A REMARK ON THE HOMOLOGY OF TEMPERLEY–LIEB ALGEBRAS

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Write TL_n for the universal Temperley–Lieb algebra, defined over the polynomial ring $\mathbb{Z}[a]$ with parameter $a \in \mathbb{Z}[a]$. If R is a commutative ring and $a \in R$ is an element then there is a homomorphism $\mathbb{Z}[a] \to R$ and $\mathrm{TL}_n(R, a) = \mathrm{TL}_n \otimes_{\mathbb{Z}[a]} R$.

In [BH20] Boyd and Hepworth study the (co)homology of the algebras $\mathrm{TL}_n(R, a)$ under various assumptions on the element $a \in R$, specifically either

(i) a is a unit in R, or
(ii) $a = v + v^{-1}$ for some unit v in R.

In this note I will explain how a universal-coefficient-type result with respect to the ring R can slightly strengthen their results.

1. Base change

The R-module $\mathrm{TL}_n(R, a)$ is finitely-generated and free, so it is in particular R-flat. Thus for any $p \geq 0$

$$\mathrm{TL}_n(R, a)^{\otimes_R p+1} = \mathrm{TL}_n(R, a) \otimes_R \mathrm{TL}_n(R, a)^{\otimes_R p}$$

is a flat left $\mathrm{TL}_n(R, a)$-module, and so the bar resolution $B_*(\mathrm{TL}_n(R, a), \mathrm{TL}_n(R, a), \mathbb{1})$ formed in the category of R-modules, i.e. having

$$B_p(\mathrm{TL}_n(R, a), \mathrm{TL}_n(R, a), \mathbb{1}) = \mathrm{TL}_n(R, a) \otimes_R \mathrm{TL}_n(R, a)^{\otimes_R p} \otimes_R \mathbb{1}$$

is a resolution by flat left $\mathrm{TL}_n(R, a)$-modules. Thus we may calculate $\mathrm{Tor}_*^{\mathrm{TL}_n(R,a)}(\mathbb{1}, \mathbb{1})$ using the bar complex $B_*(\mathbb{1}, \mathrm{TL}_n(R, a), \mathbb{1})$ formed in the category of R-modules. This is a complex of finitely-generated free R-modules.

If $R \to S$ is a ring homomorphism sending $a \in R$ to $b \in S$, then $\mathrm{TL}_n(S, b) = \mathrm{TL}_n(R, a) \otimes_R S$. It follows that

$$B_*(\mathbb{1}, \mathrm{TL}_n(S, b), \mathbb{1}) = B_*(\mathbb{1}, \mathrm{TL}_n(R, a), \mathbb{1}) \otimes_R S,$$

and as $B_*(\mathbb{1}, \mathrm{TL}_n(R, a), \mathbb{1})$ is a bounded below complex of free and hence flat R-modules there is a Base Change Spectral Sequence [Wei94, Theorem 5.6.4]

$$E^2_{p,q} = \mathrm{Tor}_p^R(\mathrm{Tor}_q^{\mathrm{TL}_n(R,a)}(\mathbb{1}, \mathbb{1}), S) \Rightarrow \mathrm{Tor}_{p+q}^{\mathrm{TL}_n(S,b)}(\mathbb{1}, \mathbb{1}). \qquad \text{(BCSS)}$$

In particular, if $R \to S$ is flat then this collapses to

$$\mathrm{Tor}_*^{\mathrm{TL}_n(R,a)}(\mathbb{1}, \mathbb{1}) \otimes_R S \xrightarrow{\sim} \mathrm{Tor}_*^{\mathrm{TL}_n(S,b)}(\mathbb{1}, \mathbb{1}).$$

If in addition $R \to S$ is faithful, then faithfully flat descent shows that

$$\mathrm{Tor}_*^{\mathrm{TL}_n(R,a)}(\mathbb{1}, \mathbb{1}) \longrightarrow \mathrm{Tor}_*^{\mathrm{TL}_n(S,b)}(\mathbb{1}, \mathbb{1}) \rightrightarrows \mathrm{Tor}_*^{\mathrm{TL}_n(S \otimes_R S,b)}(\mathbb{1}, \mathbb{1}) \qquad \text{(FFD)}$$

is an equaliser, where the parallel maps are induced by $s \mapsto 1 \otimes s, s \otimes 1 : S \to S \otimes_R S$, and we write $b = b \otimes 1 = 1 \otimes b \in S \otimes_R S$.

2. Theorem B'

The main thing I have to offer is the following strengthening of [BH20, Theorem B], which removes the assumption that $a = v + v^{-1}$.

Theorem B'. For any commutative ring R and any $a \in R$ we have

$$\mathrm{Tor}_d^{\mathrm{TL}_n(R,a)}(\mathbb{1}, \mathbb{1}) = 0$$

for $1 \leq d \leq n-2$ if n is even, and for $1 \leq d \leq n-1$ if n is odd.

Proof. Consider the R-algebra $S := R[v]/(v^2 - a \cdot v + 1)$. In the ring S we have $(a-v) \cdot v = 1$ so v is a unit with inverse $v^{-1} = a - v$, and so $a = v + v^{-1}$. Now as an R-module S is free on the basis $\{1, v\}$, so the morphism $R \to S$ is faithfully flat.

By (FFD) there is an equaliser diagram

$$\mathrm{Tor}_d^{\mathrm{TL}_n(R,a)}(\mathbb{1}, \mathbb{1}) \longrightarrow \mathrm{Tor}_d^{\mathrm{TL}_n(S,a)}(\mathbb{1}, \mathbb{1}) \rightrightarrows \mathrm{Tor}_d^{\mathrm{TL}_n(S \otimes_R S,a)}(\mathbb{1}, \mathbb{1})$$

and as $a = v + v^{-1} \in S$ it follows from [BH20, Theorem B] that the middle term vanishes for the claimed range of values of d, so the left-hand term does too. □

Remark 2.1. Alternatively, to avoid using faithfully flat descent directly one can argue that as $R \to S$ is flat we have

$$\mathrm{Tor}_d^{\mathrm{TL}_n(R,a)}(\mathbb{1}, \mathbb{1}) \otimes_R S \cong \mathrm{Tor}_d^{\mathrm{TL}_n(S,a)}(\mathbb{1}, \mathbb{1}),$$

as $a = v + v^{-1} \in S$ it follows from [BH20, Theorem B] that this vanishes for the claimed range of values of d, and that as $R \to S$ is faithful it follows that $\mathrm{Tor}_d^{\mathrm{TL}_n(R,a)}(\mathbb{1}, \mathbb{1})$ vanishes for such d too.

3. Calculations in the universal case

In this section I explain what can be said about $\mathrm{Tor}_d^{\mathrm{TL}_n}(\mathbb{1}, \mathbb{1})$ using the base change ideas of Section 1.

Lemma 3.1. *For each $d > 0$ the groups $\mathrm{Tor}_d^{\mathrm{TL}_n}(\mathbb{1}, \mathbb{1})$ are finitely-generated $\mathbb{Z}$-modules, and are (a)-torsion.*

Proof. The bar complex $B_*(\mathbb{1}, \mathrm{TL}_n, \mathbb{1})$ is a chain complex of finitely-generated $\mathbb{Z}[a]$-modules, and as this ring is noetherian it follows that the homology of this complex is degreewise a finitely-generated $\mathbb{Z}[a]$-module.

As the morphism $\mathbb{Z}[a] \to \mathbb{Z}[a, a^{-1}]$ is flat we have

$$\mathrm{Tor}_d^{\mathrm{TL}_n}(\mathbb{1}, \mathbb{1}) \otimes_{\mathbb{Z}[a]} \mathbb{Z}[a, a^{-1}] \cong \mathrm{Tor}_d^{\mathrm{TL}_n(\mathbb{Z}[a,a^{-1}],a)}(\mathbb{1}, \mathbb{1}),$$

and by [BH20, Theorem A] these groups vanish, so each of the finitely-many $\mathbb{Z}[a]$-module generators of $\mathrm{Tor}_d^{\mathrm{TL}_n}(\mathbb{1}, \mathbb{1})$ are (a)-torsion, so these are finitely-generated as $\mathbb{Z}$-modules. □

Filtering the ring $\mathbb{Z}[a]$ by powers of the ideal (a) gives an associated filtration of $\mathrm{TL}_n = \mathrm{TL}_n(\mathbb{Z}[a], a)$, having $\mathrm{gr}(\mathrm{TL}_n(\mathbb{Z}[a], a)) = \mathrm{TL}_n(\mathbb{Z}[a], 0)$. As the morphism $\mathbb{Z} \to \mathbb{Z}[a]$ is flat this gives a spectral sequence

$$E^1_{p,*} = \mathrm{Tor}_p^{\mathrm{TL}_n(\mathbb{Z},0)}(\mathbb{1}, \mathbb{1}) \otimes_{\mathbb{Z}} \mathbb{Z}[a] \Rightarrow \mathrm{Tor}_p^{\mathrm{TL}_n}(\mathbb{1}, \mathbb{1}), \tag{3.1}$$

which converges strongly as the $\mathbb{Z}[a]$-modules $\mathrm{Tor}_*^{\mathrm{TL}_n}(\mathbb{1}, \mathbb{1})$ are *a priori* known to be (a)-complete by the above lemma.

Let us investigate the groups $\mathrm{Tor}_*^{\mathrm{TL}_n(\mathbb{Z},0)}(\mathbb{1},\mathbb{1})$. By (BCSS) applied to $\mathbb{Z} \to \mathbb{F}_p$ there is a long exact sequence

$$\cdots \mathrm{Tor}_q^{\mathrm{TL}_n(\mathbb{Z},0)}(\mathbb{1},\mathbb{1}) \xrightarrow{p} \mathrm{Tor}_q^{\mathrm{TL}_n(\mathbb{Z},0)}(\mathbb{1},\mathbb{1}) \longrightarrow \mathrm{Tor}_q^{\mathrm{TL}_n(\mathbb{F}_p,0)}(\mathbb{1},\mathbb{1}) \xrightarrow{\partial} \mathrm{Tor}_{q-1}^{\mathrm{TL}_n(\mathbb{Z},0)}(\mathbb{1},\mathbb{1}) \cdots,$$

which we can consider as a Bockstein sequence.

Let $\mathbb{F}_p(i)$ denote the splitting field of the polynomial x^2+1 over $\mathbb{F}_p$ (so $\mathbb{F}_2(i) = \mathbb{F}_2$). Certainly this is $\mathbb{F}_p$-flat, so (BCSS) gives

$$\mathrm{Tor}_d^{\mathrm{TL}_n(\mathbb{F}_p,0)}(\mathbb{1},\mathbb{1}) \otimes_{\mathbb{F}_p} \mathbb{F}_p(i) \cong \mathrm{Tor}_d^{\mathrm{TL}_n(\mathbb{F}_p(i),0)}(\mathbb{1},\mathbb{1}).$$

As we have $0 = i + i^{-1} \in \mathbb{F}_p(i)$, it is a consequence of [BH20, Theorem D] that $\mathrm{Tor}_d^{\mathrm{TL}_n(\mathbb{F}_p(i),0)}(\mathbb{1},\mathbb{1}) = 0$ for all $d > 0$ if $n = 2k+1$ and p does not divide any binomial coefficient $\binom{k}{r}$. By the base change isomorphism above, $\mathrm{Tor}_d^{\mathrm{TL}_n(\mathbb{F}_p,0)}(\mathbb{1},\mathbb{1})$ vanishes under the same conditions, and so under these conditions multiplication by p acts invertibly on $\mathrm{Tor}_d^{\mathrm{TL}_n(\mathbb{Z},0)}(\mathbb{1},\mathbb{1})$.

As the $\mathrm{Tor}_d^{\mathrm{TL}_n(\mathbb{Z},0)}(\mathbb{1},\mathbb{1})$ are finitely-generated abelian groups and there exists a prime number p not dividing any $\binom{k}{r}$, it follows that these abelian groups are in fact finite, and furthermore that the primes dividing their order are factors of $\prod_{r=1}^k \binom{k}{r}$. In particular it follows that

$$\begin{aligned}
\mathrm{Tor}_d^{\mathrm{TL}_3(\mathbb{Z},0)}(\mathbb{1},\mathbb{1}) &= 0\\
\mathrm{Tor}_d^{\mathrm{TL}_5(\mathbb{Z},0)}(\mathbb{1},\mathbb{1})[\tfrac{1}{2}] &= 0\\
\mathrm{Tor}_d^{\mathrm{TL}_7(\mathbb{Z},0)}(\mathbb{1},\mathbb{1})[\tfrac{1}{3}] &= 0\\
\mathrm{Tor}_d^{\mathrm{TL}_9(\mathbb{Z},0)}(\mathbb{1},\mathbb{1})[\tfrac{1}{2\cdot 3}] &= 0\\
\mathrm{Tor}_d^{\mathrm{TL}_{11}(\mathbb{Z},0)}(\mathbb{1},\mathbb{1})[\tfrac{1}{2\cdot 5}] &= 0
\end{aligned}$$

and so on, for $d > 0$. By running the spectral sequence (3.1) the same holds for $\mathrm{Tor}_d^{\mathrm{TL}_{2k+1}}(\mathbb{1},\mathbb{1})$, and by running (BCSS) for $\mathbb{Z}[a] \to R$ the same holds for $\mathrm{Tor}_d^{\mathrm{TL}_{2k+1}(R,a)}(\mathbb{1},\mathbb{1})$ for any commutative ring R and element $a \in R$. In particular

(i) $\mathrm{Tor}_d^{\mathrm{TL}_3(R,a)}(\mathbb{1},\mathbb{1}) = 0$ for all $d > 0$ and all (R,a), and more generally

(ii) if $\prod_{r=1}^k \binom{k}{r}$ is a unit in R then $\mathrm{Tor}_d^{\mathrm{TL}_{2k+1}(R,a)}(\mathbb{1},\mathbb{1}) = 0$ for all $d > 0$.

The latter in particular applies to the classical case $R = \mathbb{C}$.

This also extends Lemma 3.1 by:

Corollary 3.2. *For each $d > 0$ and odd n the groups $\mathrm{Tor}_d^{\mathrm{TL}_n}(\mathbb{1},\mathbb{1})$ are finite abelian groups, and are (a)-torsion.*

References

[BH20] Rachael Boyd and Richard Hepworth, *On the homology of the Temperley-Lieb algebras*, arXiv:2006.04256, 2020.

[Wei94] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR 1269324

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