TOPOLOGICAL EMBEDDINGS OF S^2 IN S^4

OSCAR RANDAL-WILLIAMS

The purpose of this note is to explain the following result, on the space $\mathrm{Emb}^t(S^2, S^4)$ of locally flat topological embeddings of S^2 into S^4.

Theorem. $\pi_4(\mathrm{Emb}^t(S^2, S^4), inc)$ is not finitely-generated.

This is complementary to an analogous non-finite-generation result for the space of *smooth* embeddings by Budney–Gabai [BG19, Theorem 10.1 (4)] (though it is a different homotopy group which we show is non-finitely-generated). We will deduce the theorem from the work of Budney–Gabai [BG19, BG23] and classical tools in geometric topology.

The argument. All spaces of embeddings and so on are given by (realisation of) simplicial sets. There is a homotopy fibre sequence

$$\mathrm{Emb}^t_*(S^2, S^4) \longrightarrow \mathrm{Emb}^t(S^2, S^4) \longrightarrow S^4$$

given by restricting embeddings to a point (using the parameterised isotopy extension theorem e.g. as in [EK71] or [BL74, p. 19]). Writing $V^t_{4,2}$ for the space of germs of pointed embeddings of $\mathbb{R}^2$ into $\mathbb{R}^4$ there is a homotopy fibre sequence

$$\mathrm{Emb}^t_c(\mathbb{R}^2, \mathbb{R}^4) \longrightarrow \mathrm{Emb}^t_*(S^2, S^4) \longrightarrow V^t_{4,2}$$

given by restricting pointed embeddings to their germ near the point (again by parameterised isotopy extension). By the Alexander trick the component of the standard inclusion of the fibre of the latter fibration is contractible, giving a homotopy fibre sequence

$$V^t_{4,2} \longrightarrow \mathrm{Emb}^t(S^2, S^4)_{inc} \longrightarrow S^4. \tag{*}$$

The square

$$\begin{array}{ccc} \mathrm{Emb}^t_{(\partial D^2)\times D^2}(D^2 \times D^2, D^4) & \longrightarrow & \mathrm{Imm}^t_{(\partial D^2)\times D^2}(D^2 \times D^2, D^4) \\ \downarrow & & \downarrow \\ \mathrm{Emb}^t_\partial(D^2, D^4) & \longrightarrow & \mathrm{Imm}^t_\partial(D^2, D^4) \end{array} \tag{**}$$

is cartesian by [Las76, (a) and (b)] (Lashof supposes in this paper that the ambient dimension is ≥ 5, but for this particular result all one needs is parameterised isotopy extension). The component of the standard inclusion of the bottom left corner is contractible by the Alexander trick. The component of the standard inclusion of the top left corner may be expressed as

$$\mathrm{Emb}^t_{(\partial D^2)\times D^2}(D^2 \times D^2, D^4)_{inc} \simeq \frac{\mathrm{Homeo}_\partial(D^4)}{\mathrm{Homeo}_\partial(S^1 \times D^3)} \simeq B\mathrm{Homeo}_\partial(S^1 \times D^3), \tag{***}$$

again using the Alexander trick. By immersion theory [Lee69, BL74], in the topological category the right-hand vertical map is identified with $\Omega^2(\mathrm{Top}(4) \to V^t_{4,2})$, so

$$\mathrm{Emb}^t_{(\partial D^2)\times D^2}(D^2 \times D^2, D^4)_{inc} \simeq \Omega^2_0 \mathrm{hofib}(\mathrm{Top}(4) \to V^t_{4,2}).$$

In particular there is a map $\mathrm{Emb}^t_{(\partial D^2)\times D^2}(D^2 \times D^2, D^4)_{inc} \to \Omega^2_0 \mathrm{Top}(4)$.

Theorem (Budney–Gabai). The map

$$\pi_1(\mathrm{Emb}^d_{(\partial D^2)\times D^2}(D^2 \times D^2, D^4)) \longrightarrow \pi_1(\mathrm{Emb}^t_{(\partial D^2)\times D^2}(D^2 \times D^2, D^4))$$

has non-finitely-generated image.

Proof. We consider smooth version of the first equivalence in (***), and form the diagram

$$\begin{array}{ccccc} \pi_1(\mathrm{Emb}^d_{(\partial D^2)\times D^2}(D^2 \times D^2, D^4)) & \longrightarrow & \pi_1(B\mathrm{Diff}_\partial(S^1 \times D^3)) & \longrightarrow & \pi_1(B\mathrm{Diff}_\partial(D^4)) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_1(\mathrm{Emb}^t_{(\partial D^2)\times D^2}(D^2 \times D^2, D^4)) & \xrightarrow{\sim} & \pi_1(B\mathrm{Homeo}_\partial(S^1 \times D^3)) & \longrightarrow & 0. \end{array}$$

The top sequence is split, by choosing a $D^4 \subset S^1 \times D^3$.

In [BG19, §8] [BG23, p. 13] there are described certain diffeomorphisms $\delta_k \in \pi_0(\mathrm{Diff}_\partial(S^1 \times D^3))$, and in [BG23] there is constructed an invariant

$$W'_3 : \pi_0(\mathrm{Homeo}_\partial(S^1 \times D^3)) \otimes \mathbb{Q} \longrightarrow \text{a rational vector space}$$

under which the $W'_3(\delta_k)$ are linearly independent for $k \geq 4$. Applying the splitting of the top sequence to the elements δ_k gives an infinite sequence of elements of $\pi_1(\mathrm{Emb}^d_{(\partial D^2)\times D^2}(D^2 \times D^2, D^4))$ whose images in $\pi_1(\mathrm{Emb}^t_{(\partial D^2)\times D^2}(D^2 \times D^2, D^4)) \otimes \mathbb{Q}$ are linearly independent. □

Using this, the map from the smooth version of (**) to it gives a diagram

$$\begin{array}{ccccc} & & \pi_1(\mathrm{Emb}^d_{(\partial D^2)\times D^2}(D^2 \times D^2, D^4)) & \longrightarrow & \pi_3(\mathrm{O}(4)) \\ & & \downarrow & & \downarrow \\ \pi_4(V^t_{4,2}) & \longrightarrow & \pi_1(\mathrm{Emb}^t_{(\partial D^2)\times D^2}(D^2 \times D^2, D^4)) & \longrightarrow & \pi_3(\mathrm{Top}(4)) \end{array}$$

in which the bottom row is exact. As $\pi_3(\mathrm{O}(4))$ is finitely-generated, the result above implies that $\pi_4(V^t_{4,2})$ is non-finitely-generated. With (*) this proves the claimed Theorem.

Remark. Alexander Kupers pointed out to me the following alternative strategy. The map $\mathrm{Top}(4)/\mathrm{O}(4) \to Top/O$ is 5-connected [FQ90, Theorem 8.7A], and Top/O has finite homotopy groups [KS77, Essay IV, §10.12] so $\pi_3(\mathrm{Top}(4))$ is finitely-generated. This avoids having to introduce the space of smooth embeddings: instead one can invoke [BG23, Theorem 1.1] to see that $\pi_1(B\mathrm{Homeo}_\partial(S^1 \times D^3)) \cong \pi_1(\mathrm{Emb}^t_{(\partial D^2)\times D^2}(D^2 \times D^2, D^4)) \cong \pi_3(\mathrm{hofib}(\mathrm{Top}(4) \to V^t_{4,2}))$ is non-finitely-generated, and conclude the same for $\pi_4(V^t_{4,2})$.

Comment on other dimensions. The Theorem should not be considered to be a feature of dimension 4: rather, as a generic feature of spaces of codimension 2 embeddings.

For $d \geq 5$, where smoothing theory applies, we consider the fibre sequences

$$\Omega^{d-2}\mathrm{fib}(\tfrac{\mathrm{O}(d)}{\mathrm{O}(2)} \to V^t_{d,d-2}) \xrightarrow{\sim} \mathrm{Emb}^d_\partial(D^{d-2}, D^d) \longrightarrow \mathrm{Emb}^t_\partial(D^{d-2}, D^d) \simeq *$$

$$* \simeq \Omega^{d-2}\mathrm{O}(2) \longrightarrow \mathrm{Emb}^d_{(\partial D^{d-2})\times D^2}(D^{d-2}\times D^2, D^d) \xrightarrow{\sim} \mathrm{Emb}^d_\partial(D^{d-2}, D^d)$$

and

$$\mathrm{Emb}^d_{(\partial D^{d-2})\times D^2}(D^{d-2}\times D^2, D^d)_{inc} \simeq \frac{\mathrm{Diff}_\partial(D^d)}{\mathrm{Diff}_\partial(S^1\times D^{d-1})}.$$

For $d \neq 4,5,7$ the space $\mathrm{Diff}_\partial(D^d)$ has finitely-generated homotopy groups, by a theorem of Kupers [Kup19], and for $d \gg 0$ we have

$$\pi_1(B\mathrm{Diff}_\partial(S^1\times D^{d-1})) \supset \bigoplus_\infty \mathbb{Z}/2$$

by e.g. [Hat78] (or by [BRW22] for all $d \geq 6$ even). Thus $\pi_d(V^t_{d,d-2})$ is non-finitely-generated, so using the analogue of (*) we see that $\pi_d(\mathrm{Emb}^t(S^{d-2}, S^d))$ is too. (We also see that $\pi_1(\mathrm{Emb}^d(S^{d-2}, S^d))$ is non-finitely-generated using similar sequences.)

Acknowledgement. I am grateful to Alexander Kupers for feedback.

References

[BG19] R. Budney and D. Gabai, *Knotted 3-balls in S^4*, https://arxiv.org/abs/1912.09029, 2019.

[BG23] ———, *On the automorphism groups of hyperbolic manifolds*, https://arxiv.org/abs/2303.05010, 2023.

[BL74] D. Burghelea and R. Lashof, *The homotopy type of the space of diffeomorphisms. I, II*, Trans. Amer. Math. Soc. **196** (1974), 1–36; ibid. **196** (1974), 37–50.

[BRW22] M. Bustamante and O. Randal-Williams, *On automorphisms of high-dimensional solid tori*, https://arxiv.org/abs/2010.10887, Geometry & Topology, to appear, 2022.

[EK71] R. D. Edwards and R. C. Kirby, *Deformations of spaces of imbeddings*, Ann. Math. (2) **93** (1971), 63–88 (English).

[FQ90] M. H. Freedman and F. S. Quinn, *Topology of 4-manifolds*, Princeton Math. Ser., vol. 39, Princeton, NJ: Princeton University Press, 1990 (English).

[Hat78] A. E. Hatcher, *Concordance spaces, higher simple-homotopy theory, and applications*, Algebraic and geometric topology (Stanford, Calif., 1976), Part 1, Proc. Sympos. Pure Math., XXXII, Amer. Math. Soc., Providence, R.I., 1978, pp. 3–21.

[KS77] R. C. Kirby and L. C. Siebenmann, *Foundational essays on topological manifolds, smoothings and triangulations*, Ann. Math. Stud., vol. 88, Princeton University Press, Princeton, NJ, 1977 (English).

[Kup19] A. Kupers, *Some finiteness results for groups of automorphisms of manifolds*, Geom. Topol. **23** (2019), no. 5, 2277–2333.

[Las76] R. Lashof, *Embedding spaces*, Illinois J. Math. **20** (1976), no. 1, 144–154.

[Lee69] J. A. Lees, *Immersions and surgeries of topological manifolds*, Bull. Amer. Math. Soc. **75** (1969), 529–534.

Email address: o.randal-williams@dpmms.cam.ac.uk

Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WB, UK