\xi \in X$, $0 < t^1_0 < t^1_1 < \ldots < t^1_p < 1$, and $0 < t^2_0 < t^2_1 < \ldots < t^2_q < 1$ such that the "walls" $\{t^1_i\} \times [0,1]$ and $[0,1] \times \{t^2_i\}$ are all disjoint from the points ξ. The face maps forget the t^1_i and t^2_j's, and the augmentation forgets all of them.

The fibre of $|X_{\bullet,\bullet}| \to X$ over $\xi \in X$ is the product of the classifying spaces of the topological posets $(0,1) \setminus \mathrm{proj}_1(\xi)$ and $(0,1) \setminus \mathrm{proj}_2(\xi)$, both of which are totally ordered and nonempty and hence have contractible classifying spaces.[1]

[1] There is a bit more to say here, as a map having contractible fibres does not suffice for it to be a weak equivalence. One also needs to know that the fibres fit together somewhat well, but in this case they do: the map $|X_{\bullet,\bullet}| \to X$ is easily seen to be a *Serre microfibration*, which suffices using "Michael Weiss' lemma" [Wei05, Lemma 2.2].



On the other hand $B^{E_2}(\mathrm{Sym}^+_{\leq k}, \epsilon)_{p,q}(n)$ is given by $0 < t^1_0 < t^1_1 < \ldots < t^1_p < 1$, and $0 < t^2_0 < t^2_1 < \ldots < t^2_q < 1$, along with a labelling of each inner cube $[t^1_i, t^1_{i+1}] \times [t^2_j, t^2_{j+1}]$ by an element of $\mathrm{Sym}_{\leq k}(n_{i,j})$, satisfying $n = \sum_{i,j} n_{i,j}$. The face maps are given by forgetting the t^1_i and t^2_j's, merging labels using the E_2-algebra structure, and applying the augmentation to any configurations which end up in an outer cube.

The difference between these two bi-semi-simplicial spaces is whether only the inner cubes may contain configuration points or not. There is a map

$$X_{p,q} \longrightarrow B^{E_2}(\mathrm{Sym}_{\leq k})_{p,q}(n)$$

given by sending $(\xi; t^1_\bullet; t^2_\bullet)$ to the basepoint if any point of ξ lies outside the inner cubes, and otherwise labelling each $[t^1_i, t^1_{i+1}] \times [t^2_j, t^2_{j+1}]$ with $[t^1_i, t^1_{i+1}] \times [t^2_j, t^2_{j+1}] \cap \xi$. This respects the face maps in both directions, and is an equivalence because if $(\xi; t^1_\bullet; t^2_\bullet)$ has some point of ξ not in an inner cube then it can be canonically contracted to the basepoint by pushing such a point outwards until it lies outside $\dot{D}^2_{10}$ □

Corollary 8.2.2. *$B^{E_2}(\mathrm{Sym}^+_{\leq k}, \epsilon)(n) \simeq *$ if $n > k$.*

Proof. If $n > k$ then a configuration of n points of multiplicity $\leq k$ must consist of at least two distinct points. All n points cannot therefore be at the origin, so scaling radially outwards from the centre gives a canonical path from any configuration to one with a point outside $\dot{D}^2_{10}$, i.e. to the basepoint. □

Corollary 8.2.3. *$B^{E_2}(\mathrm{Sym}^+_{\leq k}, \epsilon)(n) \simeq S^{2n}$ if $n \leq k$.*

Proof. If $n \leq k$ then the requirement that configurations have multiplicity $\leq k$ is redundant, so this space is given by configurations of n points in R^2 modulo those where some point lies outside $\dot{D}^2_{10}$. Equivalently, it is given by the quotient of the nth symmetric power of $D^2_{10}/\partial D^2_{10}$ by the subspace of those tuples having some point at $\partial D^2_{10}/\partial D^2_{10}$. Equivalently, identifying $D^2_{10}/\partial D^2_{10} \cong \mathbb{CP}^1$ it is the quotient of $(\mathbb{CP}^1)^n/\mathfrak{S}_n$ by $(\mathbb{CP}^1)^{n-1}/\mathfrak{S}_{n-1}$, included by adding $\infty \in \mathbb{CP}^1$ to the configuration.

The Fundamental Theorem of Algebra gives a homeomorphism

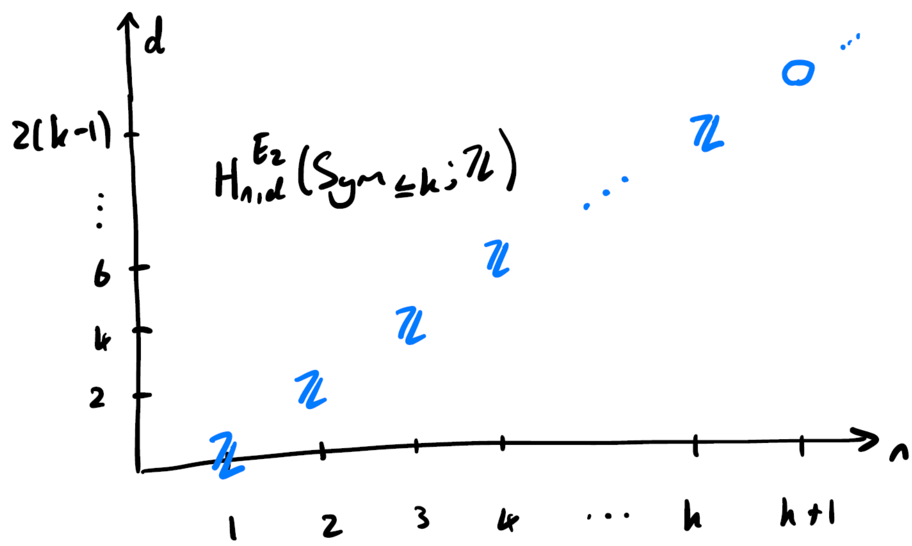
$$\mathbb{CP}^n \xrightarrow{\cong} (\mathbb{CP}^1)^n/\mathfrak{S}_n,$$

by considering $\mathbb{CP}^n$ as the projectivisation of the $(n+1)$-dimensional vector space of polynomials of degree n, and assigning to such a polynomial its unordered set of roots. (As the polynomials are not required to be monic, some of their roots may be ∞.) Using $\mathbb{CP}^n/\mathbb{CP}^{n-1} = S^{2n}$ gives the claimed result. □

Putting this together with $S^2 \wedge Q^{E_2}_{\mathbb{L}}(\mathrm{Sym}_{\leq k}) \simeq \tilde{B}^{E_2}(\mathrm{Sym}^+_{\leq k}, \epsilon)$ gives:

Theorem 8.2.4. *We have*

$$H^{E_2}_{n,*}(\mathrm{Sym}_{\leq k}; \mathbb{Z}) = \begin{cases} \mathbb{Z} & * = 2(n-1) \text{ and } n \leq k, \\ 0 & \text{otherwise.} \end{cases}$$



It is interesting to think what this means from the point of view of E_2-cells. The E_2-homology class in bidegree $(0, 1)$ corresponds to forming the free E_2-algebra on one point σ, so it gives the configuration space of distinct points. In grading 2 this has a nontrivial cycle given by interchanging two points, but as long as $k \geq 2$ this cycle is null in $\mathrm{Sym}_{\leq k}(2)$ by merging the points together: the E_2-homology class in bidegree $(2, 2)$ is an E_2-2-cell which trivialises this cycle.

8.3 Addendum: a more algebraic perspective

The only place we have used that we are working with specifically with the 2-dimensional cube is Corollary 8.2.3, but the geometric argument there can be replaced with the following algebraic argument which applies more generally.

For $n \leq k$ we have $\mathrm{Sym}_{\leq k}(n) \simeq *$, because we can scale radially *inwards* to the centre, until all n points collide: this is allowed as $n \leq k$. Thus

$$H_{n,*}(\mathrm{Sym}^+_{\leq k}; \mathbb{Z}) = \begin{cases} \mathbb{Z}\{\sigma^n\}[0] & n \leq k, \\ ? & n > k. \end{cases}$$

Using this, we can calculate $H_{n,*}(B^{E_2}(\mathrm{Sym}^+_{\leq k}, \epsilon); \mathbb{Z})$ for $n \leq k$ directly by iterating the bar construction. Consider the two bar spectral sequences

$$
\begin{aligned}
{}^{I}E^2_{n,s,t} &= \mathrm{Tor}_s^{H_{*,*}(\mathrm{Sym}^+_{\leq k};\mathbb{Z})}(\mathbb{Z}, \mathbb{Z})_{n,t} \Longrightarrow H_{n,s+t}(B^{E_1}(\mathrm{Sym}^+_{\leq k}, \epsilon); \mathbb{Z}) \\
{}^{II}E^2_{n,s,t} &= \mathrm{Tor}_s^{H_{*,*}(B^{E_1}(\mathrm{Sym}^+_{\leq k},\epsilon);\mathbb{Z})}(\mathbb{Z}, \mathbb{Z})_{n,t} \Longrightarrow H_{n,s+t}(B^{E_2}(\mathrm{Sym}^+_{\leq k}, \epsilon); \mathbb{Z}).
\end{aligned}
$$

Step 1. We have

$$H_{*,*}(\mathrm{Sym}^+_{\leq k}; \mathbb{Z}) = \mathbb{Z}[\sigma]$$

in gradings $\leq k$, so in this range of gradings we may work with the polynomial ring instead. It is well-known that

$$\mathrm{Tor}_*^{\mathbb{Z}[\sigma]}(\mathbb{Z}, \mathbb{Z})_{*,*} = \Lambda_{\mathbb{Z}}[s\sigma]$$

where $s\sigma$ has tridegree $(1, 1, 0)$, and so the first spectral sequence collapses in gradings $\leq k$, giving

$$H_{*,*}(B^{E_1}(\mathrm{Sym}^+_{\leq k}, \epsilon); \mathbb{Z}) = \Lambda_{\mathbb{Z}}[s\sigma]$$

in gradings $\leq k$, where $s\sigma$ has bidegree $(1, 1)$.

Step 2. It is also well-known that

$$\mathrm{Tor}_*^{\Lambda_{\mathbb{Z}}[s\sigma]}(\mathbb{Z}, \mathbb{Z})_{*,*} = \Gamma_{\mathbb{Z}}[s^2\sigma],$$

the free divided power algebra on a class $s^2\sigma$ of tridegree $(1, 1, 1)$, and so the second spectral sequence collapses in gradings $\leq k$, giving

$$H_{*,*}(B^{E_2}(\mathrm{Sym}^+_{\leq k}, \epsilon); \mathbb{Z}) = \Gamma_{\mathbb{Z}}[s^2\sigma]$$

in gradings $\leq k$, where $s^2\sigma$ has bidegree $(1, 2)$. Forgetting the multiplicative structure (which has no meaning anyway), this is $\mathbb{Z}$ in bidegrees $(r, 2r)$, and zero otherwise.

Putting this together with $S^0 \vee (S^2 \wedge Q^{E_2}_{\mathbb{L}}(\mathrm{Sym}_{\leq k})) \simeq B^{E_2}(\mathrm{Sym}^+_{\leq k}, \epsilon)$ and Corollary 8.2.2 gives another proof of Theorem 8.2.4.

However, working with bounded symmetric powers of $\mathrm{int}(I^d)$ instead of $\mathrm{int}(I^2)$ we still get $B^{E_d}(\mathrm{Sym}^+_{\leq k}, \epsilon)(n) \simeq *$ if $n > k$, and continuing to calculate with the bar spectral sequences gives, although these spectral sequences no longer collapse, the vanishing range

$$H_{n,*}(B^{E_d}(\mathrm{Sym}^+_{\leq k}, \epsilon); \mathbb{Z}) = 0 \text{ for } * < 2(n-1) + d \text{ and } 0 < n \leq k,$$

and so although we do not know the E_d-homology explicitly, we do get the vanishing range

$$H^{E_d}_{n,*}(\mathrm{Sym}_{\leq k}; \mathbb{Z}) = 0 \text{ for } * < 2(n-1).$$

Chapter 9

Generic homological stability II: E_2-algebras from braided monoidal groupoids

9.1 Constructing E_2-algebras

Let $(\mathsf{G}, \oplus, \mathbb{1}_{\mathsf{G}})$ be a (small) braided monoidal groupoid and $r\colon \mathsf{G} \to \mathbb{N}$ be a braided monoidal functor. For an object $x \in \mathsf{G}$, let $G_x \coloneqq \mathsf{G}(x,x)$ be its group of automorphisms. We wish to endow its classifying space

$$B\mathsf{G} \simeq \bigsqcup_{[x]\in\pi_0(\mathsf{G})} BG_x$$

with the structure of a unital $\mathbb{N}$-graded E_2-algebra.

To do so, we make some simplifying assumptions:

(i) that $r(x) = 0$ if and only if $x \cong \mathbb{1}_{\mathsf{G}}$,

(ii) that $G_{\mathbb{1}_{\mathsf{G}}}$ is trivial.

We then work in the category $\mathsf{sSet}^{\mathsf{G}} = \mathsf{Fun}(\mathsf{G}, \mathsf{sSet})$. As G is braided monoidal, the Day convolution monoidal structure on $\mathsf{sSet}^{\mathsf{G}}$ is braided too, which gives enough structure to construct the E_2-monad $E_2(-)$ on this category, and hence to discuss E_2-algebras in it. We can form the object

$$\underline{*}_{>0} : \mathsf{G} \longrightarrow \mathsf{sSet}$$
$$x \longmapsto \begin{cases} \varnothing & x \cong \mathbb{1}_{\mathsf{G}} \\ * & \text{otherwise.} \end{cases}$$

As $\varnothing$ is initial and $*$ is terminal, the endomorphism operad of $\underline{*}_{>0}$ is the terminal operad, so $\underline{*}_{>0}$ is an algebra over any operad: in particular it is an E_2-algebra. This is of course very far from being cofibrant as an E_2-algebra: the action of G_x on $* = \underline{*}_{>0}(x)$ will be free only if G_x is trivial, so it will usually not even be cofibrant in $\mathsf{sSet}^{\mathsf{G}}$. However, we can take a cellular approximation $\mathbf{T} \xrightarrow{\sim} \underline{*}_{>0}$ in $\mathsf{Alg}_{E_2}(\mathsf{sSet}^{\mathsf{G}})$.

Now using the braided monoidal functor $r : \mathsf{G} \to \mathbb{N}$ we can form the left Kan extension

$$\mathbf{R} := r_*(\mathbf{T}) \in \mathsf{Alg}_{E_2}(\mathsf{sSet}^{\mathbb{N}}),$$

which will again be cellular. Unwrapping the definition of Kan extension, for $n > 0$ we have

$$\mathbf{R}(n) = \operatorname{colim}_{\substack{x \in \mathsf{G} \\ r(x)=n}} \mathbf{T}(x) \cong \coprod_{\substack{[x]\in\pi_0(\mathsf{G}) \\ r(x)=n}} \mathbf{T}(x)/G_x \simeq \coprod_{\substack{[x]\in\pi_0(\mathsf{G}) \\ r(x)=n}} BG_x,$$

where the last identification is because $\mathbf{T}(x) \simeq *$ (as $\mathbf{T} \xrightarrow{\sim} \underline{*}_{>0}$) and because $\mathbf{T}(x)$ is a cofibrant G_x-space (as $\mathbf{T}$ is cofibrant in $\mathsf{Alg}_{E_2}(\mathsf{sSet}^{\mathsf{G}})$ and so in particular cofibrant in $\mathsf{sSet}^{\mathsf{G}}$). This construction has therefore endowed the homotopy type $B\mathsf{G}$ with the structure of a non-unital $\mathbb{N}$-graded E_2-algebra.

Remark 9.1.1. If G is *symmetric* monoidal, then we can make sense of the E_∞-monad on $\mathsf{sSet}^{\mathsf{G}}$ and repeat the above to get an $\mathbb{N}$-graded E_∞-algebra structure on $B\mathsf{G}$.

9.2 Homological stability

Suppose for simplicity that G has objects $\mathbb{N}$. To discuss $\Bbbk$-homology of the groups G_x we may as well linearise $[-]_\Bbbk : \mathsf{sSet} \to \mathsf{sMod}_\Bbbk$ and work with $\mathbf{R}_\Bbbk \in \mathsf{Alg}_{E_2}(\mathsf{sMod}_\Bbbk^{\mathbb{N}})$ and its unitalisation $\mathbf{R}_\Bbbk^+$, so that

$$H_{1,0}(\mathbf{R}_\Bbbk^+) = H_0(\mathbf{R}_\Bbbk^+(1)) = H_0(BG_1; \Bbbk).$$

Let σ denote the canonical generator of this group. It gives a map $\sigma : S_\Bbbk^{1,0} \to \mathbf{R}_\Bbbk^+$, which with the E_2-structure allows us to form

$$\sigma \cdot - : S_\Bbbk^{1,0} \otimes \mathbf{R}_\Bbbk^+ \xrightarrow{\sigma \otimes \mathrm{Id}} \mathbf{R}_\Bbbk^+ \otimes \mathbf{R}_\Bbbk^+ \xrightarrow{\cdot} \mathbf{R}_\Bbbk^+,$$

and we write $\mathbf{R}_\Bbbk^+/\sigma$ for its homotopy cofibre in $\mathsf{sMod}_\Bbbk^{\mathbb{N}}$. Unwrapping the definitions, we have

$$H_{n,d}(\mathbf{R}_\Bbbk^+/\sigma) \cong H_d(G_n, G_{n-1}; \Bbbk).$$

Thus proving ($\Bbbk$-)homological stability for the groups G_n corresponds to proving a vanishing range for the bigraded homology groups of $\mathbf{R}_\Bbbk^+/\sigma$.

9.3 Derived indecomposables.

The next thing I want to do is to obtain a "formula" for the derived E_1-indecomposables of $\mathbf{R}$. We have the formula

$$S^0 \vee S^1 \wedge Q_{\mathbb{L}}^{E_1}(\mathbf{R}) \simeq B^{E_1}(\mathbf{R}_+^+, \epsilon)$$

expressing these derived indecomposables in terms of the bar construction (after adding a basepoint and unit and taking the canonical augmentation), and as $r_* : \mathsf{sSet}^{\mathsf{G}} \to \mathsf{sSet}^{\mathbb{N}}$ is symmetric monoidal and preserves colimits we can write the latter as $r_* B^{E_1}(\mathbf{T}^+, \epsilon)$. Thus we may analyse $B^{E_1}(\mathbf{T}_+^+, \epsilon) \in \mathsf{sSet}_*^{\mathsf{G}}$.

To do so, we make a further simplifying assumption:

(iii) that $- \oplus - \colon G_x \times G_y \to G_{x \oplus y}$ is injective.

Definition 9.3.1. For $x \in \mathsf{G}$ let the *E_1-splitting complex* $S^{E_1}_\bullet(x)$ be the semi-simplicial set having

$$\mathcal{S}^{E_1}_p(x) = \operatorname*{colim}_{\substack{x_0,\ldots,x_{p+1}\\ r(x_i)>0}} \mathsf{G}(x_0 \oplus \cdots \oplus x_{p+1}, x),$$

with face maps given by using the monoidal structure to merge adjacent x_i's.

Example 9.3.2. If the objects of G are the natural numbers, then we can write this quite concretely as

$$\mathcal{S}^{E_1}_p(n) = \bigsqcup_{\substack{n_0+\cdots+n_{p+1}=n\\ n_i>0}} \frac{G_n}{G_{n_0} \times G_{n_1} \times \cdots \times G_{n_{p+1}}},$$

where we use assumption (iii) to consider $G_{n_0} \times G_{n_1} \times \cdots \times G_{n_{p+1}}$ as a subgroup of G_n.

Theorem 9.3.3. *There is a G_x-equivariant homotopy equivalence*

$$B^{E_1}(\mathbf{T}^+_+, \epsilon)(x) \simeq \Sigma^2 |S^{E_1}_\bullet(x)|.$$

Proof sketch. We have $B^{E_1}_p(\mathbf{T}^+_+, \epsilon) \simeq (\mathbf{T}^+_+)^{\otimes p}$, so that

$$B^{E_1}_p(\mathbf{T}^+_+, \epsilon)(x) \simeq \operatorname{colim}_{x_1,\ldots,x_p \in \mathsf{G}} \mathsf{G}(x_1 \oplus \cdots \oplus x_p, x)_+ \wedge \mathbf{T}^+(x_1)_+ \wedge \cdots \wedge \mathbf{T}^+(x_p)_+.$$

By assumption (iii) the group $G_{x_1} \times \cdots \times G_{x_p}$ acts freely on the set $\mathsf{G}(x_1 \oplus \cdots \oplus x_p, x)$, so as the $\mathbf{T}^+(x_i)$ are contractible it follows that

$$B^{E_1}_p(\mathbf{T}^+_+, \epsilon)(x) \simeq \operatorname{colim}_{x_1,\ldots,x_p \in \mathsf{G}} \mathsf{G}(x_1 \oplus \cdots \oplus x_p, x)_+,$$

a discrete set. Thus the semi-simplicial space $B^{E_1}_\bullet(\mathbf{T}^+_+, \epsilon)(x)$ is levelwise equivalent to a semi-simplicial set. It is not hard to identify this up to homotopy with the double suspension of $S^{E_1}_\bullet(x)$, by recognising it as a double suspension and then removing degenerate simplices. □

Corollary 9.3.4. *If $\widetilde{H}_*(|S^{E_1}_\bullet(x)|; \Bbbk) = 0$ for $* < r(x) - 2$, then $H^{E_2}_{n,d}(\mathbf{R}; \Bbbk) = 0$ for $d < n - 1$*

Proof. Forming the Kan extension, it follows that for $n > 0$ there is an equivalence

$$S^1 \wedge Q^{E_1}_{\mathbb{L}}(\mathbf{R})(n) \simeq \bigvee_{\substack{[x]\in\pi_0(\mathsf{G})\\ r(x)=n}} \Sigma^2 |S^{E_1}_\bullet(x)| /\!\!/ G_x,$$

where the homotopy orbits are formed in sSet_*. Thus under the given assumption $H^{E_1}_{n,d}(\mathbf{R})(n) = H_d(Q^{E_1}_{\mathbb{L}}(\mathbf{R})(n)) = 0$ for $d < n - 1$. The claim about E_2-homology then follows by "transferring vanishing lines up", i.e. write $B^{E_2}(\mathbf{R}^+_+, \epsilon)$ as the bar construction of $B^{E_1}(\mathbf{R}^+_+, \epsilon)$ and run the bar spectral sequence. □

Chapter 10

Generic homological stability III: a generic stability result

10.1 A generic homological stability theorem

The following is [GKRW18a, Theorem 18.2]. It concerns E_2-algebras in simplicial $\mathbb{k}$-modules, so applies to $\mathbf{R}_{\mathbb{k}}$ from Chapter 9, but could also be applied to other E_2-algebras which do not arise in that way (e.g. which do not arise by $\mathbb{k}$-linearising an E_2-algebra in sSet).

Theorem 10.1.1. *Let $\mathbf{R}$ be a non-unital E_2-algebra in $\mathbb{N}$-graded simplicial $\mathbb{k}$-modules, such that $H_{*,0}(\mathbf{R}^+) = \mathbb{k}[\sigma]$ with $|\sigma| = (1, 0)$.*

(i) If $H^{E_2}_{n,d}(\mathbf{R}) = 0$ for $d < n - 1$, then $H_{n,d}(\mathbf{R}^+/\sigma) = 0$ for $2d \leq n - 1$.

(ii) If in addition $\sigma \cdot - : H_{1,1}(\mathbf{R}) \to H_{2,1}(\mathbf{R})$ is surjective, then $H_{n,d}(\mathbf{R}^+/\sigma) = 0$ for $3d \leq 2n - 1$.

Example 10.1.2. Let us return to the $\mathbb{N}$-graded non-unital E_2-algebra $\mathrm{Sym}_{\leq k}$ given by bounded symmetric powers. We calculated $H^{E_2}_{n,d}(\mathrm{Sym}_{\leq k}; \mathbb{Z})$ outright, and saw that in fact this vanishes for $d < 2(n - 1)$, a much larger range than the theorem requires. In addition

$$H_{2,1}(\mathrm{Sym}_{\leq k}; \mathbb{Z}) = H_1(\mathrm{Sym}_{\leq k}(2); \mathbb{Z}) = \begin{cases} 0 & \text{if } k \geq 2 \\ \mathbb{Z} & \text{if } k = 1. \end{cases}$$

So, as long as $k \geq 2$ it follows that

$$H_d(\mathrm{Sym}_{\leq k}(n), \mathrm{Sym}_{\leq k}(n-1); \mathbb{Z}) = H_{n,d}(\mathrm{Sym}^+_{\leq k}/\sigma; \mathbb{Z}) = 0$$

for $3d \leq 2n - 1$.

Problem 10.1.3. The fact that the E_2-homology in fact vanishes in a much larger range than the Theorem requires suggests a better stability range is possible. Experiment with the proof of the Theorem and see what improvements you can make.

Proof of Theorem 10.1.1 (i).

Reduce to finitely-generated free algebras. From the estimate on homology we may find a CW-approximation $\mathbf{Z} \xrightarrow{\sim} \mathbf{R}$ such that $\mathbf{Z}$ only has (n,d)-E_2-cells with $d \geq n-1$, and has a single 0-dimensional cell σ, in bidegree $(1,0)$. The CW object comes with a skeletal filtration, having associated graded the free E_2-algebra on its cells,

$$\mathrm{gr}(\mathbf{Z}) = \mathbf{E}_2\left(S^{1,0,0}_{\Bbbk}\{\sigma\} \oplus \bigoplus_{\alpha\in I} S^{n_\alpha,d_\alpha,d_\alpha}_{\Bbbk}\{\alpha\}\right)$$

with $d_\alpha > 0$ and $d_\alpha \geq n_\alpha - 1$. Neglecting the filtration degree this satisfies the same hypotheses of $\mathbf{R}$, and there is a spectral sequence

$$E^1_{n,p,q} = H_{n,p+q,p}(\mathbf{E}_2^+\left(S^{1,0,0}_{\Bbbk}\{\sigma\} \oplus \bigoplus_{\alpha\in I} S^{n_\alpha,d_\alpha,d_\alpha}_{\Bbbk}\right)/\sigma) \Rightarrow H_{n,p+q}(\mathbf{R}^+/\sigma),$$

so it suffices to treat the free algebra. Writing the free algebra as the colimit over its finitely-generated subalgebras (always including the generator σ), it suffices to suppose I is finite.

Reduce to working over $\mathbb{Z}$. We have

$$\mathbf{E}_2^+\left(S^{1,0}_{\Bbbk} \oplus \bigoplus_{\alpha\in I} S^{n_\alpha,d_\alpha}_{\Bbbk}\right)/\sigma = \mathbf{E}_2^+\left(S^{1,0}_{\mathbb{Z}} \oplus \bigoplus_{\alpha\in I} S^{n_\alpha,d_\alpha}_{\mathbb{Z}}\right)/\sigma \otimes_{\mathbb{Z}} \Bbbk. \tag{10.1}$$

By the universal coefficient sequence it suffices to treat the case $\Bbbk = \mathbb{Z}$.

Reduce to working over $\mathbb{F}_\ell$. Let ℓ be a prime number. The universal coefficient sequence for (10.1) with $\Bbbk = \mathbb{F}_\ell$, writing $\mathbf{R} = \mathbf{E}_2\left(S^{1,0}_{\mathbb{Z}} \oplus \bigoplus_{\alpha\in I} S^{n_\alpha,d_\alpha}_{\mathbb{Z}}\right)$, is

$$0 \longrightarrow H_{n,d}(\mathbf{R}^+/\sigma)\otimes \mathbb{F}_\ell \longrightarrow H_{n,d}(\mathbf{R}^+/\sigma\otimes\mathbb{F}_\ell) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(H_{n,d-1}(\mathbf{R}^+/\sigma),\mathbb{F}_\ell) \longrightarrow 0.$$

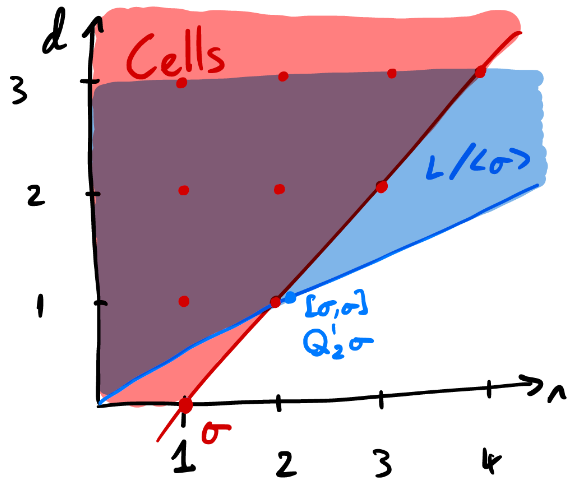
As we know that the $H_{n,d}(\mathbf{R}^+/\sigma)$ are all finitely-generated abelian groups, as we arranged the indexing set I to be finite, to show it vanishes it suffices to show that $H_{n,d}(\mathbf{R}^+/\sigma\otimes\mathbb{F}_\ell)$ does.

Do it. We have reduced to the case

$$\mathbf{R} = \mathbf{E}_2\left(S^{1,0}_{\mathbb{F}_\ell}\{\sigma\} \oplus \bigoplus_{\alpha\in I} S^{n_\alpha,d_\alpha}_{\mathbb{F}_\ell}\{\alpha\}\right)$$

with $d_\alpha > 0$ and $d_\alpha \geq n_\alpha - 1$. By Cohen's theorem we have a formula for the homology of $\mathbf{R}$: it is the free (graded) commutative algebra on a bigraded vector space L with basis (certain) Dyer–Lashof operations applied to (certain) Lie words in $\{\sigma\}\cup I$. Thus the homology of $\mathbf{R}^+/\sigma$ is the free (graded) commutative algebra on $L/\langle\sigma\rangle$. What is left?

The bracket of two elements has slope larger than the smaller of the two slopes of these elements, and the operations Q^s_ℓ and βQ^s_ℓ both increase slope. Thus the smallest slope of an element of $L/\langle\sigma\rangle$ is $\frac{1}{2}$ (realised by $Q^1_2(\sigma)$, $[\sigma,\sigma]$, or by an α of bidegree $(2,1)$). But then the free (graded) commutative algebra on $L/\langle\sigma\rangle$ vanishes in bidegrees (n,d) with $\frac{d}{n} < \frac{1}{2}$, as required. □



Proof of Theorem 10.1.1 (ii). The difficulty with getting an improved range in the above argument is that $L/\langle\sigma\rangle$ does contain elements of slope $\frac{1}{2}$, namely $Q^1_2(\sigma)$, $[\sigma,\sigma]$, or an α of bidegree $(2,1)$. Apart from these three classes though, the rest of $L/\langle\sigma\rangle$ consist of classes of slope $\geq \frac{2}{3}$. The strategy will be to

(a) show that the assumption means that we need no $(2,1)$-cells,

(b) show that the assumption means that $Q^1_2(\sigma)$ and $[\sigma,\sigma]$ are d^1-boundaries in the spectral sequence

$$E^1_{n,p,q} = H_{n,p+q,p}(\mathbf{E}_2^+\left(S^{1,0,0}_{\Bbbk}\{\sigma\} \oplus \bigoplus_{\alpha\in I} S^{n_\alpha,d_\alpha,d_\alpha}_{\Bbbk}\right)/\sigma) \Rightarrow H_{n,p+q}(\mathbf{R}^+/\sigma),$$

and that everything left has slope $\geq \frac{2}{3}$.

Together these imply the improved range. It is not necessary, but let us suppose for simplicity that $\Bbbk = \mathbb{F}_\ell$.

Claim. $H^{E_2}_{2,1}(\mathbf{R}) = 0$.

Proof of claim. Using the map $\mathbf{E}_2(S^{1,0}_{\mathbb{F}_\ell}) \to \mathbf{R}$ given by σ, form the diagram

$$\begin{array}{ccccccccc} & & H_{2,1}(\mathbf{R}) & \twoheadrightarrow & H_{2,1}(\mathbf{R},\mathbf{E}_2(S^{1,0}_{\mathbb{F}_\ell})) & \xrightarrow{0} & H_{2,0}(\mathbf{E}_2(S^{1,0}_{\mathbb{F}_\ell})) & \xrightarrow{\sim} & H_{2,0}(\mathbf{R}) \\ & & \downarrow & & \downarrow\wr & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^{E_2}_{2,1}(\mathbf{R}) & \xrightarrow{\sim} & H^{E_2}_{2,1}(\mathbf{R},\mathbf{E}_2(S^{1,0}_{\mathbb{F}_\ell})) & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

given by the map on long exact sequences induced by the Hurewicz map. The top row is as indicated, as the two rightmost terms are $\mathbb{F}_\ell\{\sigma^2\}$. Because we have $H_{*,0}(\mathbf{R},\mathbf{E}_2(S^{1,0}_{\mathbb{F}_\ell}\{\sigma\})) = 0$, it follows from the Hurewicz theorem for E_2-homology that the second vertical map is an isomorphism: thus, the first vertical map is surjective. But then the composition

$$H_{1,1}(\mathbf{R}) \xrightarrow{\sigma\cdot -} H_{2,1}(\mathbf{R}) \longrightarrow H^{E_2}_{2,1}(\mathbf{R})$$

is surjective, as the first map is by assumption, but also zero, as $\sigma \cdot -$ by definition gives something decomposable. □

We can therefore find a CW-approximation $\mathbf{Z} \xrightarrow{\sim} \mathbf{R}$ such that $\mathbf{Z}$ has no E_2-$(2,1)$-cells. The skeletal filtration then has

$$\mathrm{gr}(\mathbf{Z}) = \mathbf{E}_2 \left(S^{1,0,0}_{\mathbb{F}_\ell}\{\sigma\} \oplus \bigoplus_{\alpha \in I} S^{n_\alpha,d_\alpha,d_\alpha}_{\mathbb{F}_\ell}\{\alpha\} \right)$$

with $d_\alpha > 0$, $d_\alpha \geq n_\alpha - 1$, and $(n_\alpha, d_\alpha) \neq (2,1)$. Consider the spectral sequences

$$\begin{array}{ccc} F^1_{n,p,q} = H_{n,p+q,p}(\mathbf{E}_2^+ \left(S^{1,0,0}_{\mathbb{F}_\ell}\{\sigma\} \oplus \bigoplus_{\alpha\in I} S^{n_\alpha,d_\alpha,d_\alpha}_{\mathbb{F}_\ell}\{\alpha\}\right)) & \Longrightarrow & H_{n,p+q}(\mathbf{R}^+) \\ \downarrow & & \downarrow \\ E^1_{n,p,q} = H_{n,p+q,p}(\mathbf{E}_2^+ \left(S^{1,0,0}_{\mathbb{F}_\ell}\{\sigma\} \oplus \bigoplus_{\alpha\in I} S^{n_\alpha,d_\alpha,d_\alpha}_{\mathbb{F}_\ell}\{\alpha\}\right)/\sigma) & \Longrightarrow & H_{n,p+q}(\mathbf{R}^+/\sigma). \end{array}$$

As $\mathbf{Z}$ has no $(2,1)$-cells, in total bidegree $(2,1)$ of $F^1_{*,*,*}$ all classes are linear combinations of $\sigma \cdot \alpha$ for $|\alpha| = (1,1)$, and

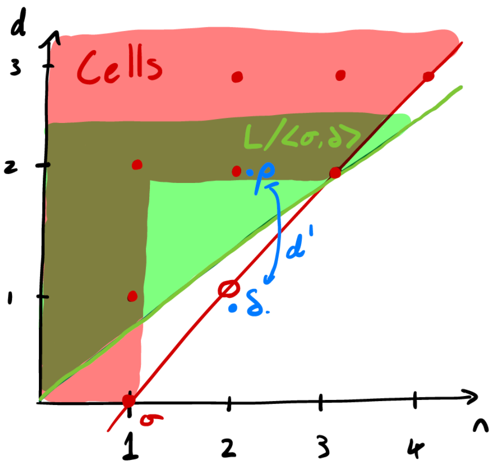
$$\delta := \begin{cases} Q^1_2(\sigma) & \text{if } \ell = 2 \\ [\sigma,\sigma] & \text{if } \ell \text{ is odd.} \end{cases}$$

As we discussed in the last proof, $F^1_{*,*,*} = \Lambda[L]$ with

$$L = \langle \sigma, \delta \rangle \oplus \langle \text{elements of slope } \geq \tfrac{2}{3} \rangle.$$

On the other hand, as $\sigma \cdot - : H_{1,1}(\mathbf{R}) \to H_{2,1}(\mathbf{R})$ is surjective by assumption we must have $\delta = \sigma \cdot x \in H_{2,1}(\mathbf{R})$, for some $x \in H_{1,1}(\mathbf{R})$. As this spectral sequence converges to $H_{*,*}(\mathbf{R}^+)$ it must have a differential of the form $d^1(\rho) = \delta - \sigma \cdot x$.

Using $E^1_{*,*,*} = F^1_{*,*,*}/(\sigma)$, we see that in total bidegree $(2,1)$ of $E^1_{*,*,*}$ there is only $\delta \in E^1_{2,0,1}$, and that there is also a differential $d^1(\rho) = \delta$ for a $\rho \in E^1_{2,1,1}$.



Using this we proceed similarly to the first case: filtering away all other parts of the d^1-differential, we get the associated graded

$$\mathrm{gr}(E^1_{*,*,*}, d^1) = (\Lambda[\delta, \rho], d\rho = \delta) \otimes (\Lambda[L/\langle \sigma, \delta, \rho \rangle], d = 0)$$

where the first term has homology $\mathbb{F}_\ell[0,0,0]$, and the second has $L/\langle \sigma, \delta, \rho \rangle$ vanishing in degrees (n, d) with $\frac{d}{n} < \frac{2}{3}$ (δ is the class of lowest slope in $L/\langle \sigma \rangle$, and all others have slope $\geq \frac{2}{3}$). It follows that $E^2_{*,*,*}$ vanishes in this range of bidegrees, and so $H_{*,*}(\mathbf{R}^+/\sigma)$ does too. $\square$

Chapter 11

Secondary homological stability for mapping class groups I

In this first lecture about secondary homological stability for mapping class groups, Theorem 1.1.6, we will give the proof with rational coefficients. Along the way we will prove ordinary homological stability for mapping class groups, Theorem 1.1.3.

11.1 Homological stability for mapping class groups

We first prove homological stability for mapping class groups, by showing that it satisfies the criteria for the generic homological stability explained in Chapter 10.

11.1.1 A braided monoidal groupoid of mapping class groups

Firstly, we need to construct an E_2-algebra $\mathbf{R} \in \mathsf{Alg}_{E_2}(\mathsf{sSet}^{\mathbb{N}})$ from a braided monoidal groupoid such that for all $g > 0$ we have $\mathbf{R}(g) \simeq B\Gamma_{g,1}$. In our case, we will use the groupoid MCG has objects given by the natural numbers and morphisms

$$\mathsf{MCG}(g,h) \coloneqq \begin{cases} \Gamma_{g,1} & \text{if } g = h, \\ \varnothing & \text{otherwise.} \end{cases}$$

The monoidal structure $\oplus$ on MCG is given by addition on objects, and on morphisms $\varphi \in \Gamma_{g,1}$ and $\psi \in \Gamma_{h,1}$ by $\varphi \oplus \psi = \phi \cup (\psi + g \cdot e_1)$ as a diffeomorphism $\Sigma_{g+h,1} = \Sigma_{g,1} \cup (\Sigma_{h,1} + g \cdot e_1)$. The braiding is given by the half right-handed Dehn twist. That the homomorphism $- \oplus -\colon \Gamma_{g,1} \times \Gamma_{h,1} \to \Gamma_{g+h,1}$ is injective is a classical result, boiling down to the result of Gramain that spaces of arcs in a surface have contractible path components.

Then $\mathbf{R} \in \mathsf{Alg}_{E_2}(\mathsf{sSet}^{\mathbb{N}})$ is the derived pushforward of the canonical (non-unital) E_2-algebra $\underline{*}_{>0}$ in $\mathsf{sSet}^{\mathsf{MCG}}$ along the unique functor $\mathsf{MCG} \to \mathbb{N}$ that is the identity on objects. By construct we have that

$$\mathbf{R}(g) = \begin{cases} B\Gamma_{g,1} & \text{if } g > 0, \\ \varnothing & \text{if } g = 0. \end{cases}$$

In terms of this E_2-algebra structure, the stabilisation map $\sigma_*\colon H_d(B\Gamma_{g-1,1};\mathbb{Z}) \to H_d(B\Gamma_{g,1};\mathbb{Z})$ is the map $H_{d,g-1}(\mathbf{R};\mathbb{Z}) \to H_{d,g}(\mathbf{R};\mathbb{Z})$ induced by multiplication with your favorite point $\sigma \in \mathbf{R}(1)$.

Example 11.1.1. This is weakly equivalent to the geometric model given in the introduction.

11.1.2 Applying the generic homological stability result

In Chapter 9, we learned that to prove the vanishing line $H^{E_2}_{g,d}(\mathbf{R};\mathbb{Z}) = 0$ for $d \leq g-2$ it suffices to prove that the semi-simplicial set—the E_1-splitting complex $S^{E_1}_\bullet(g)$—is $(g-3)$-connected:

$$[p] \longmapsto \bigsqcup_{g_0+\cdots+g_{p+1}=g} \frac{\Gamma_{g,1}}{\Gamma_{g_0,1}\times\cdots\times\Gamma_{g_{p+1},1}},$$

where each g_i is positive. The ith face map is induced by the inclusion $\Gamma_{g_i,1}\times\Gamma_{g_{i+1},1} \to \Gamma_{g_i+g_{i+1},1}$. We have already done this in Chapter 3:

Lemma 11.1.2. *There is an isomorphism of semi-simplicial sets*

$$S^{E_1}_\bullet(g) \cong S(\Sigma_{g,1}, b_0, b_1)_\bullet$$

and hence the left side is $(g-3)$*-connected.*

Here $S(\Sigma_{g,1}, b_0, b_1)_\bullet$ is the semi-simplicial set with p-simplices given by isotopy classes of $(p+1)$-tuples of arcs from a point $b_0 \in \partial\Sigma_{g,1}$ to $b_1 \in \partial\Sigma_{g,1}$ that are (i) disjoint except at endpoints, (ii) whose order agrees with the clockwise order at b_0, (iii) the arcs split the surface into $p+2$ regions of positive genus.

Proof sketch. Fix a decomposition $g_0+\cdots+g_{p+1}=g$ with g_i positive, we get a decomposition of $\Sigma_{g,1}$ into the standard pieces $\Sigma_{g_i,1}$ and a preferred p-simplex of arcs connecting b_0 to b_1. Then acting by $\Gamma_{g,1}$ on this collection yields a map $\Gamma_{g,1} \to S(\Sigma_{g,1}, b_0, b_1)_p$ which is surjective onto the p-simplices whose regions have genus $g_0,\ldots,g_{p+1}$, by the classification of surfaces. The stabiliser of the preferred p-simplex is $\Gamma_{g_0,1}\times\cdots\times\Gamma_{g_{p+1},1}$. Varying over all sums gives a bijection

$$S^{E_1}_p(g) = \bigsqcup_{g_0+\cdots+g_{p+1}=g} \frac{\Gamma_{g,1}}{\Gamma_{g_0,1}\times\cdots\times\Gamma_{g_{p+1},1}} \xrightarrow{\cong} S(\Sigma_{g,1}, b_0, b_1)_p.$$

An inspection of these bijections shows that they commute with the face maps. □

This gives condition (i) of Theorem 10.1.1. For condition (ii) of Theorem 10.1.1, we need some input about the low-degree low-genus homology of mapping class groups, going back to Chapter 6.

Lemma 11.1.3. *The stabilisation map* $H_{1,1}(\mathbf{R};\mathbb{Z}) \to H_{2,1}(\mathbf{R};\mathbb{Z})$ *is surjective.*

Proof. Equivalently, $H_1(B\Gamma_{1,1};\mathbb{Z}) \to H_1(B\Gamma_{2,1};\mathbb{Z})$ is surjective. By Theorem 6.1.1 (i) by the left side is $\mathbb{Z}$ generated by τ (the image of the Dehn twist) and by Theorem 6.1.1 (ii) the right side is $\mathbb{Z}/10$ generated by $\sigma\tau$. □

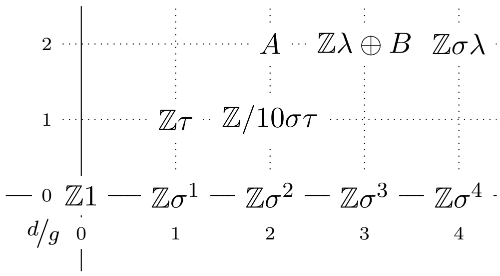


Figure 11.1: Summary of $H_{g,d}(\mathbf{R}_\mathbb{Z})$ as described in Theorem 6.1.1, where A and B are torsion and B is the image of A under multiplication by σ. Empty entries are 0.

Having verified the conditions of Theorem 10.1.1, we conclude

$$H_{g,d}(\mathbf{R}^+/\sigma;\mathbb{Z}) = 0 \qquad \text{for } 3d \leq 2g-1.$$

Here $\mathbf{R}^+$ is the unitalisation of $\mathbf{R}$, and $\mathbf{R}^+/\sigma$ is the cofiber of multiplication by σ, satisfying $H_{g,d}(\mathbf{R}^+/\sigma;\mathbb{Z}) = H_d(B\Gamma_{g,1}, B\Gamma_{g-1,1};\mathbb{Z})$, the relative homology of the stabilisation map. This is thus expressing that $\sigma_*\colon H_d(B\Gamma_{g-1,1};\mathbb{Z}) \to H_d(B\Gamma_{g,1};\mathbb{Z})$ is a surjection for $d \leq \frac{2g-1}{3}$ and an isomorphism for $d \leq \frac{2g-4}{3}$. This was the statement of Theorem 1.1.3.

11.2 Rational secondary homological stability for mapping class groups

Our next goal is to prove rational secondary homological stability for mapping class groups. Here the rational case is easier in at least two ways:

(i) it is easier to construct the secondary stability maps,

(ii) the computations in free E_2^+-algebras are simpler rationally since there are no Dyer–Lashof operations.

We will thus study the $\mathbb{Q}$-linearisation $\mathbf{R}_\mathbb{Q}$ of $\mathbf{R}$ (obtained by applying the functor $\mathbb{Q}[-]\colon \mathsf{sSet}^\mathbb{N} \to \mathsf{sMod}_\Bbbk^\mathbb{N}$) and apply a more refined version of the argument for the generic homological stability result: instead of directly invoking CW-approximation for $\mathbf{R}_\mathbb{Q}$, as we did in the proof of Theorem 10.1.1, we will invoke the relative version for a map $\mathbf{A} \to \mathbf{R}_\mathbb{Q}$ in $\mathsf{Alg}_{E_2}(\mathsf{sMod}_\mathbb{Q}^\mathbb{N})$, where $\mathbf{A}$ is a "small model" that contains all necessary low-degree low-genus E_2-cells. We will then prove by direct calculation the secondary homological stability result for $\mathbf{A}$, and next prove that it transfers to $\mathbf{R}_\mathbb{Q}$.

Homological stability with rational coefficients concerns the vanishing of the homlogy of $\mathbf{R}_\mathbb{Q}^+/\sigma$, as its homology groups are the relative rational homology groups of the stabilisation map. On this mapping cone, we can use an adapter to still produce a multiplication-by-λ map, which gives a map

$$\lambda \cdot -\colon H_{g-3,d-2}(\mathbf{R}_\mathbb{Q}^+/\sigma) \longrightarrow H_{g,d}(\mathbf{R}_\mathbb{Q}^+/\sigma).$$

It is this map that we will prove is an isomorphism or surjection in a range. Equivalently, we may form the iterated mapping cone $\mathbf{R}_\mathbb{Q}^+/(\sigma,\lambda)$, and to get secondary homological stability as stated in Theorem 1.1.6 we need to prove that

$$H_{g,d}(\mathbf{R}_\mathbb{Q}^+/(\sigma,\lambda)) = 0 \qquad \text{for } 4d \leq 3g-1.$$

Remark 11.2.1. Here "small model" does not mean that $\mathbf{A} \to \mathbf{R}_{\mathbb{Q}}$ induces an isomorphism on homology in low homological degree d, nor that it does so in low genus g; it only does so when both g and d are low. Due to the vanishing line in E_2-homology for $\mathbf{R}$, this is enough for it to "capture in a range the stability phenomena present in $\mathbf{R}_{\mathbb{Q}}$." In particular, you can't compute the stable homology of mapping class groups from $\mathbf{A}$.

Construction of A

At this point we use crucially the computations of Chapter 6. Rationally, Fig. 11.1 summarising Theorem 6.1.1 simplifies to Fig. 11.2. From it, we obtain the following:

- A map $S^{1,0}_{\mathbb{Q}}\sigma \to \mathbf{R}_{\mathbb{Q}}$ representing σ.
- A map $S^{3,2}_{\mathbb{Q}}\lambda \to \mathbf{R}_{\mathbb{Q}}$ representing λ.

These combine to a map $\mathbf{E}_2(S^{1,0}_{\mathbb{Q}}\sigma \oplus S^{3,2}_{\mathbb{Q}}\lambda) \to \mathbf{R}_{\mathbb{Q}}$ in $\mathsf{Alg}_{E_2}(\mathsf{sMod}^{\mathbb{N}}_{\mathbb{Q}})$ and since $[\sigma, \sigma] = 0$ in the target, picking a null-homotopy of the map $S^{2,1}_{\mathbb{Q}} \to \mathbf{R}_{\mathbb{Q}}$ representing this Browder bracket we get an extension of this map to

$$\mathbf{A} \coloneqq \mathbf{E}_2(S^{1,0}_{\mathbb{Q}}\sigma \oplus S^{3,2}_{\mathbb{Q}}\lambda) \cup_{[\sigma,\sigma]} D^{2,2}_{\mathbb{Q}}\rho \longrightarrow \mathbf{R}_{\mathbb{Q}}. \tag{11.1}$$

Remark 11.2.2. $\mathbf{A}$ is essentially a (rationalised version of) the bounded symmetric power $\mathrm{Sym}_{\leq 2}$ with additional free E_2-cells on the generator λ. Does this E_2-algebra have a geometric interpretation in terms of moduli spaces?

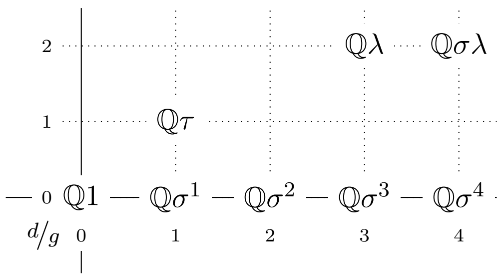


Figure 11.2: Summary of $H_{g,d}(\mathbf{R}_{\mathbb{Q}})$. Compare to Fig. 1.2.

Remark 11.2.3. Why did we not add in an E_2-cell for τ? We could have as doing so, not affect the estimate in the next lemma. I have kept it out to make the smaller model even smaller. In general, a small model capturing stability phenomena up to slope λ only needs to include the cells that appear in a minimal CW-approximation to $\mathbf{R}$ that have bidegree (g, d) with $d \leq \lambda g$. That is, *slope* $\frac{d}{g}$ is the crucial quantity.

Lemma 11.2.4. $H^{E_2}_{g,d}(\mathbf{R}_{\mathbb{Q}}, \mathbf{A}) = 0$ *for* $4d \leq 3g - 1$.

Proof. We use the long exact sequence of a pair to deal with the cases $g \geq 4$:

- $H^{E_2}_{g,d}(\mathbf{R}_{\mathbb{Q}}) = 0$ for $d \leq g - 2$ by Section 11.1.2.
- $H^{E_2}_{g,d}(\mathbf{A}) = 0$ except in bidegrees $(g, d) = (1, 0), (3, 2), (2, 2)$, by construction; in particular this vanishes for $d \leq g - 2$.

From the long exact sequence

$$\cdots \longrightarrow H^{E_2}_{g,d}(\mathbf{A}) \longrightarrow H^{E_2}_{g,d}(\mathbf{R}_{\mathbb{Q}}) \longrightarrow H^{E_2}_{g,d}(\mathbf{R}_{\mathbb{Q}}, \mathbf{A}) \longrightarrow \cdots,$$

we conclude that $H^{E_2}_{g,d}(\mathbf{R}_{\mathbb{Q}}, \mathbf{A}) = 0$ for $d \leq g-2$. As long as $g \geq 4$, this implies we can have a non-zero class only if $d \geq g-1$ or equivalently $4d \geq 4g-4 \geq 3g$ so get vanishing if $4d \leq 3g-1$.

For $g \leq 3$, we build a slightly larger model

$$\mathbf{A}' \coloneqq \mathbf{A} \coloneqq \mathbf{E}_2(S^{1,0}_{\mathbb{Q}}\sigma \oplus S^{1,1}_{\mathbb{Q}}\tau \oplus S^{3,2}_{\mathbb{Q}}\lambda) \cup^{E_2}_{[\sigma,\sigma]} D^{2,2}_{\mathbb{Q}}\rho_1 \cup^{E_2}_{\sigma\tau} D^{2,2}_{\mathbb{Q}}\rho_2,$$

which fits into a factorisation $\mathbf{A} \to \mathbf{A}' \to \mathbf{R}_{\mathbb{Q}}$. We next use the Hurewicz theorem concerning the map

$$H_{g,d}(\mathbf{R}_{\mathbb{Q}}, \mathbf{A}') \longrightarrow H^{E_2}_{g,d}(\mathbf{R}_{\mathbb{Q}}, \mathbf{A}');$$

it is a little computation that $\mathbf{A}' \to \mathbf{R}_{\mathbb{Q}}$ is an isomorphism on ordinary homology in bidegrees (g,d) with $g \leq 3$ and $d \leq 1$ (this was the reason for adding in τ and ρ_2) and surjective with $g \leq 3$ and $d = 2$, so the same is true on E_2-homology. Finally, we use the long exact sequence of triple

$$\cdots \longrightarrow H^{E_2}_{g,d}(\mathbf{A}', \mathbf{A}) \longrightarrow H^{E_2}_{g,d}(\mathbf{R}_{\mathbb{Q}}, \mathbf{A}) \longrightarrow H^{E_2}_{g,d}(\mathbf{R}_{\mathbb{Q}}, \mathbf{A}') \longrightarrow \cdots,$$

to deduce the result for $H^{E_2}_{g,d}(\mathbf{R}_{\mathbb{Q}}, \mathbf{A})$ with $g \leq 3$. □

Proof of secondary homological stability for A

We start by performing the same iterated mapping cone construction for $\mathbf{A}$ to obtain $\mathbf{A}^+/(\sigma,\lambda)$, which maps to $\mathbf{R}^+_{\mathbb{Q}}/(\sigma,\lambda)$. The corresponding secondary homological stability result is true in this case:

Lemma 11.2.5. *$H_{g,d}(\mathbf{A}^+/(\sigma,\lambda))$ for $4d \leq 3g-1$.*

Proof. Let $n_*(-)$ denote an object made filtered by putting it in filtration degree n. We can lift $\mathbf{A}$ to a filtered E_2-algebra by taking

$$\mathrm{sk}\mathbf{A} \coloneqq \mathbf{E}_2(0_* S^{1,0}_{\mathbb{Q}}\sigma \oplus 2_* S^{3,2}_{\mathbb{Q}}\lambda) \cup_{[\sigma,\sigma]} 2_* D^{2,2}_{\mathbb{Q}}\rho,$$

that is, recognising it is a CW-algebra and consider its skeletal filtration. The associated graded is simply a free E_2-algebra

$$\mathrm{gr}(\mathrm{sk}\mathbf{A}) = \mathbf{E}_2(S^{1,0,0}_{\mathbb{Q}}\sigma \oplus S^{3,2,2}_{\mathbb{Q}}\lambda \oplus D^{2,2,2}_{\mathbb{Q}}\rho),$$

so we get a spectral sequence (we are unitalising because this simplifies the homology of free E_2-algebras)

$$E^1_{g,p,q} = H_{g,p+q,p}(\mathbf{E}_2(S^{1,0,0}_{\mathbb{Q}}\sigma \oplus S^{3,2,2}_{\mathbb{Q}}\lambda \oplus D^{2,2,2}_{\mathbb{Q}}\rho)^+) \Longrightarrow H_{g,p+q}(\mathbf{A}^+)$$

with left side is the free graded-commutative algebra on all iterated bracketings of σ, λ, and ρ. More explicitly, it is given by

$$E^1_{g,p,q} = (\Lambda^*(L), d^1)$$

with Λ^* denoting the free graded commutative algebra (using only homological grading for the Koszul sign, and genus grading coming along for the ride), L is a graded vector space of brackets of σ, λ, ρ. The d^1-differential is a derivation and comes from the attaching map of the cell ρ so satisfies $d^1(\rho) = [\sigma, \sigma]$. It is a small computation that the only generators of slope $\frac{p+q}{g} < \frac{3}{4}$ are $\sigma, \lambda, [\sigma, \sigma], \rho$; as in the proof of Theorem 10.1.1 the crucial observation is that a bracket has slope strictly larger than the smallest slope of its two inputs.

The filtration sk$\mathbf{A}$ of $\mathbf{A}$ induces a filtration sk$\mathbf{A}^+/(\sigma, \lambda)$ of $\mathbf{A}^+/(\sigma, \lambda)$. This yields a spectral sequence

$$F^1_{g,p,q} = H_{g,p+q,p}(\mathbf{E}_2(S^{1,0,0}_{\mathbb{Q}}\sigma \oplus S^{3,2,2}_{\mathbb{Q}}\lambda \oplus D^{2,2,2}_{\mathbb{Q}}\rho)^+/(\sigma, \lambda)) \Longrightarrow H_{g,p+q}(\mathbf{A}^+/(\sigma, \lambda)),$$

with left side more explicitly given by

$$F^1_{g,p,q} = (\Lambda^*(L/\langle \sigma, \lambda \rangle), d^1),$$

the free graded-commutative algebra on all iterated bracketings of σ, λ, and ρ, *except* σ and λ. The map of spectral sequences

$$E^1_{*,*,*} \longrightarrow F^1_{*,*,*}$$

makes the latter into a module spectral sequence over the previous one, which determines that the d^1-differential still is a derivation satisfying $d^1(\rho) = [\sigma, \sigma]$. You might be worried about the understanding value of the differential on other bracketings, but this can be filtered away as in proof of Theorem 10.1.1. The conclusion is that an upper bound of the E^2-page is given by homology of the complex

$$(\Lambda^*(\rho, [\sigma, \sigma]), d(\rho) = [\sigma, \sigma]) \otimes (\Lambda^*(\text{other generators}), 0),$$

where explicitly the other generators is given by $L/(\sigma, \lambda, [\sigma, \sigma], \rho)$. But this vanishes in the range $\frac{d}{g} < \frac{3}{4}$ as the left side will be $\mathbb{Q}$ in degree 0 after taking homology and the right side vanishes in this range. $\square$

Let us investigate this proof a bit closer and observe that we may as well have added more freely attached E_2-cells of slope $\geq \frac{3}{4}$. The proof goes through in the same manner; there are just more "other generators."

Proposition 11.2.6. *If* $\widetilde{\mathbf{A}}$ *is of the form* $\mathbf{A} \cup^{E_2} \mathbf{E}_2(\bigoplus_\alpha S^{g_\alpha, d_\alpha}_{\mathbb{Q}})$ *with* (g_α, d_α) *satisfying* $4d_\alpha \geq 3g_\alpha$, *then* $H_{g,d}(\widetilde{\mathbf{A}}^+/(\sigma, \lambda))$ *for* $4d \leq 3g - 1$.

Proof of secondary homological stability for $\mathbf{R}_{\mathbb{Q}}$

We will use Proposition 11.2.6 to prove that $\mathbf{R}^+_{\mathbb{Q}}/(\sigma, \lambda)$ has the same vanishing line as $\mathbf{A}^+/(\sigma, \lambda)$. Applying the CW-approximation theorem in combination with Lemma 11.2.4 we get a factorisation

$$\mathbf{A} \longrightarrow \mathbf{B} \xrightarrow{\sim} \mathbf{R}_{\mathbb{Q}},$$

where $\mathbf{B}$ is obtained by attaching only E_2-cells in bidegrees (g_α, d_α) with $4d_\alpha \geq 3g_\alpha$. Since the left map is a weak equivalence, to prove Theorem 1.1.6 with rational coefficients it suffices to prove that $\mathbf{B}^+/(\sigma, \lambda)$ has the desired vanishing range:

Theorem 11.2.7. *$H_{g,d}(\mathbf{B}^+/(\sigma,\lambda)) = 0$ for $4d \leq 3g-1$.*

Proof. Taking the filtration on $\mathbf{B}^+/(\sigma,\lambda)$ induced by the skeletal filtration, we get a spectral sequence

$$E^1_{g,p,q} = H_{g,p+q,p}(\mathbf{A}[0] \cup^{E_2} \mathbf{E}_2(\bigoplus_\alpha S^{g_\alpha,d_\alpha,d_\alpha})^+/(\sigma,\lambda)) \Longrightarrow H_{p+q}(\mathbf{B}^+/(\sigma,\lambda)).$$

But since upon forgetting the additional grading, we get $\mathbf{A} \cup^{E_2} \mathbf{E}_2(\bigoplus_\alpha S^{g_\alpha,d_\alpha})$ which is of the form required in Proposition 11.2.6, we know that the E^1-page has the desired vanishing line in Proposition 11.2.6. □

This concludes the proof of Theorem 1.1.6 with rational coefficients. Next lecture we will discuss the case of integer coefficients.

Remark 11.2.8. Here is a different take on the same argument: (11.1) is the beginning of a CW-approximation of $\mathbf{R}_\mathbb{Q}$, only containing the ≤ 2-dimensional cells; it satisfies the induction hypothesis for $\epsilon = 2$ in the proof of Theorem 11.21 of [GKRW18a]. Thus we can extend it to a CW-approximation

$$\mathbf{A} \longrightarrow \mathbf{B} \xrightarrow{\sim} \mathbf{R}_\mathbb{Q},$$

where $\mathbf{B}$ is a CW-E_2-algebra that has the same E_2-cells as $\mathbf{A}$ and all further E_2-cells of bidegree (g_α, d_α) satisfying $4d_\alpha \leq 3g_\alpha$. Now take the filtration on $\mathbf{B}^+/(\sigma,\lambda)$ induced by the skeletal filtration and argue as in Lemma 11.2.5.

Remark 11.2.9. The argument in Section 5.2 of [GKRW19] is different than the one given above; it replaces the proofs of the Proposition and the Theorem by an appeal to a comparison result for relative E_2-cells for $\mathbf{A} \to \mathbf{R}_\mathbb{Q}$ to relative $\mathbf{A}$-module cells for $\mathbf{R}_\mathbb{Q}$ made into an $\mathbf{A}$-module through this map. This comparison result is Theorem 15.4 of [GKRW18a].

Chapter 12

Outlook I: General linear groups

In analogy to the case of mapping class groups, we want to study the homology of general linear groups of a field $\mathbb{F}$, using an E_∞ structure on

$$\coprod_{n=1}^{\infty} B\mathrm{GL}_n(\mathbb{F}).$$

This is the content of [GKRW18b, GKRW20], though we will be focusing on the latter.

12.1 An E_∞-algebra of general linear groups

We fix once and for all a field $\mathbb{F}$. We proceed as in Chapter 9, working in a category $s\mathrm{Sets}^{\mathsf{G}}$, where G has objects given by $\mathbb{N}$ and morphisms given by

$$\mathsf{G}(n,m) = \begin{cases} \mathrm{GL}_n(\mathbb{F}) & \text{if } n = m, \\ \varnothing & \text{otherwise.} \end{cases}$$

Then $B\mathsf{G} \simeq \coprod_{n\geq 1} B\mathrm{GL}_n(\mathbb{F})$, and the E_∞ structure on this space corresponds to the *symmetric monoidal structure* on G given by direct sum of vector spaces (G is a skeleton of the groupoid of finite-dimensional $\mathbb{F}$-vector spaces, and hence is equivalent to it) or equivalently to block sum of square matrices.

As in Chapter 9, there is an object $\underline{*}_{>0} \in s\mathrm{Sets}^{\mathsf{G}}$ taking the value $\varnothing$ at $n = 0$ and a point $*$ at any $n > 0$. This has a unique E_∞-structure in $s\mathrm{Sets}^{\mathsf{G}}$ equipped with the Day convolution symmetric monoidal structure, and as in that chapter we obtain $\mathbf{T} \in \mathrm{Alg}_{E_\infty}(s\mathrm{Sets}^{\mathsf{G}})$ as a cofibrant approximation. Writing $r\colon \mathsf{G} \to \mathbb{N}$ for the evident functor, we then set $\mathbf{R} \coloneqq r_*(\mathbf{T}) \in \mathrm{Alg}_{E_\infty}(s\mathrm{Sets}^{\mathsf{G}})$. Then

$$\mathbf{R}(n) \simeq B\mathrm{GL}_n(\mathbb{F}).$$

12.2 E_∞-homology

Then E_k-homology of $\mathbf{T} \in s\mathrm{Sets}^{\mathsf{G}}$ will push forward to E_k-homology of $\mathbf{R} \in s\mathrm{Sets}^{\mathbb{N}}$, so we may start by calculating the former. The interesting values of k seem to be $k = 1, 2, \infty$; our end-goal is to obtain the last of these.

12.2.1 Splitting complexes

Moving to the pointed setting by adding basepoints $(s\mathrm{Sets}, \times) \to (s\mathrm{Sets}_*, \wedge)$ gives an E_∞-algebra object $\mathbf{T}_+ \in s\mathrm{Sets}_*^{\mathsf{G}}$, and as in Chapter 9 its derived indecomposables may be calculated for $0 \neq n \in \mathsf{G}$ as the bar construction of its unitalisation $\mathbf{T}_+^+$:

$$B(\mathbf{T}_+^+, \epsilon_{\mathrm{can}})(n) \simeq \Sigma Q_{\mathbb{L}}^{E_1}(\mathbf{T}_+)(n) \simeq \Sigma^2 \mathcal{S}^{E_1}(n),$$

where $\epsilon_{\mathrm{can}} \colon \mathbf{T}_+^+ \to \mathbb{1}$ is the canonical augmentation. As in Theorem 9.3.3, $\mathcal{S}^{E_1}(n)$ is the semi-simplicial set given by

$$\mathcal{S}^{E_1}(n) = [p] \mapsto \coprod_{\substack{n_0+\cdots+n_{p+1}=n \\ n_i>0}} \frac{\mathrm{GL}_n(\mathbb{F})}{\mathrm{GL}_{n_0}(\mathbb{F}) \times \cdots \times \mathrm{GL}_{n_{p+1}}(\mathbb{F})},$$

and $B(\mathbf{T}_+^+, \epsilon_{\mathrm{can}})(n)$ has a very similar looking description but without the outer terms indexed by n_0 and n_{p+1}.

Let us look at 0-simplices first: this is the $\mathrm{GL}_n(\mathbb{F})$-set

$$\mathcal{S}_0^{E_1}(n) = \coprod_{n_0=1}^{n-1} \frac{\mathrm{GL}_n(\mathbb{F})}{\mathrm{GL}_{n_0}(\mathbb{F}) \times \mathrm{GL}_{n-n_0}(\mathbb{F})},$$

which can be identified with the set of pairs (P_0, P_1) of non-zero subspaces $P_0, P_1 \subset \mathbb{F}^n$ such that $P_0 \oplus P_1 \to \mathbb{F}^n$ is an equivalence. (The equivariant bijection can be seen by inspecting orbits and stabilizers.)

Similarly, p-simplices can be $\mathrm{GL}_n(\mathbb{F})$-equivariantly identified with tuples $(P_0, \ldots, P_{p+1})$ of non-zero subspaces of $\mathbb{F}^n$ forming a direct sum decomposition. This semisimplicial set can also be identified with the nerve of the *poset* with objects (P_0, P_1) and where $(P_0, P_1) \leq (P_0', P_1')$ means $P_0 \subset P_0'$ and $P_1 \supset P_1'$. Notice the similarity with the arc complex considered in Chapter 3 and Chapter 11: this was a poset of ways to cut a surface into two non-trivial pieces, we have a poset of ways of cutting a vector space into two non-trivial pieces.

This semi-simplicial set is closely related to the *Tits building*, which is the nerve of the poset whose objects are non-trivial proper subspaces $0 \subsetneq P \subsetneq \mathbb{F}^n$, ordered by inclusion. Writing $\mathcal{T}(n)$ for the nerve of this poset, we have an evident surjection

$$\mathcal{S}^{E_1}(n) \longrightarrow \mathcal{T}(n) \tag{12.1}$$

induced by $(P_0, P_1) \mapsto P_0$.

It is a classical result that $|\mathcal{T}(n)|$ is $(n-3)$-connected, and hence homotopy equivalent to a wedge of $(n-2)$-spheres [Sol69]. The homology group

$$\mathrm{St}_n \coloneqq \widetilde{H}_{n-2}(\mathcal{T}(n))$$

is an (infinite dimensional) representation of $\mathrm{GL}_n(\mathbb{F})$, called the *Steinberg module*. The Solomon–Tits theorem gives an explicit $\mathbb{Z}$-basis for it by so-called apartments [Sol69].

The map (12.1) is evidently not a homeomorphism, since non-empty proper subspaces have many different complements. It does not even induce an isomorphism on homology, but Charney proved that the complex $S^{E_1}(n)$ is also $(n-3)$-connected, so has the homotopy type of a wedge of $(n-2)$-spheres [Cha80]. It is sometimes called the *split building*, and its $\widetilde{H}_{n-2}$ the *split Steinberg module.*

Using this, the situation is formally similar to the previous lectures about the E_2-algebra of mapping class groups, in that Charney's connectivity implies

$$H^{E_1}_{g,d}(\mathbf{R}) = 0 \quad \text{for } d < g-1,$$

where $\mathbf{R} \in s\mathrm{Sets}^{\mathbb{N}}$ is $(r_*\mathbf{T}) \simeq (n \mapsto B\mathrm{GL}_n(\mathbb{F}))$. We could therefore try to proceed in the same way as for mapping class group, using the bar spectral sequences of Chapter 5 and low-dimensional calculations of homology of general linear groups.

The rest of this talk is about the situation (or at least our knowledge of the situation) being *strictly better* for general linear groups of *infinite fields* than for mapping class groups. Hence from now on we assume that $\mathbb{F}$ is infinite. Firstly, we have isomorphisms

$$H^{E_1}_{g,d}(\mathbf{R}) = H_{d-(n-1)}(B\mathrm{GL}_n(\mathbb{F}); \widetilde{H}_{n-2}(\mathcal{S}^{E_1}(n))) \xrightarrow{\cong} H_{d-(n-1)}(B\mathrm{GL}_n(\mathbb{F}); \mathrm{St}_n).$$

The first of these just uses that the suspension of $n \mapsto S^{E_1}(n)$ is the indecomposables in $s\mathrm{Sets}^{\mathsf{G}}$ by Chapter 9, which pushes forward by left Kan extension to $n \mapsto (S^{E_1}(n))_{h\mathrm{GL}_n(\mathbb{F})}$ in $s\mathrm{Sets}^{\mathbb{N}}$. The homology of this Borel construction is calculated by a spectral sequence which gives the first isomorphism. The second isomorphism is more special: it uses a trick due to Nesterenko and Suslin, which implies that even though the Tits building and Charney's split building are quite different, their homotopy orbits by $\mathrm{GL}_n(\mathbb{F})$ have the same homology. We will not elaborate on this Nesterenko–Suslin argument, but see the original paper [NS89] or [GKRW20, Section 5.3].

We could now in principle try to use knowledge about the Steinberg module, for example the basis given by Solomon–Tits, to calculate E_1 homology as $H_*(B\mathrm{GL}_n(\mathbb{F}); \mathrm{St}_n)$. For instance, Lee and Szczarba computed that the coinvariants of the Steinberg module vanish [LS76]. We will instead move on to E_2 and E_∞ homology.

12.2.2 E_2 and E_∞ homology

The E_1 indecomposables of $\mathbf{R} \in s\mathrm{Sets}^{\mathbb{N}}$ can be computed as the left Kan extension of the indecomposables of $\mathbf{T} \in s\mathrm{Sets}^{\mathsf{G}}$. By Theorem 7.1.9 we have

$$S^0 \vee S^1 \wedge Q^{E_1}_{\mathbb{L}}(\mathbf{T}) \simeq B^{E_1}(\mathbf{T}^+_+, \epsilon_{\mathrm{can}}),$$

in the notation from Chapter 9. One of the +'s denotes that we passed to a pointed setting $(s\mathrm{Sets}, \times) \to (s\mathrm{Sets}_*, \wedge)$, the other denotes that we passed from non-unital to unital E_k-algebras which we kept track of using the augmentation ϵ_{can}. Similarly, by Theorem 7.1.9 we also have that E_k indecomposables are computed by the iterated bar construction

$$S^0 \vee S^k \wedge Q^{E_k}_{\mathbb{L}}(\mathbf{T}) \simeq B^{E_k}(\mathbf{T}^+_+, \epsilon_{\mathrm{can}}).$$

When spelling out the iterated bar construction, one gets a k-fold simplicial pointed set of "k-dimensional splittings" of $\mathbb{F}^n$.

Let us say that a "splitting" of $\mathbb{F}^n$ consists of a finite pointed set X and a function $f\colon X \to \mathsf{Sub}(\mathbb{F}^n)$; the target is the set of subspaces of $\mathbb{F}^n$, with the property that the natural map $\oplus_{x\in X} f(x) \to \mathbb{F}^n$ is an isomorphism. Let us write

$$\mathcal{S}(X) = \frac{\text{set of splittings } f : X \to \mathsf{Sub}(\mathbb{F}^n)}{\text{those with } f(*) \neq 0}.$$

This is also the set of splittings $X \setminus \{*\} \to \mathsf{Sub}(M)$ with an extra basepoint, but the description above makes it clear that $\mathcal{S}$ is functorial in all maps of pointed finite sets.

A concise way of explaining what the k-fold simplicial set $B^{E_k}(\mathbf{T}_+^+, \epsilon)$ is, is as the composition of $\mathcal{S}$ with

$$\begin{aligned} \Delta^{\mathrm{op}} \times \cdots \times \Delta^{\mathrm{op}} &\longrightarrow \mathrm{Sets}_* \\ ([p_1], \cdots, [p_k]) &\longmapsto S^1_{p_1} \wedge \cdots \wedge S^1_{p_k}, \end{aligned}$$

where $S^1_\bullet$ is the usual simplicial circle, with $p+1$ many p-simplices, one of which is the basepoint. We denote this k-fold pointed simplicial set by $\widetilde{D}^k(\mathbb{F}^n)$, and have a pointed homotopy equivalence

$$B^{E_k}(\mathbf{T}_+^+, \epsilon)(n) \simeq \widetilde{D}^k(\mathbb{F}^n).$$

The non-basepoint $(p_1, \cdots, p_k)$-simplices of $\widetilde{D}^k(\mathbb{F}^n)$ are $(p_1 \times \cdots \times p_k)$-tuples of subspaces of $\mathbb{F}^n$, forming a direct sum decomposition of $\mathbb{F}^n$. Face maps in this bisimplicial set either form direct sum decompositions with fewer summands by collecting some summands into one; the "outer" face maps either forgets summands which happen to be zero, or collapses to the base point.

Example 12.2.1. For $k = 1$, $\widetilde{D}^1(\mathbb{F}^n)$ is a double suspension of $\mathcal{S}^{E_1}(\mathbb{F}^n)$ so is homotopy equivalent to a wedge of n-spheres.

Let us denote by $D^1(\mathbb{F}^n)$ the nerve of the poset of subspaces of $\mathbb{F}^n$ modulo those p-simplices $P_0 \subset \cdots \subset P_p$ where $P_0 \neq 0$ and $P_p \neq \mathbb{F}^n$. There there is a map

$$\widetilde{D}^1(\mathbb{F}^n) \longrightarrow D^1(\mathbb{F}^n)$$

of pointed simplicial sets given by sending a splitting to its flag of "partial sums."

Recording the flags by summing along each of the k directions yields a map

$$\widetilde{D}^k(\mathbb{F}^n) \longrightarrow D^1(\mathbb{F}^n) \wedge \cdots \wedge D^1(\mathbb{F}^n)$$

and we define $D^k(\mathbb{F}^n)$ as the image. The Nesterenko–Suslin argument generalises to yield that

$$\widetilde{D}^k(\mathbb{F}^n) \longrightarrow D^k(\mathbb{F}^n)$$

becomes an equivalence on homotopy orbits by $\mathrm{GL}_n(\mathbb{F})$ for all k. The case $k = 2$ has a special property: in this case every two flags arise from a splitting, so the inclusion

$$D^2(\mathbb{F}^n) \longrightarrow D^1(\mathbb{F}^n) \wedge D^1(\mathbb{F}^n)$$

is the identity. The right side is a wedge products of two pointed spaces that are homotopy equivalent to wedges of n-spheres; hence it is homotopy equivalent to a wedge of $2n$-spheres. Putting back in the double suspension on the right side of Theorem 7.1.9 we obtain [GKRW19, Theorem 6.5]:

$$H^{E_2}_{n,d}(\mathbf{R}) = 0 \quad \text{for } d < 2n-2, \text{ and} \qquad H^{E_2}_{n,2n-2}(\mathbf{R}) = (\mathrm{St}_n \otimes \mathrm{St}_n)_{\mathrm{GL}_n(\mathbb{F})}.$$

Thus is naturally leads to the question of the $\mathrm{GL}_n(\mathbb{F})$-coinvariants of the tensor square of the Steinberg module.

By explicit argument using matrix manipulations, we proved that a natural pairing on St_n induces an isomorphism [GKRW20, Theorem A]

$$(\mathrm{St}_n \otimes \mathrm{St}_n)_{\mathrm{GL}_n(\mathbb{F})} \xrightarrow{\cong} \mathbb{Z}.$$

This is interesting in its own right, proving that the Steinberg module is indecomposable (i.e. not a direct sum of two non-zero $\mathbb{Z}[\mathrm{GL}_n(\mathbb{F})]$-modules). It further implies that the E_2-homology vanishes below the line $d = 2g-2$ and is given by $\mathbb{Z}$'s on this line.

12.2.3 The maps $\coprod BS_n \to \coprod B\mathrm{GL}_n \to \mathbb{N}$

Our goal is to understand these $\mathbb{Z}$'s on the line $d = 2g-2$. Let us now consider the maps between the following three non-unital E_∞-algebras

$$\coprod_{n=1}^{\infty} BS_n \longrightarrow \coprod_{n=1}^{\infty} B\mathrm{GL}_n(\mathbb{F}) \longrightarrow \mathbb{N}.$$

As outlined above, the middle one has $H^{E_2}_{n,2n-2} = \mathbb{Z}$ and vanishing E_2-homology below that. As Chapter 8 explained, the same is true for $\mathbb{N}$ (which also has vanishing E_2 homology above this line, but we shall not use this). Entirely analogous argument show that this is also true for the first one, at least up to isomorphism

Let us now work in $s\mathrm{Sets}^{\mathbb{N}}$, and write (cofibrant replacements of) these as

$$\mathbf{S} \longrightarrow \mathbf{R} \longrightarrow \mathbf{N}.$$

Then it turns out that there are abstract isomorphisms

$$H^{E_2}_{n,2n-2}(\mathbf{S}) \cong H^{E_2}_{n,2n-2}(\mathbf{R}) \cong H^{E_2}_{n,2n-2}(\mathbf{N}) \cong \mathbb{Z},$$

and all three have $H^{E_2}_{n,d} = 0$ for $d < 2n-2$. However, the natural maps are not all isomorphisms. The easiest to show is that in rank n, the composition $\mathbf{S} \to \mathbf{N}$ induces multiplication by $\pm n!$, and with more work we show that $\mathbf{R} \to \mathbf{N}$ induces an isomorphism $\mathbb{Z} \to \mathbb{Z}$. Hence the first map also induces multiplication by $\pm n!$.

By continuing applying bar constructions, one sees

$$H^{E_k}_{n,2n-2}(\mathbf{R}) = H^{E_k}_{n,2n-2}(\mathbf{N}) = \begin{cases} \mathbb{Z} & n = 1, \\ \mathbb{Z}/p\mathbb{Z} & n = p^k \text{ with } p \text{ prime}, \\ 0 & \text{otherwise}, \end{cases}$$

for $k \geq 3$. In particular we see that although the connectivity of E_k homology went from "slope 1" to "slope 2" by passing from $k = 1$ to $k = 2$, it does not get much better by further increasing k. However, if we work rationally, the connectivity does improve by 1 in rank > 1 for $k \geq 3$.

12.2.4 E_∞-homology

Rognes defined a filtration on the algebraic theory spectrum $K(\mathbb{F})$ and showed that the associated graded could be written as $\mathbf{D}(\mathbb{F}^n)_{h\mathrm{GL}_n(\mathbb{F}^n)}$ for a certain "stable building" $\mathbf{D}(\mathbb{F}^n)$ that he defined [Rog92]. He also conjectured that this stable building should have the homotopy type of a wedge of $(2n-2)$-spheres. We prove that the homotopy orbits by $\mathrm{GL}_n(\mathbb{F})$ are as highly connected as Rognes' conjecture would imply, which may be sufficient for intended applications.

We have explained why the map $H^{E_\infty}_{n,d}(\mathbf{R}) \to H^{E_\infty}_{n,d}(\mathbf{N})$ is an isomorphism for $d \leq 2n-2$, including $2n-2$ where it may be non-zero. One may also show it is surjective for $d = 2n-1$, so that in relative homology we have

$$H^{E_\infty}_{n,d}(\mathbf{N}, \mathbf{R}) = 0 \text{ for } d < 2n.$$

These groups measure how one builds $\mathbf{N}$ from $\mathbf{R}$ by attaching E_∞-cells. The first possibly non-trivial groups

$$H^{E_\infty}_{n,2n}(\mathbf{N}, \mathbf{R})$$

look very interesting: for $n = 1$, it is $\mathbb{F}^\times$, for $n = 2$ it is the so-called *pre-Bloch group* which (up to 2-torsion) sits in an exact sequence

$$0 \longrightarrow \mu(\mathbb{F})^{\otimes 2} \longrightarrow K_3^{\mathrm{ind}}(\mathbb{F}) \longrightarrow \mathfrak{p}(\mathbb{F}) \longrightarrow \Lambda^2_{\mathbb{Z}}\mathbb{F}^\times \longrightarrow K_2(\mathbb{F}) \longrightarrow 0,$$

where $K_3^{\mathrm{ind}}(\mathbb{F})$ is defined in terms of algebraic K-theory groups of $\mathbb{F}$ and Milnor K-theory groups of $\mathbb{F}$ by the short exact sequence

$$0 \longrightarrow K_3^M(\mathbb{F}) \longrightarrow K_3(\mathbb{F}) \longrightarrow K_3^{\mathrm{ind}}(\mathbb{F}) \longrightarrow 0.$$

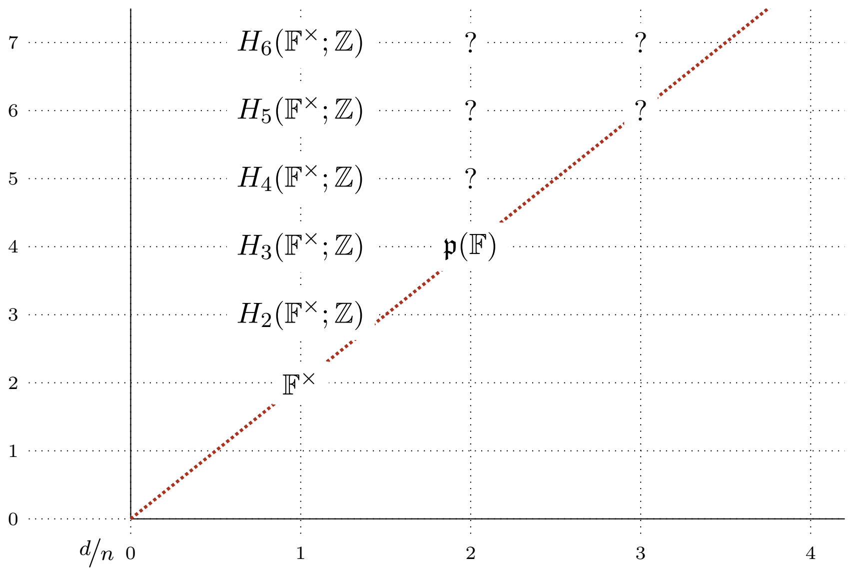


Figure 12.1: The E_∞-homology of the pair $(\mathbf{N}, \mathbf{R}_{\mathbb{Z}})$, which vanishes below the dotted line.

See Fig. 12.1 for an overview of these relative E_∞ homology groups. The groups $H_{3,6}^{E_\infty}(\mathbf{N},\mathbf{R})$ and higher are also functors of $\mathbb{F}$, it seems interesting to understand the nature of these functors.

Chapter 13

Secondary homological stability for mapping class groups II

We now finally prove secondary homological stability for mapping class groups with integer coefficients, Theorem 1.1.6. Our focus on this lecture will be to explain the features that are qualitatively different in the case of integer coefficients in comparison to that for rational coefficients as in Chapter 11.

13.1 The secondary stabilisation maps

The integral argument is supposed to be like the rational argument in Chapter 11, but complicated by a more refined "small model" $\mathbf{A}$ for the $\mathbb{Z}$-linearisation $\mathbf{R}_\mathbb{Z} \in \mathsf{Alg}_{E_2}(\mathsf{sMod}_\mathbb{Z}^\mathbb{N})$ of $\mathbf{R}$ and the presence of Dyer–Lashof operations in addition to a product and Browder bracket (after a reduction to field coefficients in $\mathbb{Q}$ and all $\mathbb{F}_\ell$). However, a significant wrinkle is the construction of the secondary stability maps, which is *not* given by multiplication-by-λ. This will have ramifications throughout the proof.

In the rational case, the map $\lambda \cdot - \colon S^{3,2} \otimes \mathbf{R}^+_\mathbb{Q}/\sigma \to \mathbf{R}^+_\mathbb{Q}/\sigma$ was obtained multiplication by a class λ on $\mathbf{R}$. Working over the integers, this is a bad idea. Firstly, it is ambiguous because there may be (and in fact, is) non-zero torsion in $H_2(B\Gamma_{3,1};\mathbb{Z})$ so there may be multiple choices of λ. Secondly, the theorem would be false with this definition. This is because the map

$$\lambda \cdot - \colon \mathbb{Z}\{1\} = H_{0,0}(\mathbf{R}^+_\mathbb{Z}) \longrightarrow H_{3,2}(\mathbf{R}^+/\sigma) = \mathbb{Z}[\mu]$$

sends 1 to 10μ, so is not surjective (as it needs to be)!

A first thought is to multiply with μ instead of a choice of λ, but $\mathbf{R}^+_\mathbb{Z}/\sigma$ is not an algebra anymore, but only a $\mathbf{R}^+_\mathbb{Z}$-module, so this is not possible. However, it does tell us that as long as we can live with coefficients in $\mathbb{Z}[\frac{1}{10}]$ rather than $\mathbb{Z}$, we *can* multiply with $\frac{\lambda}{10}$, and accept that there may be multiple choices of $\frac{\lambda}{10}$ (in fact there will not be because the torsion in $H_2(B\Gamma_{3,1};\mathbb{Z})$ is 2-torsion). The argument then goes through as in the rational case, with a more refined $\mathbf{A}$ and Dyer–Lashof operations to worry about.

Problem 13.1.1. Work out the details of this secondary homological stability theorem with coefficients in $\mathbb{Z}[\frac{1}{10}]$.

However, we wanted to prove an integral result and hence need to construct our secondary stabilisation maps in a different manner: by obstruction theory we will construct a map of $\mathbf{R}_{\mathbb{Z}}^{+}$-modules sending 1 to μ *which does not arise from multiplication with an element of* $\mathbf{R}_{\mathbb{Z}}^{+}$.

Lemma 13.1.2. *There are maps of* $\mathbf{R}_{\mathbb{Z}}^{+}$*-modules*

$$\varphi\colon S^{3,2} \wedge \mathbf{R}_{\mathbb{Z}}^{+}/\sigma \longrightarrow \mathbf{R}_{\mathbb{Z}}^{+}/\sigma$$

sending 1 *to* μ.

Proof. Recall that $S^{3,2} \otimes \mathbf{R}_{\mathbb{Z}}^{+}/\sigma$ fits in a cofiber sequence of $\mathbf{R}_{\mathbb{Z}}^{+}$-modules

$$S^{3,2} \otimes S^{1,0} \otimes \mathbf{R}_{\mathbb{Z}}^{+} \xrightarrow{\mathrm{id}\otimes(\mu\circ(\sigma\otimes\mathrm{id}))} S^{3,2} \otimes \mathbf{R}_{\mathbb{Z}}^{+} \xrightarrow{\mathrm{id}\otimes\mathrm{quot}} S^{3,2} \otimes \mathbf{R}_{\mathbb{Z}}^{+}/\sigma.$$

To construct a map $\varphi\colon S^{3,2} \otimes \mathbf{R}_{\mathbb{Z}}^{+}/\sigma \longrightarrow \mathbf{R}_{\mathbb{Z}}^{+}/\sigma$, it hence suffices to construct a map $\phi\colon S^{3,2} \otimes \mathbf{R}_{\mathbb{Z}}^{+} \to \mathbf{R}_{\mathbb{Z}}^{+}/\sigma$ of $\mathbf{R}_{\mathbb{Z}}^{+}$-modules and check its precomposition with the left map is null as a $\mathbf{R}_{\mathbb{Z}}^{+}$-module map. This map ϕ is

$$S^{3,2} \otimes \mathbf{R}_{\mathbb{Z}}^{+} \xrightarrow{\mu\otimes\mathbf{R}_{\mathbb{Z}}^{+}} \mathbf{R}_{\mathbb{Z}}^{+}/\sigma \otimes \mathbf{R}_{\mathbb{Z}}^{+} \xrightarrow{\beta} \mathbf{R}_{\mathbb{Z}}^{+} \otimes \mathbf{R}_{\mathbb{Z}}^{+}/\sigma \xrightarrow{\mathrm{act}} \mathbf{R}_{\mathbb{Z}}^{+}/\sigma.$$

To verify it is null as $\mathbf{R}_{\mathbb{Z}}^{+}$-module map, it suffices to verify its restriction $\phi \circ (\mathrm{id} \otimes \sigma)$ given by

$$S^{3,2} \otimes S^{1,0} \xrightarrow{\mathrm{id}\otimes\sigma} S^{3,2} \otimes \mathbf{R}_{\mathbb{Z}}^{+} \xrightarrow{\mu\otimes\mathbf{R}_{\mathbb{Z}}^{+}} \mathbf{R}_{\mathbb{Z}}^{+}/\sigma \otimes \mathbf{R}_{\mathbb{Z}}^{+} \xrightarrow{\beta} \mathbf{R}_{\mathbb{Z}}^{+} \otimes \mathbf{R}_{\mathbb{Z}}^{+}/\sigma \xrightarrow{\mathrm{act}} \mathbf{R}_{\mathbb{Z}}^{+}/\sigma. \quad (13.1)$$

is null as a map in $\mathsf{sMod}_{\mathbb{Z}}^{\mathbb{N}}$. This is an element of $H_{4,2}(\mathbf{R}_{\mathbb{Z}}^{+}/\sigma) = H_2(B\Gamma_{4,1}, B\Gamma_{3,1}; \mathbb{Z})$ which vanishes by homological stability for mapping class groups, so it is indeed null-homotopic. ☐

Remark 13.1.3. As usual in obstruction theory, the choices of φ are a torsor for $H_{4,3}(\mathbf{R}_{\mathbb{Z}}^{+}/\sigma) = H_3(B\Gamma_{4,1}, B\Gamma_{3,1}; \mathbb{Z})$, which is indeed a non-zero torsion group. Our proof will show that *any* secondary stabilisation map constructed this way is an isomorphism or surjection in a range.

Remark 13.1.4. This proof used a technique which is useful throughout topology: constructing maps by obstruction theory can often become easier by imposing more conditions on the map you are trying to construct. Here this is done by requiring it is a $\mathbf{R}_{\mathbb{Z}}^{+}$-module map.

13.2 Integral secondary homological stability for mapping class groups

Fix a φ as in Lemma 13.1.2. To prove Theorem 1.1.6 with integer coefficients we need to prove that its cofiber C_φ, which fits into a cofiber sequence

$$S^{3,2} \wedge \mathbf{R}_{\mathbb{Z}}^{+}/\sigma \xrightarrow{\varphi} \mathbf{R}_{\mathbb{Z}}^{+}/\sigma \longrightarrow \mathsf{C}_\varphi,$$

has the property that $H_{g,d}(\mathsf{C}_\varphi) = 0$ for $4d \leq 3g - 1$. As in the rational case, this is done by building a "small model" $\mathbf{A} \to \mathbf{R}_{\mathbb{Z}}$, proving the result for $\mathbf{A}$, and showing it transfers to $\mathbf{R}_{\mathbb{Z}}$. There will be some additional wrinkles due to the different nature of φ in comparison to $\lambda \cdot -$.

Construction of A

Look once more to Fig. 11.1. We will also need that From it, we obtain the following:

- A map $S^{1,0}_{\mathbb{Z}}\sigma \to \mathbf{R}_{\mathbb{Z}}$ representing σ.
- A map $S^{1,1}_{\mathbb{Z}}\tau \to \mathbf{R}_{\mathbb{Z}}$ representing τ.

These combine to a map $\mathbf{E}_2(S^{1,0}_{\mathbb{Z}}\sigma \oplus S^{1,1}_{\mathbb{Z}}\tau) \to \mathbf{R}_{\mathbb{Z}}$. Since we have relations $10\sigma\tau = 0$, $Q^1_{\mathbb{Z}}(\sigma) = m\sigma\tau$ for some $m \in \mathbb{Z}/10$ (in fact, $m = 3$ but we will not need this), and $\sigma^2\tau = 0$ in the target, picking null-homotopies we get an extension of this map to one with domain

$$\mathbf{A} \coloneqq \mathbf{E}_2(S^{1,0}_{\mathbb{Z}}\sigma \oplus S^{1,1}_{\mathbb{Z}}\tau) \cup^{E_2}_{10\sigma\tau} D^{2,2}_{\mathbb{Z}}\rho_1 \cup_{Q^1_{\mathbb{Z}}(\sigma)-m\sigma\tau} D^{2,2}_{\mathbb{Z}}\rho_2 \cup^{E_2}_{\sigma^2\tau} D^{3,2}_{\mathbb{Z}}\rho_3.$$

The reason we do not see a free generator λ as before, is that its corresponding homology class is obtained from ρ_3 (10 times its attachment map vanishes, so ρ_3 will give rise to a rational class). We could also not avoid having τ as a generator, given that ρ_3 is attached along a multiple of it (so our model is more like the $\mathbf{A}'$ of Lemma 11.2.4 than $\mathbf{A}$).

Lemma 13.2.1. *$H^{E_2}_{g,d}(\mathbf{R}_{\mathbb{Z}}, \mathbf{A}) = 0$ for $4d \leq 3g - 1$.*

Proof sketch. As in Lemma 11.2.4 for the rational case, using the vanishing of E_2-homology by arc complexes to deal with $g \geq 4$, and doing $g \leq 3$ by the Hurewicz theorem. □

Proof of secondary homological stability for A

By construction, $\mathbf{A}$ satisfies that $H^{E_2}_{g,d}(\mathbf{A}) = 0$ for $d < g - 1$ and that $H_{1,1}(\mathbf{A}) \to H_{2,1}(\mathbf{A})$ is surjective. Thus it has the same homological stability range as $\mathbf{R}_{\mathbb{Z}}$ and we conclude that $H_{4,2}(\mathbf{A}^+/\sigma) = 0$. This allows us to construct a map $\alpha\colon S^{3,2} \otimes \mathbf{A}^+/\sigma \to \mathbf{A}^+/\sigma$ and by picking the null-homotopy used in the construction of φ to arise from that in the construction of α, we may assume that there is a homotopy-commutative diagram

$$\begin{array}{ccc} S^{3,2} \otimes \mathbf{A}^+/\sigma & \xrightarrow{\alpha} & \mathbf{A}^+/\sigma \\ \downarrow & & \downarrow \\ S^{3,2} \otimes \mathbf{R}^+/\sigma & \xrightarrow{\varphi} & \mathbf{R}^+/\sigma. \end{array}$$

Let C_α denote the cofiber of α.

Lemma 13.2.2. *$H_{g,d}(\mathsf{C}_\alpha) = 0$ for $4d \leq 3g - 1$.*

Proof. We intend to proceed as in Lemma 11.2.5, by endowing $\mathbf{A}$ with its skeletal filtration to obtain $\mathrm{sk}\mathbf{A} \in \mathsf{Alg}_{E_2}(\mathsf{sMod}^{\mathbb{N}\times\mathbb{Z}_\leq}_{\mathbb{Z}})$ and performing a computation in the spectral sequence for the corresponding filtration on the mapping cone. Before we can do so, we need to prove that α lifts to a filtered map on $\mathrm{sk}\mathbf{A}$. In particular, we will need to show that it lifts to a map

$$\mathrm{sk}\alpha\colon S^{3,2,3} \otimes \mathrm{sk}\mathbf{A}^+/\sigma \longrightarrow \mathrm{sk}\mathbf{A}^+/\sigma,$$

where the sphere $S^{3,2,3}$ has this trigrading so that it will eventually yields an element whose d^1 which can hit the lift of ρ_3 in tridegree $(3,2,2)$. We could take any $S^{3,2,q}$ for $q \geq 3$ but then we would need to consider higher differentials.

The idea is to use the same construction as for φ, but replace $\mathbf{R}_{\mathbb{Z}}^+$ with $\mathrm{sk}\mathbf{A}^+$ and $S^{3,2}$ with $S^{3,2,3}$. In particular, one reduces the construction to proving that the obstruction class in $H_{4,2,3}(\mathrm{sk}\mathbf{A}^+/\sigma)$ analogous to (13.1) vanishes. In principle there could be an obstruction here, but we know there is no such obstruction when we take the colimit, so our strategy will be to reduce to this case. We first observe that $H_{4,3,q}(\mathrm{gr}(\mathrm{sk}\mathbf{A}^+/\sigma)) = 0$ when $q \geq 4$; it suffices to verify this with coefficients in a field $\mathbb{F}$ given by $\mathbb{Q}$ pr $\mathbb{F}_\ell$. Then we use F. Cohen formula's for the homology of a free E_k^+-algebra as well as the fact that $\sigma, \tau, \rho_1, \rho_2, \rho_3$ have filtration degree equal to their homological degree and all operations preserve the filtration degree and increase homological degree. By consideration of the long exact sequence of a pair for each filtration step, we see that

$$H_{4,2,q}(\mathrm{sk}\mathbf{A}_{\mathbb{F}}^+/\sigma) \longrightarrow H_{4,2,q+1}(\mathrm{sk}\mathbf{A}_{\mathbb{F}}^+/\sigma)$$

is injective for $q \geq 3$. Since the colimit vanishes, this proves that the obstruction group vanishes and thus we can find our map $\mathrm{sk}\alpha$.

Having done so, we can proceed with the proof. As in the generic homological stability result, it suffices to prove the result after tensoring with $\mathbb{Q}$ or $\mathbb{F}_\ell$. Let us focus on the latter, with ℓ odd ([GKRW19] of course contains the full argument). We consider spectral sequences for the filtered object $\mathsf{C}_{\mathrm{sk}\alpha,\mathbb{F}_\ell}$. Since $\mathrm{sk}\alpha$ strictly increases filtration and taking associated graded commutes with cofibers, we have

$$\mathrm{gr}(\mathsf{C}_{\mathrm{sk}\alpha,\mathbb{F}_\ell}) \simeq (S_{\mathbb{F}_\ell}^{0,0,0} \oplus S_{\mathbb{F}_\ell}^{3,3,3}\rho_4) \otimes \mathrm{gr}(\mathrm{sk}\mathbf{A}_{\mathbb{F}_\ell}^+/\sigma),$$

where ρ_4 denoted the generator the right term in the cofiber sequence $S^{3,2,3} \to 0S^{3,3,3}$. Hence the E^1-page of the latter spectral sequence is given by

$$E^1_{g,p,q} = ((S_{\mathbb{F}_\ell}^{0,0,0} \oplus S_{\mathbb{F}_\ell}^{3,3,3}\rho_4) \otimes S^*_{\mathbb{F}_\ell}(L/\langle\sigma\rangle), d^1)$$

where L is a graded vector space of Dyer–Lashof operations applied to bracketings of $\sigma, \tau, \rho_1, \rho_2, \rho_3$. This is a module over the corresponding spectral sequence for the filtered object $\mathrm{sk}\mathbf{A}_{\mathbb{F}_\ell}^+$, from which we deduce that the d^1-differential satisfies $d^1([\sigma,\sigma]) = 0$, $d^1(\rho_2) = -\frac{1}{2}[\sigma,\sigma]$, $d^1(\rho_4) = \rho$, and is a derivation. Filtering away the remaining differential, a short computation shows that non-zero class in its homology of lowest slope (equal to $\frac{2\ell-1}{2\ell} \geq \frac{5}{6}$) is represented by $[\sigma,\sigma] \cdot \rho_2^{\ell-1}$. ☐

Once more, an inspection of the proof show that one may as well have added added more freely attached E_2-cells of slope $\geq \frac{3}{4}$ to $\mathbf{A}$; we can still construct an α, lift it to the skeletal filtration, and prove in the same manner as in the lemma that $H_{g,d}(\mathsf{C}_\alpha) = 0$ for $4d \leq 3g - 1$. Moreover, for the first two of these steps we do not even need the E_2-cells to be trivially attached.

Proof of secondary homological stability for $\mathbf{R}_{\mathbb{Z}}$

As in the rational case we use this to prove that C_φ, the cofiber of the secondary stabilisation map on $\mathbf{R}_{\mathbb{Z}}^+$, has the same property. Applying the CW-approximation in combination with Lemma 13.2.1 we get a factorisation in $\mathsf{Alg}_{E_2}(\mathsf{sMod}_{\mathbb{Z}}^{\mathbb{N}})$

$$\mathbf{A} \longrightarrow \mathbf{B} \xrightarrow{\sim} \mathbf{R}_{\mathbb{Z}}$$

where $\mathbf{B}$ is obtained by attaching only E_2-cells in bidegrees (g_α, d_α) with $4d_\alpha \geq 3d_\alpha$. On $\mathbf{B}$ we can construct a secondary stabilisation map β and hence C_β, and since the right map is a weak equivalence it suffices to prove that C_β has the desired vanishing range. This is done as in the rational range by filtering away the attaching maps of the attaching E_2-cells to reduce to the case of $\mathbf{A}$ with freely attached E_2-cells. As in the proof of Lemma 13.2.2, there is a wrinkle in proving that β lifts to a filtered map, which is handled in exactly the same manner. This completes the proof of Theorem 1.1.6 with integer coefficients.

Remark 13.2.3. The argument in Section 5.3 of [GKRW19] is different than the one given here. It rather follows the suggestion of Remark 11.2.8 of recognising $\mathbf{A}$ is the beginning of a CW-approximation $\mathbf{B} \to \mathbf{R}_{\mathbb{Z}}$, constructing a map β which lifts to the skeletal filtration, and using the spectral sequence for the cofiber of β on $\mathrm{sk}\mathbf{B}^+/\sigma$.

13.3 Improving Theorem 1.1.6

There are two improvements which can be made to the secondary homological stability theorem.

(1) It extends to certain local coefficient systems; those arising from the tensor powers of $H_1(\Sigma_{g,1}; \Bbbk)$. This is done in Section 5.5 of [GKRW19]. This improved homological stability result can be improved with Tommasi's computation for $H^*(B\Gamma_4; \mathbb{Q})$ to obtain that $H_3(B\Gamma_{4,1}; \mathbb{Q}) = 0$. This is done in Section 6.2 of [GKRW19].

(2) An upshot of (1) is that $\mathbf{A} \to \mathbf{R}_{\mathbb{Q}}$ is a better approximation than expected: $H^{E_2}_{g,d}(\mathbf{R}_{\mathbb{Q}}, \mathbf{A}) = 0$ for $5d \leq 4g - 1$. Feeding this back into the rational secondary homological stability argument improves its range from slope $\frac{3}{4}$ to $\frac{4}{5}$. This is done in Section 6.1 of [GKRW19].

Remark 13.3.1. What is needed for further improvements?

- A computation of $H_3(B\Gamma_{4,1}; \mathbb{Z})$ would be the main input for improving the range with integer coefficients rather than with rational coefficients.
- A computation of $H_4(B\Gamma_{4,1}; \mathbb{Q})$ and $H_4(B\Gamma_{5,1}; \mathbb{Q})$ would be the main input for improving the range with rational coefficients from slope $\frac{4}{5}$ to $\frac{5}{6}$.

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