ERRATUM TO: STABLE MODULI SPACES OF HIGH DIMENSIONAL MANIFOLDS

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Abstract. We are grateful to Nathan Perlmutter for pointing out a gap in the proof of Theorem 5.14 given in Section 6.5 of the paper mentioned in the title. We fill in this gap.

In the proof of [GRW14b, Proposition 6.19] it is necessary that the "half Whitney trick" used at the top of p. 345 to cancel an intersection point between cores of e_i and e_k^j by deforming the embedding e_i does not create new intersection points between e_i and other embeddings. In order to achieve this, it suffices to pick the "half Whitney discs" labelled T in the proof of Proposition 6.19 to be disjoint from all other cores, which is easy to achieve if cores of all the embeddings e_k^j are pairwise transverse, but may not be possible to achieve in general.

With this remark, the proof of [GRW14b, Proposition 6.19] is valid if the vertices $v_1, \ldots, v_k \in \widetilde{Y}_0(a, \varepsilon, (W, \ell_W))$ are assumed to be pairwise transverse. Without this assumption it is unclear whether the conclusion of Proposition 6.19 holds and hence whether assumption (iii) of [GRW14b, Theorem 6.2] holds for the augmented bi-semi-simplicial space

$$\widetilde{D}_{\theta,L}^{n-1,\mathcal{A}}(\mathbb{R}^N)_{p,\bullet} \longrightarrow D_{\theta,L}^{n-1,n-2}(\mathbb{R}^N)_p, \tag{0.1}$$

and hence the published paper does not contain a complete proof that the induced map $|\widetilde{D}_{\theta,L}^{n-1,\mathcal{A}}(\mathbb{R}^N)_{p,\bullet}| \to D_{\theta,L}^{n-1,n-2}(\mathbb{R}^N)_p$ is a weak equivalence. To supply one, it seems best to apply [GRW14b, Theorem 6.2] in a two-step argument, instead of attempting to apply it directly to (0.1). Since the simplicial degree p, as well as the data n, $\mathcal{A}$, θ, L and N shall be completely fixed throughout the proof, we shall change notation as follows.

Definition 0.1. Let $\widetilde{D}_{\theta,L}^{n-1,\mathcal{A}}(\mathbb{R}^N)_{p,q}$ be as defined in the paper, fix p and write

$$E_{-1} = D_{\theta,L}^{n-1,n-2}(\mathbb{R}^N)_p \quad \text{and} \quad E_0 = \widetilde{D}_{\theta,L}^{n-1,\mathcal{A}}(\mathbb{R}^N)_{p,0},$$

and let $E_{\bullet,\bullet}$ be the (bi-)augmented bisemisimplicial space defined by

$$E_{s,t} \subset E_0 \times_{E_{-1}} \cdots \times_{E_{-1}} E_0,$$

the subspace of the $(s+t+2)$-fold fiber product consisting of $(v_0, \ldots, v_s, w_0, \ldots, w_t)$ satisfying

- (i) the cores of v_i and v_j are transverse for all $i \neq j$,
- (ii) the cores of v_i and w_j are disjoint for all i and j,
- (iii) the cores of w_i and w_j are disjoint for all $i \neq j$.

The map (0.1) may then be identified with $E_{-1,\bullet} \to E_{-1,-1}$. We claim that [GRW14b, Theorem 6.2] applies to both augmented semisimplicial spaces

$$E_{\bullet,t} \longrightarrow E_{-1,t} \quad \text{for } t \geq -1, \tag{0.2}$$

$$E_{s,\bullet} \longrightarrow E_{s,-1} \quad \text{for } s \geq 0. \tag{0.3}$$

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Granted this claim, [GRW14b, Theorem 6.2] gives weak equivalences $|E_{\bullet,t}| \to E_{-1,t}$ for any $t \geq -1$ and $|E_{s,\bullet}| \to E_{s,-1}$ for any $s \geq 0$. The first of these specialises to $|E_{\bullet,-1}| \to E_{-1,-1}$ and geometrically realises to $|E_{\bullet,\bullet}| \to |E_{-1,\bullet}|$ and the second geometrically realises to $|E_{\bullet,\bullet}| \to |E_{\bullet,-1}|$. These three weak equivalences fit into a commutative diagram

$$\begin{tikzcd} |E_{\bullet,\bullet}| \arrow[r, "\simeq"] \arrow[d, "\simeq"] & |E_{\bullet,-1}| \arrow[d, "\simeq"] \\ |E_{-1,\bullet}| \arrow[r] & E_{-1,-1}, \end{tikzcd}$$

from which it follows that the bottom horizontal map is also a weak equivalence, which proves [GRW14b, Theorem 5.14].

To see that [GRW14b, Theorem 6.2] applies to (0.2) and (0.3) we first note that both $E_{0,t} \to E_{-1,t}$ and $E_{s,0} \to E_{s,-1}$ have local lifts of any map from a disc, as in the proof of [GRW14b, Proposition 6.10].

Proof that [GRW14b, Theorem 6.2] *applies to* (0.2). Exactly as in the published paper, replacing "disjoint" by "transverse". In [GRW14b, Proposition 6.19] we then need to prove that for any finite collection $\{v_1, \ldots, v_k\}$ of surgery data we may produce another one which is transverse to all $v_1, \ldots, v_k$. For $k = 0$ and $t = -1$ this is produced by Lemma 6.21 and Lemma 6.22 in the published paper, for $k > 0$ it is produced by perturbing v_1, and for $k = 0$ and $t \geq 0$ it is produced by perturbing w_0, using that transversality is an open and dense relation. $\square$

Proof that [GRW14b, Theorem 6.2] *applies to* (0.3). This uses similar techniques to the proofs in the published version. We must explain how to find, given surgery data $v_0, \ldots, v_s$ which have pairwise transverse cores, and a finite collection of surgery data $\{w_1, \ldots, w_k\}$ such that v_i and w_j have disjoint cores for all j (but no relationship imposed among the w_j), a further surgery datum w which is disjoint from all of $v_0, \ldots, v_s, w_1, \ldots, w_k$.

We only need to do this when $s \geq 0$ so we may start with v_0, which is disjoint from all w_j, and first perturb it to a w' which is transverse to v_0, remains transverse to all other v_i, and remains disjoint from all w_j. We then use the "half Whitney trick" to make w' disjoint from $v_0, \ldots, v_s$. Since these are mutually transverse, and disjoint from all w_j, and since the dimension of the ambient manifold is at least 6, there is no obstruction to picking the half Whitney discs to be disjoint from all other cores. In finitely many steps we have isotoped w' to a surgery datum w whose core is disjoint from all $v_0, \ldots, v_s$ and $w_1, \ldots, w_k$. $\square$

Remark. See [GRW14a, Remark 7.13] for a minor additional correction to [GRW14b].

References

[GRW14a] Søren Galatius and Oscar Randal-Williams, *Homological stability for moduli spaces of high dimensional manifolds. I*, arXiv:1403.2334, 2014.

[GRW14b] ——, *Stable moduli spaces of high-dimensional manifolds*, Acta Math. **212** (2014), no. 2, 257–377.

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