CORRIGENDUM TO:
ON THE COHOMOLOGY OF TORELLI GROUPS

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ABSTRACT. We correct several mistakes in the paper, in particular related to Theorem 4.1 and to the application in the classical case $2n = 2$. The qualitative results are unchanged (with one small exception), and we describe the necessary changes to the quantitative results.

We are grateful to Erik Lindell and Arthur Soulié for pointing out several of the issues that we will discuss in this corrigendum, and for discussing in depth our attempts to solve them. Unless stated otherwise, references are to the published paper [KRW20b].

1. THEOREM 4.1

1.1. **Corrected statement.** The following statement replaces Theorem 4.1:

Theorem 4.1′. *If $2n \geq 6$ the ring homomorphism*

$$\frac{i^*(K^\vee) \otimes^{\mathsf{d(s)Br}} \left(\mathcal{P}(-, \mathcal{V})'_{\geq 0} \otimes \det^{\otimes n} \right)}{(\kappa_{\mathcal{L}_i} \mid 4i - 2n > 0)} \longrightarrow H^*(B\mathrm{Tor}(W_g, D^{2n}); \mathbb{Q})$$

induced by Ψ^t is an isomorphism onto the maximal algebraic G'_g-subrepresentation of $H^*(B\mathrm{Tor}(W_g, D^{2n}); \mathbb{Q})$ in a range of degrees tending to infinity with g.

If $2n = 2$ and $H^*(B\mathrm{Tor}(W_g, D^2); \mathbb{Q})$ is finite dimensional in degrees $* < N$ for all large enough g, then this homomorphism is an isomorphism onto the maximal algebraic G'_g-subrepresentation in degrees $* \leq N$ for all large enough g.

The difference is that we have omitted "and is a monomorphism in degree $N + 1$" in the final sentence. That claim was not justified by the given proof: we will comment on this below. The corresponding claim should be removed from Theorem B.

The remaining problems concern the proof rather than the statement: the most significant of these is described in the following subsection and is followed by an explanation how to resolve it. Afterwards we point out and correct other minor issues in the proof.

1.2. **The uniformity problem.** The problem arises in the proof of Theorem 4.1, circa line 12 on page 50, and concerns an implicit erroneous interchange of quantifiers stemming from our use of the phrase "for all sufficiently large g" without specifying what g is required to be large with respect to.

Concretely, Theorem 3.15 says that the map

$$(\mathrm{C}) \qquad \Phi_S \colon \mathcal{P}(S, \mathcal{V})_{\geq 0} \otimes (\det \mathbb{Q}^S)^{\otimes n} \longrightarrow H^*(B\mathrm{Diff}(W_g, D^{2n}); \mathcal{H}(g)_{\mathbb{Q}}^{\otimes S})$$

"is an isomorphism in a range of degrees tending to ∞ with g", but the discussion in Section 9.1 attempting to give a quantitative version of this statement is fallacious. It is true that the map

$$\alpha \colon B\mathrm{Diff}^{\theta \times Y}(W_g, D^{2n}; \ell_{D^{2n}})_{\ell_g} \longrightarrow \Omega_0^\infty(\mathrm{MT}\theta \wedge Y_+)$$

discussed there is an isomorphism in degrees $* \leq \frac{g-3}{2}$ when $2n \geq 6$ (or $* \leq \frac{2g-2}{3}$ when $2n = 2$) by the cited references, but implementing this in the proof of Theorem 3.15 shows that:

Proposition X.1. *The map Φ_S is an isomorphism in degrees satisfying $* + |S| \leq \frac{g-3}{2}$ when $2n \geq 6$ (or $* + |S| \leq \frac{2g-2}{3}$ when $2n = 2$).* □

This corrects the range claimed in Section 9.1. The shift of degrees is because in the proof of Theorem 3.15 the coefficient system $\mathcal{H}(n)_{\mathbb{Q}}^{\otimes S}$ arises in homological degree $|S|$, and it is the total homological degree which must fulfil the inequality $\leq \frac{g-3}{2}$ (or $\leq \frac{2g-2}{3}$). It does indeed follow that this range of degrees tends to ∞ with g for each fixed finite set S, *but it does not follow* that in a given range of homological degrees $* \leq N + 1$ one may choose g large enough such that the natural transformation

$$\Phi : \mathcal{P}(-, \mathcal{V})_{\geq 0} \otimes (\det \mathbb{Q}^-)^{\otimes n} \longrightarrow H^*(B\mathrm{Diff}(W_g, D^{2n}); \mathcal{H}(g)_{\mathbb{Q}}^{\otimes -})$$

is an isomorphism of functors in homological degrees $* \leq N + 1$, because that is asserting something about its values on *all* finite sets S at once. This is implicitly done in the proof of Theorem 4.1.

Indeed the statement that Φ_S is an isomorphism in a range of degrees independent of $|S|$ is simply false, as may be seen in cohomological degree zero because then the map Φ_S may be identified with the map (2.2), so (cf. Theorem 2.6) it is surjective but is not injective when $|S| \gg g$.

1.3. **Solving the uniformity problem.** Let us abbreviate $\mathcal{H}(g) := \mathcal{H}(g)_{\mathbb{Q}}$ from now on, as we will only discuss the rational setting. Solving the problem requires a uniform homological stability theorem in the sense recently introduced by Miller–Patzt–Petersen–Randal-Williams [MPPRW24]. In the case $2n = 2$ we will be able to simply use [MPPRW24, Theorem 1.1], after some preliminary preparation. In the case $2n \geq 6$ we will proceed differently, relying on embedding calculus to deduce uniform homological stability.

1.3.1. *Addenda to Section 2.3.* The methods discussed here are to be understood in the context of Section 2.3, and especially Proposition 2.16. They already to some degree arise in the proof of Corollary 2.18.

There is an identification

$$\mathsf{d(s)Br}(T, S) \cong \mathsf{(s)Br}_{2g}(T, S)/\langle(f, m_T, m_S) \text{ with } m_S \neq \varnothing\rangle,$$

natural with respect to T in all morphisms of $\mathsf{(s)Br}_{2g}$ and natural with respect to S only in bijections: that is, it is an isomorphism of functors $\mathsf{(s)Br}_{2g}^{\mathrm{op}} \times \mathsf{FB} \to \mathbb{Q}\text{-mod}$. For $B \in (\mathbb{Q}\text{-mod})^{\mathsf{(s)Br}_{2g}}$ we define its *reduction* as the coend

$$B_{\mathrm{red}}(S) := \int^{T \in \mathsf{(s)Br}_{2g}} \mathsf{d(s)Br}(T, S) \otimes_{\mathbb{Q}} B(T).$$

This is natural in $S \in \mathsf{FB}$, so reduction yields a functor between ($\mathbb{Q}$-linear) functor categories $(-)_{\mathrm{red}} : (\mathbb{Q}\text{-mod})^{\mathsf{(s)Br}_{2g}} \longrightarrow (\mathbb{Q}\text{-mod})^{\mathsf{FB}}$.

Lemma X.2.

(i) For $A \in (\mathbb{Q}\text{-}mod)^{\mathsf{d(s)Br}}$ there is an isomorphism $(i_*(A))_{red}(-) \cong A(-)$ in $(\mathbb{Q}\text{-}mod)^{\mathsf{FB}}$, naturally in A.

(ii) For $K \in (\mathbb{Q}\text{-}mod)^{\mathsf{(s)Br}_{2g}}$ given by $K(S) = H(g)^{\otimes S}$, there is an isomorphism $K_{red}(-) \cong H(g)_{[-]}$ in $(\mathbb{Q}\text{-}mod)^{\mathsf{FB}}$.

(iii) If $\Phi : B \to B'$ is a morphism in $(\mathbb{Q}\text{-}mod)^{\mathsf{(s)Br}_{2g}}$ which is an isomorphism when evaluated at all sets S with $|S| < N$, then the morphism $\Phi_{red} : B_{red} \to B'_{red}$ in $(\mathbb{Q}\text{-}mod)^{\mathsf{FB}}$ is also an isomorphism when evaluated at all sets S with $|S| < N$.

Proof. For (i) we calculate

$$(i_*(A))_{\mathrm{red}}(S) = \int^{T \in (\mathsf{s})\mathsf{Br}_{2g}} \mathsf{d}(\mathsf{s})\mathsf{Br}(T,S) \otimes_{\mathbb{Q}} i_*(A)(T)$$

$$\cong \int^{T \in (\mathsf{s})\mathsf{Br}_{2g}} \mathsf{d}(\mathsf{s})\mathsf{Br}(T,S) \otimes_{\mathbb{Q}} \int^{U \in \mathsf{d}(\mathsf{s})\mathsf{Br}} (\mathsf{s})\mathsf{Br}_{2g}(U,T) \otimes_{\mathbb{Q}} A(U)$$

$$\cong \int^{U \in \mathsf{d}(\mathsf{s})\mathsf{Br}} \left(\int^{T \in (\mathsf{s})\mathsf{Br}_{2g}} \mathsf{d}(\mathsf{s})\mathsf{Br}(T,S) \otimes (\mathsf{s})\mathsf{Br}_{2g}(U,T) \right) \otimes_{\mathbb{Q}} A(U)$$

$$\cong \int^{U \in \mathsf{d}(\mathsf{s})\mathsf{Br}} \mathsf{d}(\mathsf{s})\mathsf{Br}(U,S) \otimes_{\mathbb{Q}} A(U) \cong A(S)$$

using the (co)Yoneda lemma and Fubini theorem for coends.

For (ii), there is by definition an exact sequence of functors on $(\mathsf{s})\mathsf{Br}_{2g}$

$$(1.1) \qquad \bigoplus_{a \neq b \in S} (\mathsf{s})\mathsf{Br}_{2g}(-, S \setminus \{a,b\}) \longrightarrow (\mathsf{s})\mathsf{Br}_{2g}(-, S) \longrightarrow \mathsf{d}(\mathsf{s})\mathsf{Br}(-, S) \longrightarrow 0,$$

where the left map inserts the ordered matching (a,b) on the corresponding summand. Taking coend over $(\mathsf{s})\mathsf{Br}_{2g}$ of K against this gives the exact sequence

$$(1.2) \qquad \bigoplus_{a \neq b \in S} H(g)^{\otimes S \setminus \{a,b\}} \longrightarrow H(g)^{\otimes S} \longrightarrow K_{\mathrm{red}}(S) \longrightarrow 0,$$

where the left-hand map is given by inserting the form $\omega \in H(g)^{\otimes 2}$ into positions a and b. The cokernel of this map is the definition of $H(g)_{[S]}$.

For (iii), taking coends over $(\mathsf{s})\mathsf{Br}_{2g}$ of Φ against (1.1) gives a diagram

$$\begin{array}{ccccccc}
\bigoplus_{a \neq b \in S} B(S \setminus \{a,b\}) & \longrightarrow & B(S) & \longrightarrow & B_{\mathrm{red}}(S) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\bigoplus_{a \neq b \in S} B'(S \setminus \{a,b\}) & \longrightarrow & B'(S) & \longrightarrow & B'_{\mathrm{red}}(S) & \longrightarrow & 0
\end{array}$$

with exact rows. If $|S| < N$ then the left-hand and middle vertical maps are isomorphisms, so the right-hand vertical map is too. $\qquad\square$

1.3.2. *A stronger Proposition 2.16.* The following variant of Proposition 2.16 differs from that given in the published paper in that it has a hypothesis on $\phi_{\mathrm{red}}^{\mathsf{Br}_{2g}}$ rather than $\phi^{\mathsf{Br}_{2g}}$. When the latter is an isomorphism so is the former, and in this case Proposition 2.16$'$ is stronger than Proposition 2.16. (When the latter is only a monomorphism it not clear whether the former must be.)

As in the published paper, we let $G \subset \mathrm{O}_{g,g}(\mathbb{Q})$ be an arithmetic subgroup (not contained in $\mathrm{SO}_{g,g}(\mathbb{Q})$) and phrase the result in this case; the statement for arithmetic subgroups $G \subset \mathrm{Sp}_{2g}(\mathbb{Q})$ is given replacing Brauer categories by signed Brauer categories. Recall that $K(S) = H(g)^{\otimes S}$ and $K_{\mathrm{red}}(S) \cong H(g)_{[S]}$.

Proposition 2.16$'$. *Let $B \in \mathsf{Rep}(G)$, $A \in (\mathbb{Q}\text{-}mod)^{\mathsf{dBr}}$ have finite length, and there be given a map*

$$\phi^{\mathsf{Br}_{2g}} : i_*(A) \longrightarrow [K \otimes B]^G \in (\mathbb{Q}\text{-}mod)^{\mathsf{Br}_{2g}}.$$

Then there is an induced map

$$\phi : i^*(K^\vee) \otimes^{\mathsf{dBr}} (1_{\mathsf{Rep}(G)} \odot A) \longrightarrow B \in \mathsf{Rep}(G),$$

which is an isomorphism onto the maximal algebraic subrepresentation of B if

$$\phi_{red}^{\mathsf{Br}_{2g}} : A \longrightarrow [K_{red} \otimes B]^G \in (\mathbb{Q}\text{-}mod)^{\mathsf{FB}}$$

is an isomorphism, and is a monomorphism if $\phi_{red}^{\mathsf{Br}_{2g}}$ is a monomorphism.

If $\phi_{red}^{\mathsf{Br}_{2g}}$ is an isomorphism, then for a partition λ of q the multiplicity of the irreducible G-representation $V_\lambda(H(g))$ in B is the same as the multiplicity of the irreducible Σ_q-representation S^λ in $A(\{1, 2, \ldots, q\})$.[1]

Proof. For the sake of readability, we reproduce parts of the proof of Proposition 2.16 rather than explaining how to modify it. The map $\phi^{\mathsf{Br}_{2g}}$ has an adjoint $\phi^{\mathsf{dBr}} \colon A \to i^*([K \otimes B])^G = [i^*(K \otimes B)]^G$. As $[-]^G \cong \mathrm{Hom}_{\mathsf{Rep(G)}}(1_{\mathsf{Rep(G)}}, -)$, this is adjoint to a map $1_{\mathsf{Rep}(G)} \odot A \longrightarrow i^*(K \otimes B) \in \mathsf{Rep}(G)^{\mathsf{dBr}}$ which is in turn adjoint to the map ϕ of the statement. Conversely, $\phi^{\mathsf{Br}_{2g}}$ can be recovered from ϕ as

$$i_*(A)(S) \xrightarrow{\int^{T \in \mathsf{dBr}} \kappa \otimes A(T)} \int^{T \in \mathsf{dBr}} [K(S) \otimes K(T)^\vee]^G \otimes_{\mathbb{Q}} A(T) \xrightarrow{[K(S) \otimes \phi]^G} [K(S) \otimes B]^G$$

where the first map uses the definition $i_*(A)(S) = \int^{T \in \mathsf{dBr}} \mathsf{Br}_{2g}(T, S) \otimes A(T)$ and the natural transformation $\kappa \colon \mathsf{Br}_{2g}(T, S) \to [K(S) \otimes K(T)^\vee]^G$ of functors $\mathsf{Br}_{2g} \times \mathsf{Br}_{2g}^{\mathrm{op}} \to \mathbb{Q}$-mod induced by the functoriality of K.

We next prove that

$$[K_{\mathrm{red}}(S) \otimes \phi]^G \colon \int^{T \in \mathsf{dBr}} [K_{\mathrm{red}}(S) \otimes K(T)^\vee]^G \otimes A(T) \longrightarrow [K_{\mathrm{red}}(S) \otimes B]^G.$$

is a mono- or isomorphism for every finite set S if $\phi_{\mathrm{red}}^{\mathsf{Br}_{2g}}$ is. Observe there is a commutative diagram of natural transformations of functors $\mathsf{dBr}^{\mathrm{op}} \times \mathsf{FB} \to \mathbb{Q}$-mod

$$\begin{array}{ccc}
\mathsf{Br}_{2g}(T, S) & \xrightarrow{\ \kappa\ } & [K(S) \otimes K(T)^\vee]^G \\
\downarrow & & \downarrow \\
\mathsf{dBr}(T, S) & \xrightarrow{\ \kappa_{\mathrm{red}}\ } & [K_{\mathrm{red}}(S) \otimes K(T)^\vee]^G
\end{array}$$

where the vertical maps are evidently surjective and κ is surjective by Theorem 2.6, so κ_{red} is also surjective. Using the (co)Yoneda lemma and κ_{red} we can form the composition

$$A(S) \xrightarrow{\int^{T \in \mathsf{dBr}} \kappa_{\mathrm{red}} \otimes A(T)} \int^{T \in \mathsf{dBr}} [K_{\mathrm{red}}(S) \otimes K(T)^\vee]^G \otimes A(T) \xrightarrow{[K_{\mathrm{red}}(S) \otimes \phi]^G} [K_{\mathrm{red}}(S) \otimes B]^G,$$

which is the map $\phi_{\mathrm{red}}^{\mathsf{Br}_{2g}}$, by comparison with the description of $\phi^{\mathsf{Br}_{2g}}$ above. The left map is surjective, as κ_{red} is. Thus the right map $[K_{\mathrm{red}}(S) \otimes \phi]^G$ is a mono- or isomorphism if the composition $\phi_{\mathrm{red}}^{\mathsf{Br}_{2g}}$ is.

We now apply Lemma 2.12 to $\mathcal{A} = \mathsf{Rep}(G)$, $\Lambda = \mathsf{FB}$, and kernel $K_{\mathrm{red}}(S) = H(g)_{[S]}$ to conclude that ϕ is a mono- or isomorphism onto the maximal algebraic subrepresentation B^{alg} of B. Indeed, since every irreducible algebraic representation is a summand of some $K_{\mathrm{red}}(S) = H(g)_{[S]}$, $\mathcal{A}_{K_{\mathrm{red}}}$ is the category of finite-dimensional algebraic representations and $\mathcal{A}_{K_{\mathrm{red}}}^\circ$ is the category of finite-dimensional G-representations V which contain no algebraic subrepresentation. The kernel of ϕ is a subrepresentation of $i^*(K^\vee) \otimes^{\mathsf{dBr}} (1_{\mathsf{Rep}(G)} \odot A)$, which is algebraic, so $\mathrm{Ker}(\phi)$ is also algebraic: if it lies in $\mathsf{Rep}(G)_{K_{\mathrm{red}}}^\circ$ it is therefore zero, so ϕ is injective. If the cokernel of ϕ lies in $\mathsf{Rep}(G)_{K_{\mathrm{red}}}^\circ$ then it contains no algebraic subrepresentations, so the image of ϕ is the maximal algebraic subrepresentation of B.

For the last part, observe that evaluating $\phi_{\mathrm{red}}^{\mathsf{Br}_{2g}}$ at $S = \{1, \ldots, q\}$ we get an isomorphism

$$A(\{1, \ldots, q\}) \longrightarrow [H(g)_{[q]} \otimes B]^G$$

of Σ_q-representations. With the decomposition $H(g)_{[q]} = \bigoplus_{\lambda \vdash q} S^\lambda \otimes V_\lambda(H(g))$ this shows that the multiplicity of S^λ in $A(\{1, \ldots, q\})$ agrees with that $V_\lambda(H(g))$ in B. $\quad\square$

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[1]Part of the claim is that if $V_\lambda(H(g))$ is not irreducible, so is zero by Theorem 2.4, then S^λ does not occur in $A(\{1, 2, \ldots, q\})$.

1.3.3. *Aside: the end of the proof of Proposition 2.16.* Let us address an issue related to what we have discussed so far, though it becomes irrelevant once one substitutes Proposition $2.16'$ for Proposition 2.16.

At the end of the proof of Proposition 2.16 we say to extract $A(\{1, \ldots, q\})$ from $i_*(A)(\{1, \ldots, q\})$ by "Taking the kernels of all the maps induced by $(\mathrm{inc}\colon S \to S', m_S, \varnothing)$ with m_S nontrivial" but this seems hard to see this directly from the definition. Let us explain why it does work, under the assumptions of Proposition 2.16.

Let

$$N(-, S) := \bigcap_{\substack{(f\colon S \to S', m_S, \varnothing) \\ m_S \neq \varnothing}} \mathrm{Ker}\big[f_*\colon (\mathsf{s})\mathsf{Br}_{2g}(-, S) \to (\mathsf{s})\mathsf{Br}_{2g}(-, S')\big]$$

and define the end $B^{\mathrm{red}}(S) := \int_{T \in (\mathsf{s})\mathsf{Br}_{2g}} N(T, S) \otimes_{\mathbb{Q}} B(T)$. Then the analogue of Lemma X.2 (ii) shows that $K^{\mathrm{red}}(S) \cong H(g)^{[S]}$. Let us now apply $(-)^{\mathrm{red}}$ and $(-)_{\mathrm{red}}$ to the isomorphism $\phi^{\mathsf{Br}_{2g}}\colon i_*(A) \longrightarrow [K \otimes B]^G$, to obtain a commutative diagram

$$\begin{array}{ccccc}
i_*(A)^{\mathrm{red}} & \longrightarrow & i_*(A) & \longrightarrow & i_*(A)_{\mathrm{red}} \cong A \\
{\scriptstyle \phi^{\mathsf{Br}_{2g}, \mathrm{red}}}\downarrow{\scriptstyle \cong} & & {\scriptstyle \phi^{\mathsf{Br}_{2g}}}\downarrow{\scriptstyle \cong} & & {\scriptstyle \phi^{\mathsf{Br}_{2g}}_{\mathrm{red}}}\downarrow{\scriptstyle \cong} \\
[K^{\mathrm{red}} \otimes B]^G & \longrightarrow & [K \otimes B]^G & \longrightarrow & [K_{\mathrm{red}} \otimes B]^G
\end{array}$$

where we have used that $[- \otimes B]^G$ is exact on algebraic G-representations to commute it with $(-)^{\mathrm{red}}$ and $(-)_{\mathrm{red}}$, and obtain the bottom-left and bottom-right entries. The bottom composition is an isomorphism because it is induced by the compositions $H(g)^{[S]} \to H(g)^{\otimes S} \to H(g)_{[S]}$, which are isomorphisms ([FH91, eq. (17.12)]), and hence so is $i_*(A)^{\mathrm{red}}(S) \to i_*(A)(S) \to i_*(A)_{\mathrm{red}}(S) \cong A(S)$. We conclude "Taking the kernels of all the maps induced by $(\mathrm{inc}\colon S \to S', m_S, \varnothing)$ with m_S nontrivial" does indeed recover $A(S)$ from $i_*(A)(S)$ in this case, as claimed.

1.3.4. *Reduction to uniform homological stability.* Let us consider the construction $(-)_{\mathrm{red}}$ applied to the natural transformation Φ. Using Lemma X.2 (i) with the identity $i_*(\mathcal{P}(-, \mathcal{V})'_{\geq 0} \otimes \det^{\otimes n}) = \mathcal{P}(-, \mathcal{V})_{\geq 0} \otimes \det^{\otimes n}$, which is how the primed version is defined at the top of p. 36, we obtain an identification

$$(\mathcal{P}(-, \mathcal{V})_{\geq 0} \otimes \det^{\otimes n})_{\mathrm{red}}(S) \cong (\mathcal{P}(-, \mathcal{V})'_{\geq 0} \otimes \det^{\otimes n})(S).$$

The exact sequence of algebraic G'_g-representations

$$\bigoplus_{a \neq b \in S} H(g)^{\otimes S \setminus \{a,b\}} \longrightarrow H(g)^{\otimes S} \longrightarrow H(g)_{[S]} \longrightarrow 0,$$

and the fact that the surjection is split—because algebraic G'_g-representations are semisimple—gives $H^*(B\mathrm{Diff}(W_g, D^{2n}); \mathcal{H}(g)^{\otimes -})_{\mathrm{red}}(S) \cong H^*(B\mathrm{Diff}(W_g, D^{2n}); \mathcal{H}(g)_{[S]})$. We hence obtain a map

$$(\mathrm{C_{red}}) \qquad \Phi_{\mathrm{red}, S}\colon \mathcal{P}(S, \mathcal{V})'_{\geq 0} \otimes (\det \mathbb{Q}^S)^{\otimes n} \longrightarrow H^*(B\mathrm{Diff}(W_g, D^{2n}); \mathcal{H}(g)_{[S]}).$$

Combining Lemma X.2 (iii) with Proposition X.1 gives:

Corollary X.3. *The map $\Phi_{red, S}$ is an isomorphism in degrees satisfying $* + |S| \leq \frac{g-3}{2}$ when $2n \geq 6$ (or $* + |S| \leq \frac{2g-2}{3}$ when $2n = 2$).* $\qquad\square$

The map $\Phi_{\mathrm{red},S}$ is compatible with stabilisation with respect to g on the right-hand side, so consider for $k \geq 0$ the diagram

(1.3)
$$
\begin{array}{ccc}
& & H^*(B\mathrm{Diff}(W_{g+k}, D^{2n}); \mathcal{H}(g+k)_{[S]}) \\
& \overset{\Phi_{\mathrm{red},S}}{\nearrow} & \downarrow \\
\mathcal{P}(S, \mathcal{V})'_{\geq 0} \otimes (\det \mathbb{Q}^S)^{\otimes n} & \xrightarrow[\Phi_{\mathrm{red},S}]{} & H^*(B\mathrm{Diff}(W_g, D^{2n}); \mathcal{H}(g)_{[S]})
\end{array}
$$

where the right map is induced by stabilisation. The diagonal map is an isomorphism in degrees satisfying $* + |S| \leq \frac{g+k-3}{2}$ when $2n \geq 6$ (or $* + |S| \leq \frac{2g+2k-2}{3}$ when $2n = 2$), so by taking $k \gg 0$ it is an isomorphism in an arbitrarily large range of degrees. Thus the horizontal map is an isomorphism in the same range that the vertical map is, within this range. If the vertical map is an isomorphism in a range that depends only on g and not on S, i.e. a *uniform homological stability range*, then we conclude that the horizontal map $\Phi_{\mathrm{red},S}$ is also an isomorphism in a range independent of S, and hence that Φ_{red} is a natural isomorphism in a range of homological degrees. The proof of Theorem 4.1′ is then completed in the same way as Theorem 4.1 in the published paper, using Proposition 2.16′ instead of Proposition 2.16.

We can deal straight away with the case $2n = 2$, using the uniform homological stability theorem of [MPPRW24].

Proposition X.4. *When $2n = 2$ the map $\Phi_{red,S}$ is an isomorphism in homological degrees $* \leq \frac{g-4}{4}$ independently of S, and so Φ_{red} is a natural isomorphism in this range.*

Proof. There is a decomposition $H(g)_{[S]} = \bigoplus_{\lambda \vdash S} S^\lambda \otimes V_\lambda(H(g))$ into irreducible G'_g-representations. The uniform homological stability theorem [MPPRW24, Theorem 1.1], and the discussion in the paragraphs after it, after dualising, says that

$$
H^d(B\mathrm{Diff}(W_{g+i}, D^{2n}), B\mathrm{Diff}(W_{g+i-1}, D^{2n}); V_\lambda(H(g+i)), V_\lambda(H(g+i-1))) = 0
$$

for $d < \frac{g+i}{4}$. This implies that the vertical map in (1.3) is an isomorphism in degrees satisfying $* + 1 < \frac{g+1}{4}$, i.e. in degrees $* \leq \frac{g-4}{4}$. $\qquad\square$

1.3.5. *The case $2n \geq 6$.* Our strategy in this case will be to show that the domain and codomain of $\Phi_{\mathrm{red},S}$ both vanish in degrees $* < \min\left(g-1, \frac{(n-2)|S|}{3}\right)$, where of course $n \geq 3$. Having done so, the argument is completed as follows.

Proposition X.5. *When $2n \geq 6$ the map $\Phi_{red,S}$ is an isomorphism in homological degrees $* \leq \frac{n-2}{n+1} \frac{g-3}{2}$ independently of S, and so Φ_{red} is a natural isomorphism in this range.*

Proof. Let $d \leq \frac{n-2}{n+1} \frac{g-3}{2}$ and consider $\Phi_{\mathrm{red},S}$ in this degree. If $|S| > \frac{3}{n-2}d$ then its domain and codomain both vanish, so the map is an isomorphism. If $|S| \leq \frac{3}{n-2}d$ then $d + |S| \leq \frac{n+1}{n-2}d \leq \frac{g-3}{2}$ and we have explained that $\Phi_{\mathrm{red},S}$ is an isomorphism under this condition as a consequence of Proposition X.1. $\qquad\square$

Combining this with (1.3) and the decomposition of $H(g)_{[S]}$ into irreducibles, we obtain the following uniform homological stability theorem.

Corollary X.6. *For $2n \geq 6$ and $\lambda \vdash q$ any partition, the map*

$$
H^*(B\mathrm{Diff}(W_{g+1}, D^{2n}); V_\lambda(\mathcal{H}(g+1))) \longrightarrow H^*(B\mathrm{Diff}(W_g, D^{2n}); V_\lambda(\mathcal{H}(g)))
$$

is an isomorphism in homological degrees $* \leq \frac{n-2}{n+1} \frac{g-3}{2}$. $\qquad\square$

It remains to establish a vanishing range for the domain and codomain of $\Phi_{\mathrm{red},S}$. The following is our vanishing estimate for the domain of $\Phi_{\mathrm{red},S}$.

Theorem X.7. *Let $\mathcal{V}$ be a graded algebra which is supported in even degrees and in degrees $\leq n$ consists only of the unit (cf. Assumption 3.13). In homological degree d, the object $\mathcal{P}(S, \mathcal{V})'_{\geq 0}$ vanishes as long as $|S| > \frac{2n+1}{n} d$.*

Proof. As in Remark 3.12, choosing a homogeneous basis $\mathcal{B}$ for the graded vector space $\mathcal{V}$ containing the multiplicative unit $1 \in \mathcal{V}$, $\mathcal{P}(S, \mathcal{V})'_{\geq 0}$ has a homogeneous basis given by by partitions of S whose parts are labelled by $\mathcal{B}$ having no parts (i) of size 0 with label of degree $\leq 2n$, (ii) of size 1 with label of degree $< n$ (hence of degree $\leq n$, given our assumption on $\mathcal{V}$), (iii) of size 2 labelled by $1 \in \mathcal{B}$. A labelled partition $(\{P_i\}, \{c_i\})$ is given homological degree $\sum_i n(|P_i| - 2) + |c_i|$. The conditions (i)–(iii) ensure that each labelled part (P, c) contributes $n(|P| - 2) + |c| \geq 1$ to the homological degree.

If $|S| > \frac{2n+1}{n} d$ then a labelled partition $(\{P_i\}, \{c_i\})$ of S must either have $> d$ parts, and hence contribute to homological degree $> d$, or have $\leq d$ parts and hence contribute to homological degree $\sum_i n(|P_i| - 2) + |c_i| \geq n|S| - 2nd + \sum_i |c_i| \geq n|S| - 2nd > d$. $\square$

The following is our vanishing estimate for the codomain of $\Phi_{\mathrm{red}, S}$.

Theorem X.8. *For $2n \geq 6$ and any finite set S,*

$$H^*(B\mathrm{Diff}(W_g, D^{2n}); \mathcal{H}(g)_{[S]}) = 0$$

in degrees $* < \min(g - 1, \frac{n-2}{3} |S|)$.

Proof. We first proceed as in the beginning of the proof of Theorem 4.1. The Serre spectral sequence for (4.1) and an appeal to Theorem 2.3 and Lemma 4.3 gives an identification of $H^d(B\mathrm{Diff}(W_g, D^{2n}); \mathcal{H}(g)_{[S]})$ with

$$\bigoplus_{p+q=d} H^p(G'_\infty; \mathbb{Q}) \otimes \left[H^q(B\mathrm{Tor}(W_g, D^{2n}); \mathbb{Q}) \otimes H(g)_{[S]} \right]^{G'_g}$$

in total degrees $d < g - 1$. The claim will then follow as long as irreducible algebraic G'_g-representations $V_\lambda(H(g))$ with $|\lambda| > \frac{3}{n-2} d$ do not arise inside the G'_g-representation $H^{* \leq d}(B\mathrm{Tor}(W_g, D^{2n}); \mathbb{Q})$.

In the companion paper [KRW20a] we proved that each $H^i(B\mathrm{Tor}(W_g, D^{2n}); \mathbb{Q})$ is an algebraic G'_g-representation. There, as well as here, there is a need to discuss homotopy groups, which are only *gr*-algebraic representations in the sense of [KRW20a, Definition 2.3]: these are representations of $\Gamma_g := \pi_0(\mathrm{Diff}(W_g, D^{2n}))$ (or later its quotient $\Lambda_g := \pi_0(\mathrm{Emb}^{\cong}_{1/2\partial}(W_{g,1})) \cong \Gamma_g / \Theta_{2n+1}$) with a finite filtration whose associated graded consists of restrictions of algebraic G'_g-representations. Let us say that a *gr*-algebraic G'_g-representation has *weight* $\leq k$ if, after passing to the associated graded of the aforementioned filtration, it is isomorphic to a sum of $V_\lambda(H(g))$'s with $|\lambda| \leq k$. Our goal is therefore to show that $H^{* \leq d}(B\mathrm{Tor}(W_g, D^{2n}); \mathbb{Q})$ has weight $\leq \frac{3}{n-2} d$. We can do this by keeping track of weights in the argument given in [KRW20a] with some additional computational input from [KRW25]. (Though the latter in places cites the former as well as [KRW20b], the results we need from it do not rely on these citations. We explain in Remark X.9 that it is possible to obtain a similar result with a worse range using only [KRW20a].)

We first make some elementary observations about weight. Firstly, the representations of weight $\leq k$ are closed under forming subquotient, extensions, and duals, and weight is (sub)additive under tensor products of representations (cf. [KRW20a, Lemma 2.5]). Secondly, if X is a 0-connected simple space with an unbased G'_g-action and $A \geq 0$, then

$$\pi_j(X) \otimes \mathbb{Q} \text{ is algebraic of weight } \leq Aj \text{ for all } j \geq 1$$

$$\implies H^i(X; \mathbb{Q}) \text{ is algebraic of weight } \leq Ai \text{ for all } i \geq 1.$$

This is proved in the same way as [KRW20a, Lemma 2.11], keeping track of weight. Other then additivity of weight under tensor products, one uses that $H^*(K(W,n);\mathbb{Q}) \cong S^*(W^\vee[n])$ in degree kn has weight $\leq k \cdot \text{weight } W$.

It suffices to estimate the weight of $\pi_j(B\text{Tor}(W_g, D^{2n})) \otimes \mathbb{Q}$ as $\leq \frac{j}{n-2} + 2$—this in turn yields a bound of the weight of the form $\leq Aj$ for all $j \geq 1$, with $A = \frac{3}{n-2}$ using that $n \geq 3$—which we do in the following steps:

(1) Using the long exact sequence of homotopy groups associated to the Weiss fibre sequence [KRW20a, (3)] with trivial action on the fibre

$$B\text{Diff}_\partial(D^{2n}) \longrightarrow B\text{Diff}_\partial(W_g, D^{2n}) \longrightarrow B\text{Emb}_{1/2\partial}^{\cong}(W_{g,1}),$$

it suffices to estimate the weight of $\pi_j(\text{Emb}_{1/2\partial}^{\text{id}}(W_{g,1})) \otimes \mathbb{Q}$, which we do by considering the long exact sequences coming from the (convergent) embedding calculus tower and estimating the weight of the homotopy groups $\pi_j(L_k\text{Emb}_{1/2\partial}^{\text{id}}(W_{g,1})) \otimes \mathbb{Q}$ of each of the layers.

(2) For the first layer, i.e. $k = 1$, the proof of [KRW20a, Proposition 4.5] implies that $\pi_j(L_1\text{Emb}_{1/2\partial}^{\text{id}}(W_{g,1})) \otimes \mathbb{Q}$ has weight $\leq \frac{j}{n-1} + 2$.

(3) For the higher layers, i.e. $k > 1$, we estimate the weights using [KRW20a, (13), (14)]. Following the proof of [KRW20a, Proposition 5.11], this requires estimating the weights of the homotopy groups $\pi_{j+r}(L_k\text{Emb}_{1/2\partial}^{\text{id}}(W_{g,1})) \otimes \mathbb{Q}$ and the relative cohomology groups $H^r(W_{g,1}^k, \Delta_\partial; \mathbb{Q})$.

(4) By the proof of [KRW25, Proposition 5.17], $H^r(W_{g,1}^k, \Delta_\partial; \mathbb{Q})$ has weight $\leq \frac{r-k}{n-1} + 1$ and vanishes for $r > kn$.

(5) By the proof of [KRW25, Proposition 5.31], $\pi_s(\text{tohofib}_{I \subset x_{\underline{k}}} \text{Emb}(I, W_{g,1})) \otimes \mathbb{Q}$ has weight $\leq \frac{s-1}{n-1} - 2(k-1)$ and vanishes for $s \leq 2(k-1)(n-1)$.

Combining (4) and (5) we obtain in (3) that $\pi_j(L_k\text{Emb}_{1/2\partial}^{\text{id}}(W_{g,1})) \otimes \mathbb{Q}$ has weight $\leq \frac{j-1+k}{n-1} + 2$ and the lowest degree in which it can be non-zero is $j = k(n-2) - 2n + 3$. We obtain, using this and (2) in (1), that

$$\text{weight } \pi_j(\text{Emb}_{1/2\partial}^{\text{id}}(W_{g,1})) \otimes \mathbb{Q} \leq \max_{k \text{ with } k \leq \frac{j+2n-3}{n-2}} \left\{ \frac{j-1+k}{n-1} + 2 \right\} \leq \frac{j+1}{n-2} + 2$$

and we deduce the result using the Weiss fibre sequence and shifting degrees by one. □

Remark X.9. It follows from the computations in [KRW25] that the bounds obtained in the previous proof are not far from optimal. If one prefers not to rely on that paper, one can make the following substitutions in the above argument to obtain a worse bound that is sufficient to establish Theorem 4.1′:

(4′) From the proof of [KRW20a, Proposition 5.7], one can obtain that $H^r(W_{g,1}^k, \Delta_\partial; \mathbb{Q})$ has weight $\leq \frac{r-1}{n-1} + 1$ and vanishes for $r > kn$.

(5′) By the proof of [KRW20a, Proposition 5.10] we have that

$$\pi_s(\text{tohofib}_{I \subset x_{\underline{k}}} \text{Emb}(I, W_{g,1})) \otimes \mathbb{Q} \longrightarrow \pi_s(\text{Emb}(I, W_{g,1})) \otimes \mathbb{Q}$$

is the inclusion of a summand. Keeping track of weights in the proof of [KRW20a, Lemma 2.12], one can obtain that $\pi_s(\text{Emb}(I, W_{g,1})) \otimes \mathbb{Q}$ has weight $\leq \frac{s-1}{n-1} + 1$. It vanishes for $s \leq 2(k-1)(n-1)$ by the higher Blakers–Massey theorem (see e.g. [MV15, Theorem 10.3.3]).

1.4. **A finiteness issue.** The last paragraph of the proof of Theorem 4.1 of [KRW20b] is too brief. There we apply Proposition 2.16 with $B = H^i(B\text{Tor}(W_g, D^{2n}); \mathbb{Q})$ and $i \leq N$, but have only assumed that these are finite-dimensional for $i < N$ and the statement of Proposition 2.16 asks for B to be a finite-dimensional G-representation. Nonetheless the conclusion is valid, by the following discussion.

Consider the setting of Proposition 2.16′ but with B a not necessarily finite-dimensional G-representation: we claim that the conclusion of this Proposition still holds. To see this, let $B^{\mathrm{alg}} \leq B$ denote the maximal algebraic subrepresentation, i.e. the union of its algebraic subrepresentations. The induced map $[K \otimes B^{\mathrm{alg}}]^G \to [K \otimes B]^G$ is then an isomorphism. As A is assumed to have finite length and $\phi_{\mathrm{red}}^{\mathsf{Br}_{2g}} : i_*(A) \to [K_{\mathrm{red}} \otimes B^{\mathrm{alg}}]^G$ is assumed to be an isomorphism, it follows that $[H(g)_{[S]} \otimes B^{\mathrm{alg}}]^G$ is finite-dimensional for every finite set S, and hence that $\mathrm{Hom}_G(V_\lambda, B^{\mathrm{alg}})$ is finite-dimensional for each irreducible algebraic G-representation V_λ. The evaluation map

$$\bigoplus_{\substack{\text{irreducible algebraic} \\ G\text{-representations } V_\lambda}} V_\lambda \otimes \mathrm{Hom}_G(V_\lambda, B^{\mathrm{alg}}) \longrightarrow B^{\mathrm{alg}}$$

is tautologically surjective, and there are finitely-many irreducibles so the domain is finite-dimensional, and hence B^{alg} is in fact finite-dimensional. One can then apply Proposition 2.16′ to B^{alg}.

2. The case $2n = 2$

2.1. **The monomorphism assertion.** In our new statement of Theorem 4.1′ we have omitted, in the case $2n = 2$, the claim that the map is a monomorphism in degree $N + 1$. The reason that this is not justified in the proof is that the kernel of the edge homomorphism

$$H^{N+1}(B\mathrm{Diff}(W_g, D^{2n}); \mathcal{H}(g)_{\mathbb{Q}}^{\otimes S}) \longrightarrow [H^{N+1}(B\mathrm{Tor}(W_g, D^{2n}); \mathbb{Q}) \otimes H(g)^{\otimes S}]^{G'_g}$$

has a contribution from $E_\infty^{1,N}$, and as we have not assumed anything about the finiteness of $H^N(B\mathrm{Tor}(W_g, D^{2n}); \mathbb{Q})$ we cannot use Theorem 2.3 to say anything about $E_2^{1,N}$.

The new statement of Theorem 4.1′ means that in Theorem 8.1 only the calculation of $H^2(B\mathrm{Tor}(W_g, D^2); \mathbb{Q})^{\mathrm{alg}}$ can be obtained by employing Johnson's theorem that $H^1(B\mathrm{Tor}(W_g, D^2); \mathbb{Q})$ is finite-dimensional for $g \geq 3$.

However, the second part of Theorem 8.1 can be rescued and even strengthened by applying the recent theorem of Minahan [Min23] that $H^2(B\mathrm{Tor}(W_g, D^2); \mathbb{Q})$ is finite-dimensional for $g \geq 51$. Minahan's theorem has been improved upon by Minahan–Putman [MP25, Theorem B] to say that $H^2(B\mathrm{Tor}(W_g, D^2); \mathbb{Q})$ is finite-dimensional for $g \geq 5$ and algebraic for $g \geq 6$. Using this, Theorem 4.1′ describes $H^2(B\mathrm{Tor}(W_g, D^2); \mathbb{Q})$ for all large enough g, and implies that for all large enough g

$$\begin{aligned}
H^3(B\mathrm{Tor}(W_g, D^2); \mathbb{Q})^{\mathrm{alg}} = {} & V_1 + V_{2,1} + 3V_{1^3} + 2V_{2^2,1} + 3V_{2,1^3} + V_{3,2,1^2} \\
& + 2V_{2^3,1} + V_{3,2^3} + 4V_{1^5} + 2V_{2^2,1^3} + V_{3^2,1^3} \\
& + 2V_{2,1^5} + V_{2^3,1^3} + 2V_{1^7} + V_{2^2,1^5} + V_{1^9}.
\end{aligned}$$

2.2. **A typo.** On pp. 75-76 of [KRW20b] (p. 52 of the arXiv version) we mistranscribed computer-calculated Poincaré series for $H^*(B\mathrm{Tor}^+(W_g, *); \mathbb{Q})^{\mathrm{alg}}$ and $H^*(B\mathrm{Tor}^+(W_g); \mathbb{Q})^{\mathrm{alg}}$. In both cases the term $2s_{\langle 2^3, 1^3 \rangle}$ should instead be $s_{\langle 2^3, 1^3 \rangle}$. This now makes Remark 8.2 irrelevant: there is nothing to explain, as our expression now agrees with Sakasai's computation in [Sak05] (with the V_1 term present). Using Minahan's theorem as described above, this calculation completely describes $H^3(B\mathrm{Tor}(W_g, *); \mathbb{Q})^{\mathrm{alg}}$ and $H^3(B\mathrm{Tor}(W_g); \mathbb{Q})^{\mathrm{alg}}$ for all large enough g.

2.3. **Relation to Sakasai's result.** On pp. 76-77 of [KRW20b] (pp. 52-53 of the arXiv version) we described how to settle the ambiguity in Sakasai's paper [Sak05], but the argument given is fallacious. Even assuming Minahan's theorem, so that our calculations in degree 3 are valid, the image of the composition

$$\Lambda^3(V_{1^3}) \xrightarrow{\tau^*} H^3(B\mathrm{Tor}^+(W_g); \mathbb{Q}) \longrightarrow H^3(B\mathrm{Tor}(W_g, D^2); \mathbb{Q})$$

after applying $[- \otimes V_1]^{\mathrm{Sp}_{2g}(\mathbb{Z})}$ is *not* the subspace of those elements which can be represented by trivalent graphs with one leg, three internal vertices, and no loops as claimed, but is instead something more complicated: see [RW24, Section 3.4].

Nonetheless the conclusion is correct, as follows. With the correction indicated in the previous subsection our expression for $H^3(B\mathrm{Tor}(W_g); \mathbb{Q})^{\mathrm{alg}}$ agrees with Sakasai's expression for $\tau^*(\Lambda^3(V_{1^3}))$ with the V_1-term present, so showing that it should be present in Sakasai's paper is equivalent to showing that the $V_1 \leq H^3(B\mathrm{Tor}(W_g); \mathbb{Q})^{\mathrm{alg}}$ lies in the subspace spanned by products of degree-1 cohomology classes. It follows from Minahan's theorem and Theorem A, the discussion after it, and Section 5.2 of [RW24] that in fact all of $H^3(B\mathrm{Tor}(W_g); \mathbb{Q})^{\mathrm{alg}}$ is spanned by products of degree-1 classes (in the language of that paper, this is equivalent to the fact that $\mathcal{G}\mathrm{raph}_g(S)$ is spanned by trivalent graphs with all labels equal to 1, for any finite set S). Thus indeed the V_1-term should be present in Sakasai's result, and therefore $\kappa_{e^3} - (2 - 2g)e^2 \neq 0 \in H^4(B\mathrm{Tor}(W_g, *); \mathbb{Q})$ holds.

The argument given for Corollary 8.3 is correct, again invoking Minahan's theorem.

3. Section 9: Explicit ranges

As a convenient reference, we collect here the new status of all the explicit ranges.

Ranges for Theorem 3.15. The corrected range is described in Proposition X.1.

Ranges for Theorem 4.1$'$. Using Propositions X.4 and X.5 the corrected explicit range is degrees $* \leq \frac{n-1}{n+2}\frac{g-3}{2}$ when $2n \geq 6$ (and $* \leq \frac{g-4}{4}$ when $2n = 2$, in the range of degrees $* \leq N$ in which it applies).

Ranges for Theorem 8.1. Using Johnson's theorem that $H^1(B\mathrm{Tor}(W_g, D^2); \mathbb{Q})$ is finite-dimensional for $g \geq 3$, Theorem 4.1$'$ describes $H^2(B\mathrm{Tor}(W_g, D^2); \mathbb{Q})^{\mathrm{alg}}$ for $g \geq 12$. Using in addition [MP25] this in fact describes $H^2(B\mathrm{Tor}(W_g, D^2); \mathbb{Q})$ in this range, and furthermore the calculation of $H^3(B\mathrm{Tor}(W_g, D^2); \mathbb{Q})^{\mathrm{alg}}$ is valid for $g \geq 16$.

Ranges for Proposition 2.17 and Corollary 2.18. These are unchanged.

Ranges for Theorem 5.1. This is unchanged.

References

[FH91]   W. Fulton and J. Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991. 5

[KRW20a] A. Kupers and O. Randal-Williams, *The cohomology of Torelli groups is algebraic*, Forum of Mathematics, Sigma **8** (2020), e64. 7, 8

[KRW20b] _____, *On the cohomology of Torelli groups*, Forum of Mathematics, Pi **8** (2020), e7. 1, 7, 8, 9

[KRW25]   _____, *On diffeomorphisms of even-dimensional discs*, J. Amer. Math. Soc. **38** (2025), no. 1, 63–178. 7, 8

[Min23]   D. Minahan, *The second rational homology of the Torelli group is finitely generated*, https://arxiv.org/abs/2307.07082, 2023. 9

[MP25]   D. Minahan and D. Putman, *The second rational homology of the Torelli group*, https://arxiv.org/abs/2504.00211, 2025. 9, 10

[MPPRW24] J. Miller, P. Patzt, D. Petersen, and O. Randal-Williams, *Uniform twisted homological stability*, https://arxiv.org/abs/2402.00354, 2024. 2, 6

[MV15]   B. A. Munson and I. Volić, *Cubical homotopy theory*, New Mathematical Monographs, vol. 25, Cambridge University Press, Cambridge, 2015. 8

[RW24]   O. Randal-Williams, *On the cohomology of Torelli groups. II*, Int. Math. Res. Not. IMRN (2024), no. 3, 2339–2381. 10

[Sak05]   T. Sakasai, *The Johnson homomorphism and the third rational cohomology group of the Torelli group*, Topology Appl. **148** (2005), no. 1-3, 83–111. 9

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