CORRIGENDUM TO: CONFIGURATION SPACES AS COMMUTATIVE MONOIDS

SAMUEL A. MOORE AND OSCAR RANDAL-WILLIAMS

Abstract. We correct the statement of Theorem 1.2 and its proof. In full generality Theorem 1.2 is only valid rationally, but in some key cases the integral statement is still valid. We also give the analogous corrections for spaces of 0-cycles in Appendix A.

1. The issue

We refer throughout to the paper [RW24]. The first named author has observed that the statement of Theorem 1.2 cannot possibly be true, by direct calculation with $\mathbb{F}_p$-homology:

Counterexample. Let $M = \{\text{pt.}\}$ be the 1-point manifold, $\pi : L \to M$ be the trivial rank d vector bundle with $d \geq 3$ odd, and p be an odd prime. Write $\mathrm{Sym}_*^k(-) = (-)^{\wedge k}_{\mathfrak{S}_k}$ for the symmetric powers in based spaces.

According to [JTTW63, Corollary 2.4], the space $(L \oplus L)^+ = \mathrm{Sym}_*^2(S^d)$ is homeomorphic to $\Sigma^{d-1}\mathbb{R}\mathrm{P}^{d-1}$, and so, by our assumptions on d and p, it has trivial reduced $\mathbb{F}_p$-homology. It is a theorem of Dold [Dol58, Theorem 7.2] that the homology (with coefficients in a PID R) of a symmetric power of a space depends only on the R-homology of the space; a simple consequence is that the same is true for based symmetric powers (of well-based spaces) and reduced homology. As $(L \oplus L)^+_{\mathfrak{S}_2}$ has trivial reduced $\mathbb{F}_p$-homology, Dold's result means that $\mathrm{Sym}_*^k([(L \oplus L)^+]_{\mathfrak{S}_2})$ has trivial $\mathbb{F}_p$-homology for each $k > 0$.

The map $\epsilon : \mathbf{Com}([(L \oplus L)^+]_{\mathfrak{S}_2}[2]) \to S^0[0]$ is an isomorphism in odd gradings and grading 0, and is $\mathrm{Sym}_*^k([(L \oplus L)^+]_{\mathfrak{S}_2}) \to \{\text{pt.}\}$ in grading $2k > 0$, so it follows from the above that it is a $\mathbb{F}_p$-homology equivalence. If the square in [RW24, Theorem 1.2] were a homotopy pushout of commutative monoids, then it would follow that the map $\mathbf{Com}(L^+[1]) \to \mathbf{C}(\{\text{pt.}\}; L)$ is a $\mathbb{F}_p$-homology equivalence too. In gradings $k > 1$ this would say that $\mathrm{Sym}_*^k(S^d)$ has trivial reduced $\mathbb{F}_p$-homology, and as this is the cofibre of the stabilisation map $\mathrm{Sym}^{k-1}(S^d) \to \mathrm{Sym}^k(S^d)$ between unbased symmetric powers, it would follow that $S^d = \mathrm{Sym}^1(S^d) \to \mathrm{Sym}^\infty(S^d)$ is a $\mathbb{F}_p$-homology equivalence. But $\mathrm{Sym}^\infty(S^d) \simeq K(\mathbb{Z}, d)$ by the Dold–Thom theorem, and this Eilenberg–MacLane space has $\mathbb{F}_p$-homology in degrees $> d$.

The error is the claim in the proof of Lemma 3.2 that the first displayed square is a pushout in $\mathsf{Top}_*^{\mathbb{N}}$. This claim is true when $\pi : L \to M$ is the trivial rank 0 vector bundle, which is the case corresponding to Theorem 1.1: hence Theorem 1.1 seems to be correct as stated. But it is not true when the bundle $\pi : L \to M$ has strictly positive rank. (A further error is that $\mathbf{S}$ is not a subobject of $\mathbf{R}$ in this case.) The same issue arises in the setting treated in Appendix A; we comment on it later.

2. The correction

We will explain how to modify the argument to prove the following variant of (the incorrect) Theorem 1.2.

Theorem 1.2′. *There is a pushout square*

$$\begin{tikzcd} \mathbf{Com}([(L \oplus L)^+]_{\mathfrak{S}_2}[2]) \arrow[r, "\epsilon"] \arrow[d, "\Delta"] & S^0[0] \arrow[d] \\ \mathbf{Com}(L^+[1]) \arrow[r] & \mathbf{C}(M; L) \end{tikzcd}$$

of unital commutative monoids in $\mathsf{Top}_*^{\mathbb{N}}$, where ϵ is the augmentation and Δ is induced by the diagonal inclusion $[(L \oplus L)^+]_{\mathfrak{S}_2} \to [L^+ \wedge L^+]_{\mathfrak{S}_2} = \mathbf{Com}(L^+[1])(2)$. The induced map from the homotopy pushout of commutative monoids

$$\mathbf{Com}(L^+[1]) \otimes^{\mathbb{L}}_{\mathbf{Com}([(L\oplus L)^+]_{\mathfrak{S}_2}[2])} S^0[0] \longrightarrow \mathbf{C}(M; L)$$

is a rational homology equivalence, and is an integral homology equivalence if L has rank at most 1.

The applications discussed in Section 2 of the published paper are all in rational (co)homology, and so follow from Theorem 1.2′ with no changes required. We can choose L of rank at most one such that each $C_n(M; L)$ is orientable and use Poincaré duality to express $H^*(C_n(M))$ as a regrading of $H_{n,*}(\mathbf{C}(M; L))$, unless M is non-orientable and even-dimensional: so even with integer coefficients, Theorem 1.2′ can be used to describe $H^*(C_n(M))$ for most manifolds of interest.

3. Proof of Theorem 1.2′

Let us first comment on how it can be that the first displayed square in the proof of Lemma 3.2 is a pushout in $\mathsf{Top}_*^{\mathbb{N}}$ when $L \to M$ is the trivial rank 0 vector bundle, but not more generally. For a vector bundle $L \to M$, we think of elements of $\mathbf{Com}(L^+[1])$ as (the point at ∞ together with) unordered configurations of points in M, possibly with repeats, where a point at location $m \in M$ is equipped with a label in the fibre L_m. When L has rank 0, so the label data is trivial, every element can be *canonically* expressed as $\mu + 2\lambda$, where the unordered tuple μ of points in M has no repeats. But when the rank of L is positive this is no longer possible: although the underlying points in M can be canonically partitioned in this way, if there is an odd number of labels at the same point of M then there is no way to decide which of these labels should be attributed to μ and which to 2λ. We will solve this by analysing, in a sense, the spaces of such expressions.

Define a filtration $F_\bullet\mathbf{R}$ of $\mathbf{R} := \mathbf{Com}(L^+[1])$ by first letting $F_0\mathbf{R}$ be the image of the map $\mathbf{Com}([(L \oplus L)^+]_{\mathfrak{S}_2}[2]) =: \mathbf{S} \to \mathbf{R}$; explicitly, this is the closed subspace of $\mathbf{R}$ consisting of those labelled configurations where each point of M carries an even number of labels.[1] We then let

$$F_p\mathbf{R}(n) := F_{p-1}\mathbf{R}(n) \cup \mathrm{Im}\big((L^+)^{\wedge p} \wedge ((L \oplus L)^+)^{\wedge(n-p)/2} \to \mathbf{R}(n)\big)$$

when $n - p$ is even, and put $F_p\mathbf{R}(n) :- F_{p-1}\mathbf{R}(n)$ otherwise. Concretely, $F_p\mathbf{R}$ is the closed subspace of $\mathbf{R}$ consisting of those labelled configuration where $\leq p$ points of M carry an odd number of labels.

[1]This is modified from the filtration in the proof of Lemma 3.2, as the map $\mathbf{S} \to F_0\mathbf{R}$ is not injective if L does not have rank 0. This is because in the domain the even number of labels at each point of M are grouped together in pairs—and there can be several ways to do this—whereas in the target they are not.

As explained in Lemma 3.2, for each $p \geq 1$ there is a commutative square of right $\mathbf{S}$-modules

$$\begin{array}{ccc} F_{p-1}\mathbf{R}(p)[p] \otimes \mathbf{S} & \longrightarrow & F_{p-1}\mathbf{R} \\ \downarrow & & \downarrow \\ \mathbf{R}(p)[p] \otimes \mathbf{S} & \longrightarrow & F_p\mathbf{R} \end{array} \tag{3.1}$$

but this is *not* a pushout square when $\mathrm{rank}(L) > 0$: we therefore let $F_p'\mathbf{R} \to F_p\mathbf{R}$ denote the induced map from the pushout. Let us write $F_0'\mathbf{R} := \mathbf{S}$, so there is an induced map $F_0'\mathbf{R} \to F_0\mathbf{R}$ too. If L has rank 0 then the primed and unprimed versions agree and the argument below simplifies to that in the published paper: we therefore focus on the case that L has positive rank.

Lemma 3.6′. *The spaces $\mathbf{R}(n)$, $\mathbf{S}(n)$, $F_p\mathbf{R}(n)$, and $F_p'\mathbf{R}(n)$ are all compact Hausdorff.*

Proof. Abbreviate "compact Hausdorff" to cH. We have $\mathbf{R}(n) = \mathrm{Sym}^n_*(L^+)$. As L is a manifold, it is locally compact Hausdorff, so its one-point compactification L^+ is cH. The collection of cH spaces is closed under smash products quotient by finite groups (like $\mathfrak{S}_n$). Similar reasoning shows that $[(L \oplus L)^+]_{\mathfrak{S}_2}$ is cH and hence that $\mathbf{S}(n)$ is too.

Now $F_p\mathbf{R}(n)$ is a closed subspace of the cH space $\mathbf{R}(n)$, so is again cH. Finally, $F_p'\mathbf{R}(n)$ is the pushout,

$$\mathbf{R}(p) \wedge \mathbf{S}(n-p) \longleftarrow F_{p-1}\mathbf{R}(p) \wedge \mathbf{S}(n-p) \longrightarrow F_{p-1}\mathbf{R}(n)$$

where the left hand map is the inclusion of a closed subspace and all spaces are cH and so normal: thus the pushout is normal by [Dug66, VII 3.4], and in particular Hausdorff. It is a quotient of a compact space so is also compact. □

The augmentation $\epsilon : B(\mathbf{S}, \mathbf{S}, S^0) \xrightarrow{\sim} S^0$ induces a map

$$\mathbf{R} \otimes_{\mathbf{S}} B(\mathbf{S}, \mathbf{S}, S^0) \longrightarrow \mathbf{R} \otimes_{\mathbf{S}} S^0 \overset{\text{Lemma 3.4}}{=} \mathbf{C}(M; L)$$

and to prove Theorem 1.2′ we must show that this is a rational (or integral if $\mathrm{rank}(L) = 1$) homology equivalence. To do so we consider the auxiliary statements

(I_p) $\quad F_p\mathbf{R} \otimes_{\mathbf{S}} B(\mathbf{S}, \mathbf{S}, S^0) \to F_p\mathbf{R} \otimes_{\mathbf{S}} S^0$ is a homology equivalence

(I_p') $\quad F_p'\mathbf{R} \otimes_{\mathbf{S}} B(\mathbf{S}, \mathbf{S}, S^0) \to F_p'\mathbf{R} \otimes_{\mathbf{S}} S^0$ is a homology equivalence.

The statements (I_p) prove Theorem 1.2′, by taking the filtered colimit over p, using that the inclusion $F_p\mathbf{R} \to \mathbf{R}$ is an isomorphism in gradings $\leq p$. It remains to prove these statements, which we shall do by induction on p. The key calculation will be the following:

Lemma 3.7′. *Let $k \geq 2$. The natural maps*

$$\mathrm{Sym}^a_*(S^d) \wedge \mathrm{Sym}^b_*(\mathrm{Sym}^k_*(S^d)) \longrightarrow \mathrm{Sym}^{a+kb}_*(S^d)$$

are rational homology equivalences, and are homotopy equivalences if $d = 1$.

Proof. For a well-pointed space X there are canonical isomorphisms

$$\widetilde{H}_*(\mathrm{Sym}^i_*(X); \mathbb{Q}) \cong \mathrm{Sym}^i(\widetilde{H}_*(X; \mathbb{Q})).$$

If d is odd this shows that $\mathrm{Sym}^i_*(S^d)$ has trivial rational homology for all $i \geq 2$; if d is even then it shows that the natural quotient map $S^{id} = (S^d)^{\wedge i} \to \mathrm{Sym}^i_*(S^d)$ is a rational homology equivalence.

Suppose first that d is odd. If $b = 0$ then the map is a homeomorphism, so suppose $b > 0$. As $k \geq 2$ it then follows that the domain and codomain both have trivial homology.

Suppose now that d is even. Then the natural quotient map from S^{ad+kbd} to both domain and codomain is a rational homology equivalence, so the map is a rational homology equivalence.

If $d = 1$ then $\mathrm{Sym}^i_*(S^1)$ is contractible for all $i \geq 2$. This follows by recognising it as the 1-point compactification of $\mathrm{Sym}^i(\mathbb{R})$, then using that points in $\mathbb{R}$ are canonically ordered to obtain a homeomorphism $\mathrm{Sym}^i(\mathbb{R}) \cong \mathbb{R} \times [0, \infty)^{i-1}$, whose 1-point compactification is $S^1 \wedge [0, \infty]^{\wedge i-1} \simeq *$. Using this we see that if $b > 0$ then the domain and codomain are both contractible; if $b = 0$ the map is a homeomorphism. □

Proposition 3.8′. *For each $p \geq 0$, the map $F'_p\mathbf{R} \to F_p\mathbf{R}$ is a rational homology equivalence; if* $\mathrm{rank}(L) = 1$*, it is moreover an integral homology equivalence.* $\mathrm{rank}(L) = 1$.

Proof. Fix some $n \geq 0$. We first record that the spaces $F'_p\mathbf{R}(n)$ and $F_p\mathbf{R}(n)$ are compact Hausdorff by Lemma 3.6′. They are also easily seen to be locally contractible, so we may replace singular (co)homology by sheaf cohomology in what follows.

We can remove the points at ∞ in the square (3.1) defining $F'_p\mathbf{R}(n)$, and equip everything in sight with the filtration $F^\#_\bullet(-)$ by closed subspaces, where $F^\#_q(-)$ is the subspace where there are $\leq q$ distinct points in M carrying labels. Let $S^\#_q(-)$ denote the q^{th} strata of these filtrations, i.e. the loci where there are precisely q points in M with labels. It suffices to show that the induced maps on strata induce isomorphisms on compactly-supported sheaf cohomology: the map of spectral sequences in compactly supported sheaf cohomology for these filtrations then gives the desired conclusion.

Having restricted to the q^{th} strata, everything fibres over the space $S^\#_q\mathrm{Sym}^n(M)$ by recording the underlying points. We claim that the comparison maps $S^\#_q F'_p(M) \to S^\#_q F_p(M)$ induce isomorphisms on compactly-supported sheaf cohomology over each fibre; this will prove the claim by the Leray–Serre spectral sequence.

We first consider the map $S^\#_q F'_0\mathbf{R}(n) \to S^\#_q F_0\mathbf{R}(n)$. The only nonempty fibres over $S^\#_q\mathrm{Sym}^n(M)$ are over arrangements $x = \sum_i c_i m_i$ where the m_i are pairwise distinct and the c_i are all even. On such fibres, the map identifies with

$$\prod_{i=1}^{q} \mathrm{Sym}^{b_i}(\mathrm{Sym}^2(L_{m_i})) \longrightarrow \prod_{i=1}^{q} \mathrm{Sym}^{2b_i}(L_{m_i}),$$

where we emphasise that $\mathrm{Sym}^i(L_{m_i})$ is the symmetric product of the topological space L_{m_i}, and has nothing to do with its vector space structure. This induces an isomorphism on compactly-supported (sheaf) cohomology by Lemma 3.7′.

For $p \geq 1$, we turn to the the map

$$S^\#_q F'_p\mathbf{R}(n) \longrightarrow S^\#_q F_p\mathbf{R}(n). \tag{3.2}$$

The vertical maps in the (restriction to q^{th} strata of the) square (3.1) are the identity on all fibres over $S^\#_q\mathrm{Sym}^n(M)$, except over the points $x = \sum_i c_i m_i \in C_q(M)$ where exactly p of the c_i are odd. Over such points, the map $S^\#_q F'_p\mathbf{R}(n) \longrightarrow S^\#_q F_p\mathbf{R}(n)$ takes the form

$$\prod_{i=1}^{q} \mathrm{Sym}^{a_i}(L_{m_i}) \times \mathrm{Sym}^{b_i}(\mathrm{Sym}^2(L_{m_i})) \longrightarrow \prod_{i=1}^{q} \mathrm{Sym}^{c_i}(L_{m_i}).$$

By Lemma 3.7′, this induces an isomorphism on compact-supported sheaf cohomology. □

Proposition 3.9′. *With rational homology, or integral homology if* $\mathrm{rank}(L) = 1$*:*

(i) The statement (I_0') holds.
(ii) For $p \geq 0$, (I_p') implies (I_p).
(iii) For $p \geq 1$, (I_{p-1}) implies (I_p').

Proof. Throughout, we take (co)homology with rational or integral coefficients as the case may be.

(i) As $F_0'\mathbf{R} = \mathbf{S}$ the map $F_0'\mathbf{R} \otimes_{\mathbf{S}} B(\mathbf{S}, \mathbf{S}, S^0) \longrightarrow F_0'\mathbf{R} \otimes_{\mathbf{S}} S^0$ is $B(\mathbf{S}, \mathbf{S}, S^0) \to S^0$ and is an equivalence.

(ii) Consider the square

$$\begin{tikzcd}
B(F_p'\mathbf{R}, \mathbf{S}, S^0) \arrow[r, equal] \arrow[d] & F_p'\mathbf{R} \otimes_{\mathbf{S}} B(\mathbf{S}, \mathbf{S}, S^0) \arrow[r] \arrow[d] & F_p'\mathbf{R} \otimes_{\mathbf{S}} S^0 \arrow[d] \\
B(F_p\mathbf{R}, \mathbf{S}, S^0) \arrow[r, equal] & F_p\mathbf{R} \otimes_{\mathbf{S}} B(\mathbf{S}, \mathbf{S}, S^0) \arrow[r] & F_p\mathbf{R} \otimes_{\mathbf{S}} S^0
\end{tikzcd}$$

induced by $F_p'\mathbf{R} \to F_p\mathbf{R}$. The top map is a homology equivalence by (I_p'). As $F_p'\mathbf{R} \to F_p\mathbf{R}$ is a homology equivalence by Proposition 3.8′, the bar spectral sequence shows that the left-hand map is a homology equivalence. We claim that the induced map $F_p'\mathbf{R} \otimes_{\mathbf{S}} S^0 \to F_p\mathbf{R} \otimes_{\mathbf{S}} S^0$ is an isomorphism. Granted this claim, commutativity of the square shows that the bottom map is a homology equivalence, proving (I_p).

The claim for $p = 0$ is immediate as both objects are S^0. To prove the claim for $p \geq 1$, apply $- \otimes_{\mathbf{S}} S^0$ to the pushout square defining $F_p'\mathbf{R}$, giving a pushout square

$$\begin{tikzcd}
F_{p-1}\mathbf{R}(p)[p] \arrow[r] \arrow[d] & F_{p-1}\mathbf{R} \otimes_{\mathbf{S}} S^0 \arrow[d] \\
\mathbf{R}(p)[p] \arrow[r] & F_p'\mathbf{R} \otimes_{\mathbf{S}} S^0
\end{tikzcd}$$

in $\mathsf{Top}_*^{\mathbb{N}}$. Now $F_{p-1}\mathbf{R} \otimes_{\mathbf{S}} S^0$ is trivial when evaluated in gradings $> p-1$, and agrees with $\mathbf{R} \otimes_{\mathbf{S}} S^0 = \mathbf{C}(M; L)$ in gradings $\leq p - 1$. This implies that

$$(F_p'\mathbf{R} \otimes_{\mathbf{S}} S^0)(n) = \begin{cases} (F_{p-1}\mathbf{R} \otimes_{\mathbf{S}} S^0)(n) = \mathbf{C}(M; L)(n) & n < p \\ \mathbf{R}(p)/F_{p-1}\mathbf{R}(p) = \mathbf{C}(M; L)(p) & n = p \\ * & n > p, \end{cases}$$

which is the same as $(F_p\mathbf{R} \otimes_{\mathbf{S}} S^0)(n)$, proving the claim.

(iii) Applying the map $- \otimes_{\mathbf{S}} B(\mathbf{S}, \mathbf{S}, S^0) \to - \otimes_{\mathbf{S}} S^0$ of colimit-preserving functors to the pushout square defining the right $\mathbf{S}$-module $F_p'\mathbf{R}$ gives a commutative cube

$$\begin{tikzcd}
F_{p-1}\mathbf{R}(p)[p] \otimes B(\mathbf{S}, \mathbf{S}, S^0) \arrow[rr] \arrow[dr] \arrow[dd] & & F_{p-1}\mathbf{R}(p)[p] \otimes S^0 \arrow[dr] \arrow[dd] & \\
& F_{p-1}\mathbf{R} \otimes_{\mathbf{S}} B(\mathbf{S}, \mathbf{S}, S^0) \arrow[rr] \arrow[dd] & & F_{p-1}\mathbf{R} \otimes_{\mathbf{S}} S^0 \arrow[dd] \\
\mathbf{R}(p)[p] \otimes B(\mathbf{S}, \mathbf{S}, S^0) \arrow[rr] \arrow[dr] & & \mathbf{R}(p)[p] \otimes S^0 \arrow[dr] & \\
& F_p'\mathbf{R} \otimes_{\mathbf{S}} B(\mathbf{S}, \mathbf{S}, S^0) \arrow[rr] & & F_p'\mathbf{R} \otimes_{\mathbf{S}} S^0
\end{tikzcd}$$

in $\mathsf{Top}_*^{\mathbb{N}}$, where the left and right faces are pushout squares. The back vertical maps are closed cofibrations by Lemma 3.1, so the left and right faces are in fact homotopy pushout squares. The back horizontal maps are both equivalences, as they are obtained by applying $F_{p-1}\mathbf{R}(p)[p] \otimes -$ and $\mathbf{R}(p)[p] \otimes -$ to an equivalence, and the front top horizontal map is a homology equivalence by (I_{p-1}). Since the left and right faces are homotopy pushouts, this implies that the front bottom horizontal map is also homology equivalence, proving (I'_p). $\square$

Remark 3.1. The integral result in Theorem 1.2′ is sharp in that the analogue for $\mathrm{rank}(L) = 2$ does not hold. For example, suppose the conclusion of that Theorem holds for $M = \mathbb{R}^2$ with the trivial rank 2 vector bundle $L \to M$. Fix an odd prime p and take $\mathbb{F}_p$ coefficients in the following.

One computes that $\widetilde{H}_*([(L \oplus L)^+]_{\mathfrak{S}_2}) = \Sigma^6\mathbb{F}_p$ and $\widetilde{H}_*(L^+) = \Sigma^4\mathbb{F}_p$; the latter implies that $\widetilde{H}_*(\mathrm{Sym}^2_*(L^+)) \simeq \Sigma^8\mathbb{F}_p$ since $2 \in \mathbb{F}_p^\times$. The map $\widetilde{C}_*([(L \oplus L)^+]_{\mathfrak{S}_2}) \to \widetilde{C}_*(\mathrm{Sym}^2_*(L^+))$ therefore factors (up to homotopy) through 0. Following Dold [Dol58, Section 7], we may identify $\widetilde{C}_*(\mathbf{Com}(L^+[1]))$ with the free simplicial commutative $\mathbb{F}_p$-algebra $\mathbb{L}\mathrm{Sym}^*(\widetilde{C}_*(L^+)[1])$, and hence see that the induced map $\widetilde{C}_*(\mathbf{R}) \to \widetilde{C}_*(\mathbf{S})$ factors (up to homotopy of maps of simplicial commutative $\mathbb{F}_p$-algebras) through $\mathbb{F}_p = \mathbb{L}\mathrm{Sym}^*(0)$. It follows that we may write

$$\widetilde{C}_*(\mathbf{C}(M; L); \mathbb{F}_p) \simeq (\mathbb{F}_p \otimes^{\mathbb{L}}_{\mathbb{L}\mathrm{Sym}^*(\Sigma^6\mathbb{F}_p[2])} \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{L}\mathrm{Sym}^*(\Sigma^4\mathbb{F}_p[1]),$$

where we have implicitly chosen formality equivalences for the chains. In particular, $\mathbb{L}\mathrm{Sym}^*(\Sigma^4\mathbb{F}_p[1])$ is a homotopy retract of $\widetilde{C}_*(\mathbf{C}(M; L); \mathbb{F}_p)$. But one computes that $H_{2p+2}(\mathbb{L}\mathrm{Sym}^p(\Sigma^4\mathbb{F}_p)) \neq 0$ (for example by considering the homology of $\mathrm{Sym}^p_*(S^4)$ [Nak61, Theorem 6.7], or applying [Bou, Theorem 8.11]), so also

$$H^{2p-2}(C_p(\mathbb{R}^2); \mathbb{F}_p) \simeq H_{2p+2}(C_p(\mathbb{R}^2; L)^+; \mathbb{F}_p) \neq 0,$$

where the first isomorphism is an application of Poincaré duality (as in [RW24, §2.1]). This contradicts the theorem of Arnol'd that $H^i(C_n(\mathbb{R}^2); \mathbb{Z}) = 0$ for $i \geq n$ [Arn70].

4. Analogous corrections to Appendix A

The same oversight arises in the claimed equivalence

$$\text{(A.2)} \quad \mathbf{Com}(\vee_{i=1}^m L^+[1_i]) \otimes^{\mathbb{L}}_{\mathbf{Com}([(L^{\oplus mk})^+]_{\mathfrak{S}_k^m}[k,\ldots,k])} S^0[0, \ldots, 0] \longrightarrow \mathbf{Z}^{m,k}(M; L),$$

which is indeed an equivalence when L is the trivial rank 0 vector bundle by the argument in the published paper, but when L has strictly positive rank has to be analysed by an argument similar to that of the previous section. However in this case the conclusion is not quite as general as claimed in the published paper, even rationally.

Theorem 4.1′. *The map (A.2) is a rational homology equivalence if* $\mathrm{rank}(L)$ *is even, or if* $\mathrm{rank}(L)$ *is odd and* $k \geq 2$*; furthermore it is an integral homology equivalence if* $\mathrm{rank}(L) = 0$*, or if* $\mathrm{rank}(L) = 1$ *and* $m = 1$*.*

The argument we will give below fails in the case $\mathrm{rank}(L)$ odd and $k = 1$, even rationally. This is just as well, as the claim is false in this case:

Counterexample. Consider $Z^1_{1,2}(\mathbb{R}^d)$ with d odd. This is the space of one red and two blue points in $\mathbb{R}^d$, where the blue points may collide with each other but not with the red point. Recording the position of the red point provides a fibre sequence $\mathrm{Sym}^2(\mathbb{R}^d \setminus 0) \to Z_{1,2}(\mathbb{R}^d) \to \mathbb{R}^d$. In particular, we have homotopy equivalences $Z^1_{1,2} \simeq \mathrm{Sym}^2(\mathbb{R}^d \setminus \{0\}) \simeq \mathrm{Sym}^2(S^{d-1})$ and so $H_*(Z^1_{1,2}(\mathbb{R}^d)) = \mathbb{Q}[0] \oplus \mathbb{Q}[d-1] \oplus \mathbb{Q}[2(d-1)]$.

On the other hand, if (A.2) were a rational homology equivalence with $L \to \mathbb{R}^d$ the trivial d-dimensional bundle then, using that the map $S^{3d} = (L^{\oplus 2})^+ \to$

$\mathbf{Com}(L^+[1_1] \vee L^+[1_2])(1,1) = S^{2d} \wedge S^{2d}$ is nullhomotopic, the bar spectral sequence would collapse to give

$$H_{*,*}(\mathbf{Z}^{2,1}(\mathbb{R}^d; L) = \mathbb{Q}[r_{2d,(1,0)}, b_{2d,(0,1)}] \otimes \Lambda[x_{3d+1,(1,1)}].$$

Then in bigrading $(1,2)$ we would have

$$H^{6d-*}(Z^1_{1,2}(\mathbb{R}^d)) \cong H_{*,(1,2)}(\mathbf{Z}^{2,1}(\mathbb{R}^d; L)) = \mathbb{Q}\{bx, rb^2\} = \mathbb{Q}[5d+1] \oplus \mathbb{Q}[6d],$$

which would give $H_*(Z^1_{1,2}(\mathbb{R}^d)) = \mathbb{Q}[0] \oplus \mathbb{Q}[d-1]$, contradicting the above. □

This means that all claims in Appendix A about the rational homology of $Z^k_{n_1,\ldots,n_m}(M)$ for $k=1$ and odd-dimensional M should be withdrawn. In particular, the discussion in Section A.2.3 should be withdrawn. Other than this, the applications discussed in Section A.2 are in rational homology, so are unchanged; the applications discussed in Section A.3 treat the case L has rank 0, so are unchanged.

Remark 4.1. As recorded in Section A.2.1, it is still the case that [FWW19, Theorem 1.4] is false in the case $n = 1$ (corresponding to our $k = 1$). This is because [FWW19, Theorem 6.1 1.] is false in this case, which follows e.g. from the calculation in the counterexample above: the map $\mathrm{Sym}^2(S^{d-1}) \simeq Z^1_{1,2}(\mathbb{R}^d) \to \mathrm{Sym}^1(\mathbb{R}^d) \times \mathrm{Sym}^2(\mathbb{R}^d) \simeq *$ is not a rational homology equivalence. Alternatively, it also follows from the fact that $\mathbf{Com}(\vee_{i=1}^m L^+[1_i]) \to \mathbf{Z}^{m,1}(M; L)$ factors through the map (A.2) so cannot (when M has nontrivial cup products) induce an isomorphism on rational homology.

To prove Theorem 4.1′, we use the $\mathbb{N}^m$-graded topological commutative monoids

$$\mathbf{S} := \mathbf{Com}([(L^{\oplus mk})^+]_{\mathfrak{S}^m_k}[k, k, \ldots, k]) \text{ and } \mathbf{R} := \mathbf{Com}(\vee_{i=1}^m L^+[1_i])$$

and the natural map $\mathbf{S} \to \mathbf{R}$ between them. We think of elements of $\mathbf{R}(n_1, \ldots, n_m)$ as configurations in M labelled by L, where the configurations have m colours and there are n_i points having colour i. We define a filtration $F_\bullet\mathbf{R}$ of $\mathbf{R}$ by letting $F_p\mathbf{R}$ be the right $\mathbf{S}$-module given by the image of the multiplication map

$$\bigvee_{\sum n_i \leq p} \mathbf{R}(n_1, \ldots, n_m)[n_1, \ldots, n_m] \otimes \mathbf{S} \longrightarrow \mathbf{R}.$$

In particular, $F_0\mathbf{R} = \mathrm{im}(\mathbf{S} \to \mathbf{R})$ as before. For each $p \geq 1$, the relevant square then takes the form

$$\begin{array}{ccc} \bigvee_{\sum n_i = p} F_{p-1}\mathbf{R}(n_1, \ldots, n_m)[n_1, \ldots, n_m] \otimes \mathbf{S} & \longrightarrow & F_{p-1}\mathbf{R} \\ \downarrow & & \downarrow \\ \bigvee_{\sum n_i = p} \mathbf{R}(n_1, \ldots, n_m)[n_1, \ldots, n_m] \otimes \mathbf{S} & \longrightarrow & F_p\mathbf{R} \end{array} \tag{4.1}$$

and we write $F'_p\mathbf{R} \to F_p\mathbf{R}$ for the induced map from the pushout. If $p = 0$, we put $F'_p\mathbf{R} = \mathbf{S}$. As before, the map $F'_p\mathbf{R} \to F_p\mathbf{R}$ is a homeomorphism if $\mathrm{rank}(L) = 0$ and then the argument below simplifies, but it is not a homeomorphism if $\mathrm{rank}(L) > 0$. The analogue of Lemma 3.6′ holds for these spaces, by the same kind of argument, as does the analogue of Proposition 3.9′, given that of Proposition 3.8′.

Let us now prove the analogue of Proposition 3.8′. Let us write $\underline{n} = (n_1, \ldots, n_m)$, and write arrangements as $\sum_i \underline{b}_i m_i$ where $(\underline{b}_i)_j$ denotes the number of times the point m_i appears with colour j. We again use the filtration $F^\#_q(-)$ and, having restricted to q^{th} strata, our objects now fibre over $S^\#_q\mathrm{Sym}^{n_1,\ldots,n_m}(M)$. First consider the map $S^\#_q F'_0\mathbf{R}(\underline{n}) \to S^\#_q F_0\mathbf{R}(\underline{n})$. The only nonempty fibres over $S^\#_q\mathrm{Sym}^{n_1,\ldots,n_m}(M)$ are over arrangements $\underline{x} = k\sum_i \underline{b}_i m_i$ where the m_i are pairwise distinct. On such fibres it is the map

$$\prod_{i=1}^{q} \mathrm{Sym}^{b_i}(\mathrm{Sym}^k(L_{m_i})^m) \longrightarrow \prod_{i=1}^{q} \mathrm{Sym}^{b_i k}(L_{m_i})^m.$$

Similarly, for $p \geq 1$, the vertical maps in the (restriction to q^{th} strata of the) square (4.1) are the identity on all fibres, except those of the form $\underline{x} = \sum_i(\underline{a}_i + k\underline{b}_i)m_i$ where $\sum_i (a_i)_j = p$ and for each i, there is j with $(a_i)_j < k$; these are the points which do not come from F_{p-1}. Over such points, the map on fibres is the map

$$\prod_{i=1}^{q}\left(\prod_{j=1}^{m} \mathrm{Sym}^{(\underline{a}_i)_j}(L_{m_i})\right) \times \mathrm{Sym}^{b_i}(\mathrm{Sym}^k(L_{m_i})^m) \longrightarrow \prod_{i=1}^{q}\prod_{j=1}^{m} \mathrm{Sym}^{(\underline{c}_i)_j}(L_{m_i}).$$

In either case, it follows from Lemma 4.2′ below—the analogue of Lemma 3.7′—that these maps induce isomorphisms on compactly-supported rational (or integral) sheaf cohomology under the stated conditions on $\mathrm{rank}(L)$, k, and m, and hence by the Leray–Serre spectral sequence on compactly-supported sheaf cohomology that $S_q^{\#}F_p'\mathbf{R}(\underline{n}) \to S_q^{\#}F_p\mathbf{R}(\underline{n})$ induces an isomorphism in compactly-supported sheaf cohomology, and hence by the spectral sequence of the stratification that $F_p'\mathbf{R} \to F_p\mathbf{R}$ induces an isomorphism on sheaf cohomology too. These spaces are again locally contractible, so this map induces an isomorphism on singular (co)homology too. This concludes the proof of (i) in the analogue of Proposition 3.9′.

Lemma 4.2′. *Let $k \geq 1$, $m \geq 1$, and $km \geq 2$. The natural maps*

$$\left(\bigwedge_{i=1}^{m} \mathrm{Sym}_*^{a_i}(S^d)\right) \wedge \mathrm{Sym}_*^{b}(\mathrm{Sym}_*^k(S^d)^{\wedge m}) \longrightarrow \bigwedge_{i=1}^{m} \mathrm{Sym}_*^{a_i+kb}(S^d)$$

are rational homology equivalences if any of the following hold:

(i) d is even;

(ii) d is odd and $k \geq 2$.

These maps are integral homology equivalences if $d = 1$ and $m = 1$.

Proof. We reuse the calculations from the proof of Lemma 3.7′.

Let us do the integral case first, so $d = 1$ and $m = 1$, and hence $k \geq 2$. Now $\mathrm{Sym}_*^i(S^1) \simeq *$ for $i \geq 2$, so if $b > 0$ then the domain and codomain are both contractible; if $b = 0$ then the map is a homeomorphism.

For the rational case, if d is even then the natural quotient maps from the $((bk + \sum a_i)dm)$-sphere to the domain and codomain are rational homology equivalences, so the map is too.

If d is odd and $k \geq 2$, first observe that if $b = 0$ then the map is a homeomorphism; if $b > 0$ then using the fact that $\mathrm{Sym}_*^i(S^d)$ has trivial rational homology for all $i \geq 2$, it follows that the domain and codomain both have trivial rational homology. □

If d is odd and $k = 1$ then this map is not a rational homology equivalence, in line with the Counterexample.

References

[Arn70] V. I. Arnold, *Certain topological invariants of algebraic functions*, Trudy Moskov. Mat. Obšč. **21** (1970), 27–46.

[Bou] A.K. Bousfield, *Operations on derived functors of non-additive functors*, Unpublished.

[Dol58] A. Dold, *Homology of symmetric products and other functors of complexes*, Ann. of Math. (2) **68** (1958), 54–80.

[Dug66] J. Dugundji, *Topology*, Allyn and Bacon, Inc., Boston, MA, 1966.

[FWW19] B. Farb, J. Wolfson, and M. M. Wood, *Coincidences between homological densities, predicted by arithmetic*, Adv. Math. **352** (2019), 670–716.

[JTTW63] I. M. James, Emery Thomas, H. Toda, and G. W. Whitehead, *On the symmetric square of a sphere*, J. Math. Mech. **12** (1963), 771–776.

[Nak61] M. Nakaoka, *Homology of the infinite symmetric group*, Ann. of Math. (2) **73** (1961), 229–257.

[RW24] O. Randal-Williams, *Configuration spaces as commutative monoids*, Bull. Lond. Math. Soc. **56** (2024), no. 9, 2847–2862, With an appendix by Quoc P. Ho and Randal-Williams.

Email address: `samuel.moore@maths.ox.ac.uk`

Mathematical Institute, Woodstock Road, Oxford OX2 6GG, UK

Email address: `o.randal-williams@dpmms.cam.ac.uk`

Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WB, UK