A FIBRE BUNDLE OF SIGNATURE 4

OSCAR RANDAL-WILLIAMS

In [2], Chern, Hirzebruch and Serre studied the question of multiplicativity of the signature of fibre bundles. That is, given a fibre bundle

$$F \longrightarrow E \xrightarrow{\pi} B$$

consisting of oriented compact manifolds, when is $\sigma(E) = \sigma(B)\cdot\sigma(F)$? They showed that this multiplicativity does hold if $\pi_1(B)$ acts trivially on $H^*(F;\mathbb{Q})$, but later Kodaira [7], Atiyah [1], and Hirzebruch [6] produced examples of fibre bundles for which the signature is not multiplicative. On the other hand, as the signature and Euler characteristic agree modulo 2, and the latter is multiplicative, it is true that $\sigma(E) \equiv \sigma(B)\cdot\sigma(F) \mod 2$ for all such fibre bundles. More recently, Hambleton, Korzeniewski, and Ranicki [5, Theorem A] proved that $\sigma(E) \equiv \sigma(B)\cdot\sigma(F) \mod 4$ for all such fibre bundles.

It is interesting to ask whether multiplicativity holds modulo larger powers of 2. This has been recently taken up by C. Rovi [10], who has shown that the defect $\frac{\sigma(E)-\sigma(B)\cdot\sigma(F)}{4}$ taken modulo 2 can be expressed as the Arf invariant of a quadratic form given in terms of the Pontrjagin square (this was earlier related to the signature by Morita [9]). Meyer [8] has shown that there are surface bundles over surfaces having signature 4, so Rovi's invariant is nontrivial; in this note we show there are also higher-dimensional examples.

Let B^8 be a Bott manifold of signature 0, that is, a Spin manifold of signature 0 having $\hat{A}$-genus 1; recall that B^8 and $\mathbb{HP}^2$ form a basis for 8th Spin cobordism.

Theorem 0.1. *For each element of the subgroup $\mathbb{Z}\{B^8, 4\mathbb{HP}^2\} \subset \Omega_8^{Spin}$ and for each $g \geq 7$, there is a smooth oriented fibre bundle*

$$\#^g S^3 \times S^3 \longrightarrow E^8 \xrightarrow{\pi} \Sigma^2 \tag{0.1}$$

over an orientable surface and a Spin structure $\mathfrak{s}$ on E, with $(E,\mathfrak{s})$ in the nominated Spin cobordism class.

In particular, there is such a fibre bundle having $[E] = 4[\mathbb{HP}^2]$, and so having signature 4.

The first step is the theorem of Galatius and the author [3, Theorem 1.2] describing the homology of the classifying spaces $B\mathrm{Diff}(\#^g S^3 \times S^3, D^6)$ of the group of diffeomorphisms of the manifold $\#^g S^3 \times S^3$ which fix a disc pointwise. The answer is given in terms of the infinite loop space of the Thom spectrum

$$\mathrm{MTSpin}(6) := \mathrm{Th}(-\gamma_6 \to B\mathrm{Spin}(6)),$$

and is the statement that the Pontrjagin–Thom map

$$\alpha_g : B\mathrm{Diff}(\#^g S^3 \times S^3, D^6) \longrightarrow \Omega_0^\infty \mathrm{MTSpin}(6)$$

induces an isomorphism on homology in degrees $* \leq \frac{g-3}{2}$. In particular, as long as $g \geq 7$ this map induces an isomorphism on first and second homology.

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If F denotes the homotopy fibre of the map of spectra $\mathrm{MTSpin}(6) \to \Sigma^{-6}\mathrm{MSpin}$, then Galatius and the author [4, §5.1] have shown that F is connective, that $\pi_0(F) = \mathbb{Z}$ and $\pi_1(F) = \mathbb{Z}/4$, and that the long exact sequence of homotopy groups is as shown below:

$$\pi_2(\mathrm{MTSpin}(6)) \to \Omega_8^{Spin} = \mathbb{Z}\{B^8, \mathbb{HP}^2\} \xrightarrow{\phi} \mathbb{Z}/4 \xrightarrow{0} \pi_1(\mathrm{MTSpin}(6)) \to \Omega_7^{Spin} = 0.$$

This shows that $\Omega_0^\infty \mathrm{MTSpin}(6)$ is simply-connected, and so by the Hurewicz theorem and the theorem above we have isomorphisms

$$\pi_2(\Omega_0^\infty \mathrm{MTSpin}(6)) \longrightarrow H_2(\Omega_0^\infty \mathrm{MTSpin}(6);\mathbb{Z}) \xleftarrow{(\alpha_g)_*} H_2(B\mathrm{Diff}(\#^g S^3 \times S^3, D^6);\mathbb{Z})$$

for any $g \geq 7$.

Now we use the theorem of Hambleton, Korzeniewski, and Ranicki [5, Theorem A] that the signature of fibre bundles is multiplicative modulo 4, so any fibre bundle as in (0.1) must have $\mathrm{sign}(E) \equiv 0 \mod 4$. It follows that $\mathrm{Ker}(\phi)$ must consist of manifolds of signature zero modulo 4, and the only way this can happen is if $\phi(B^8) = 0$ and $\phi(\mathbb{HP}^2) = \pm 1$, whence $\mathrm{Ker}(\phi) = \mathbb{Z}\{B^8, 4\mathbb{HP}^2\}$. Any element of this subgroup lifts to $\pi_2(\mathrm{MTSpin}(6))$ and hence to $H_2(B\mathrm{Diff}(\#^g S^3 \times S^3, D^6);\mathbb{Z})$; as any second homology class is represented by a surface, it follows that the required smooth fibre bundles exist.

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E-mail address: o.randal-williams@dpmms.cam.ac.uk

Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WB, UK